THE ARTIN-TATE PAIRING ON THE BRAUER GROUP OF A SURFACE

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ABSTRACT. There is a canonical pairing on the Brauer group of a surface over a finite field, and an old conjecture of Tate predicts that this pairing is alternating. In this talk I will present a resolution to Tate's conjecture. The key new ingredient is a circle of ideas originating in algebraic topology, centered around the Steenrod operations. The talk will advertise these new tools (while assuming minimal background in algebraic topology).

CONTENTS

1. INTRODUCTION

1.1. **Tate's question.** Let X be a smooth, projective, geometrically connected surface over $\mathbf{F}_q$. Let ℓ be a prime distinct from the characteristic of $\mathbf{F}_q$. Artin and Tate introduced a skew-symmetric pairing $\langle \cdot, \cdot \rangle_{\mathrm{AT}}$ on $\mathrm{Br}(X)[\ell^\infty]$, which I will define a little later. I call this the *Artin-Tate pairing.* The question that concerns us is very easy to state:

Conjecture 1.1 (Tate, 1966)**.** *The pairing* $\langle \cdot, \cdot \rangle_{\mathrm{AT}}$ *is alternating.*

Let me remind you that skew-symmetric means

$$\langle x, y \rangle_{\mathrm{AT}} + \langle y, x \rangle_{\mathrm{AT}} = 0$$

while alternating means

$$\langle x, x \rangle_{\mathrm{AT}} = 0.$$

They are equivalent if 2 is invertible, but otherwise the second condition is stronger.

The purpose of this talk is to present an answer to Conjecture 1.1. But before getting that I want to provide a little context and motivation for the problem.

Notes for 60 minute talk on [?].

1.2. **Context.** The motivation for studying $\mathrm{Br}(X)$ and the pairing actually comes from the Birch and Swinnerton-Dyer Conjecture, through a "geometric analogue" of BSD which was found by Artin and Tate [**?**], and is now called the *Artin-Tate Conjecture*.

The basic idea is that there is a dictionary between the invariants appearing the BSD Conjecture and certain invariants of a smooth, projective, geometrically connected surface X over $\mathbf{F}_q$. The Artin-Tate Conjecture is the translation of the BSD Conjecture under this dictionary.

BSD Conjecture	**Artin-Tate Conjecture**
$L(E,s)$	$\zeta(X,s)$
$E(K)$	$\mathrm{NS}(X)$
$Ш(E)$	$\mathrm{Br}(X)$
Cassels-Tate pairing	Artin-Tate pairing

In fact, the two conjectures are equivalent in many cases! More precisely, work of Milne and Gordon shows (as originally conjectured by Tate) that for a surface which admits a curve fibration $f\colon X \to C$, the Artin-Tate Conjecture for X is equivalent to the BSD Conjecture for the Jacobian of the generic fiber of f.

As you might guess from the analogy, $\mathrm{Br}(X)$ should be finite but we do not know how to prove it.[1] We'll set $\mathrm{Br}(X)_{\mathrm{nd}}$ to be the maximal non-divisible quotient of $\mathrm{Br}(X)$; they are conjecturally equal but $\mathrm{Br}(X)_{\mathrm{nd}}$ is unconditionally finite. The Artin-Tate pairing is constructed as a non-degenerate skew-symmetric pairing on $\mathrm{Br}(X)_{\mathrm{nd}}$.

1.3. **Some amusing history.** Tate motivates Conjecture 1.1 by stating in [**?**] that the Cassels-Tate pairing is alternating for Jacobians. This sounds pretty compelling, except for one critical flaw: the premise is false! Cassels had proved that it was alternating for elliptic curves, and Tate had generalized the alternating property to an abelian variety with principal polarization arising from a line bundle (over the ground field). Every Jacobian is canonically principally polarized, but over a general field this polarization actually need not arise from a line bundle. In fact, Poonen and Stoll eventually showed, more than 30 years later in [**?**], that the Cassels-Tate pairing need not be alternating even for Jacobians of dimension $g \geq 2$.

The history of Conjecture 1.1 is perhaps even more tortuous than that of the analogous question for the Cassels-Tate pairing. In the 1970's Manin exhibited a surface X with $\mathrm{Br}(X)_{\mathrm{nd}} = \mathbf{Z}/2\mathbf{Z}$. Since $\mathbf{Z}/2\mathbf{Z}$ cannot support a nondegenerating alternating pairing, it seems to disprove Conjecture 1.1! In fact, it is an elementary algebra exercise to check that a finite abelian group with a nondegenerate alternating pairing *must have square order.* (This is analogous to fact that a finite-dimensional vector space with a nondegenerate alternating pairing has even dimension.)

But in 1997 Urabe found mistakes in Manin's calculations. Moreover, Urabe proved [**?**] that $\#\,\mathrm{Br}(X)_{\mathrm{nd}}$ actually *is* always a perfect square![2] A similar result was proved in 2005 (using different methods) by Liu-Lorenzini-Raynaud [**?**], who showed that if $\mathrm{Br}(X)$ is finite then $\#\,\mathrm{Br}(X)$ is a perfect square.

1.4. **Results.** Our main theorem finally brings closure to this rather troubled past:

Theorem 1.2 (F). *Conjecture 1.1 is correct.*

[1]The finiteness is known to imply both the Artin-Tate Conjecture and the BSD Conjecture for Jacobians over function fields.

[2]Poonen and Stoll also showed that the analogous statement is false for $Ш_{\mathrm{nd}}$.

The rest of the talk will be concerned with the proof of Theorem 1.2. However, let me mention at the outset that proof rests on tools coming from *homotopy theory.*

2. Pairings on the cohomology of a surface

2.1. **Notation.** For an abelian group G, we write G_{nd} for the maximal non-divisible quotient of G, which is the quotient of G by its largest divisible subgroup.

Let X be a smooth, projective, geometrically connected surface over $\mathbf{F}_q$. Let $\ell \neq \operatorname{char} \mathbf{F}_q$ be a prime.

2.2. **The Brauer group.** We define the *Brauer group* of X to be $\mathrm{Br}(X) := H^2_{\text{ét}}(X; \mathbf{G}_m)$. We will define a non-degenerate pairing on $\mathrm{Br}(X)_{\mathrm{nd}}[\ell^\infty]$.

Lemma 2.1. *We have an isomorphism*

$$\mathrm{Br}(X)_{\mathrm{nd}}[\ell^\infty] \xleftarrow{\sim} H^2_{\text{ét}}(X, \mathbf{Q}_\ell/\mathbf{Z}_\ell(1))_{\mathrm{nd}}.$$

Proof. Exercise using the Kummer sequence. □

From the short exact sequence

$$0 \to \mathbf{Z}_\ell(1) \to \mathbf{Q}_\ell(1) \to \mathbf{Q}_\ell(1)/\mathbf{Z}_\ell(1) \to 0$$

we obtain a boundary map

$$\widetilde{\delta} \colon H^2_{\text{ét}}(X, \mathbf{Q}_\ell/\mathbf{Z}_\ell(1)) \xrightarrow{\sim} H^3_{\text{ét}}(X, \mathbf{Z}_\ell(1)).$$

Lemma 2.2. *The map $\widetilde{\delta}$ induces an isomorphism*

$$\widetilde{\delta} \colon H^2_{\text{ét}}(X, \mathbf{Q}_\ell/\mathbf{Z}_\ell(1))_{\mathrm{nd}} \xrightarrow{\sim} H^3_{\text{ét}}(X, \mathbf{Z}_\ell(1))_{\mathrm{tors}}.$$

Proof. Exercise in chasing long exact sequences. □

2.3. **Poincaré duality for étale cohomology.** Note that I am really using the *absolute étale cohomology* $H^*_{\text{ét}}(X)$. This is to be contrasted with *geometric étale cohomology* $H^*_{\text{ét}}(X_{\overline{\mathbf{F}}_q})$. It may be helpful to comment on what these "look like".

The geometric étale cohomology of a variety is meant to look like the singular cohomology of the analogous complex manifold. For instance, the geometric étale cohomology of a smooth projective surface should behave like a the singular cohomology of a compact orientable 4-manifold: in particular, it should enjoy a Poincaré duality of dimension 4.

On the other hand, absolute étale cohomology blends geometric étale cohomology and Galois cohomology of the ground field. In general, it does not look like the singular cohomology of any manifold. However, something special happens for $\mathbf{F}_q$, which is that $\mathrm{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q) \cong \widehat{\mathbf{Z}}$ looks very similar to $\pi_1(S^1) = \mathbf{Z}$. The upshot is that $H^*(\mathrm{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q))$ has a Poincaré duality of dimension 1. Hence the absolute étale cohomology $H^*_{\text{ét}}(X_{\mathbf{F}_q})$ has a Poincaré duality of dimension 5. More precisely, we have a fundamental class in $H^5_{\text{ét}}(X; \mathbf{Z}_\ell(2))$ which gives an isomorphism

$$\int \colon H^5_{\text{ét}}(X; \mathbf{Z}_\ell(2)) \xrightarrow{\sim} \mathbf{Z}_\ell.$$

The cup product then induces a perfect pairing

$$H^i_{\text{ét}}(X; \mathbf{Z}_\ell(1)) \times H^{5-i}_{\text{ét}}(X; \mathbf{Z}_\ell(1)) \to H^5_{\text{ét}}(X; \mathbf{Z}_\ell(2)) \xrightarrow{\sim} \mathbf{Z}_\ell.$$

2.4. **The Artin-Tate pairing.** We can now define the pairing $\langle\cdot,\cdot\rangle_{\mathrm{AT}}$ on $\mathrm{Br}(X)_{\mathrm{nd}}[\ell^\infty] = H^2_{\text{ét}}(X, \mathbf{Q}_\ell/\mathbf{Z}_\ell(1))_{\mathrm{nd}}$.

Definition 2.3. For $x, x' \in H^2_{\text{ét}}(X, \mathbf{Q}_\ell/\mathbf{Z}_\ell(1))$, we define

$$\langle x, x'\rangle_{\mathrm{AT}} := \int_X x \smile (\beta x').$$

Remark 2.4. The only property that we used to make this definition is a 5-dimensional Poincaré duality. So an analogous definition could have been made for a 5-dimensional orientable closed manifold. That is called the *linking form*, and in fact it plays an important role in the classification of 5-manifolds.

The analogy between the linking form on 5-manifolds and the Artin-Tate pairing provided the first traction for my approach towards Conjecture 1.1: my method began by trying to adapt certain technology which was useful to study the linking form. However, the argument ultimately goes beyond what was previously known in the topological setting, and yields new results *even for manifolds*. More precisely, the proof of Theorem 1.2 simultaneously establishes a necessary and sufficient criterion for the linking form (on any odd-dimensional closed orientable manifold) to be alternating, which was not previously known.

2.5. **An auxiliary pairing.** We define an auxiliary pairing on the group $H^2_{\text{ét}}(X; \mathbf{Z}/\ell^n\mathbf{Z}(1))_{\mathrm{nd}}$. Again the key point is that there is a 5-dimensional Poincaré duality for $H^*_{\text{ét}}(X; \mathbf{Z}/\ell^n\mathbf{Z}(*))$, which means in particular that there is a fundamental class inducing an isomorphism

$$\int : H^5_{\text{ét}}(X; \mathbf{Z}/\ell^n\mathbf{Z}(2)) \xrightarrow{\sim} \mathbf{Z}/\ell^n\mathbf{Z}.$$

The short exact sequence of sheaves

$$0 \to \mathbf{Z}/\ell^n\mathbf{Z}(1) \xrightarrow{\ell^n} \mathbf{Z}/\ell^{2n}\mathbf{Z}(1) \to \mathbf{Z}/\ell^n\mathbf{Z}(1) \to 0$$

inducing the *Bockstein operation*

$$\beta\colon H^i_{\text{ét}}(X; \mathbf{Z}/\ell^n\mathbf{Z}(1)) \to H^{i+1}_{\text{ét}}(X; \mathbf{Z}/\ell^n\mathbf{Z}(1)).$$

Definition 2.5. We define the pairing

$$\langle\cdot,\cdot\rangle_n\colon H^2_{\text{ét}}(X; \mathbf{Z}/\ell^n\mathbf{Z}(1)) \times H^2_{\text{ét}}(X; \mathbf{Z}/\ell^n\mathbf{Z}(1)) \to \mathbf{Z}/\ell^n\mathbf{Z}$$

by

$$\langle x, y\rangle_n := \int x \smile \beta y.$$

(Note that $\langle\cdot,\cdot\rangle_n$ need not be nondegenerate.)

What is the relation between $\langle\cdot,\cdot\rangle_n$ and the Artin-Tate pairing? The long exact sequence associated to

$$0 \to \mathbf{Z}_\ell(1) \to \mathbf{Z}_\ell(1) \to \mathbf{Z}/\ell^n\mathbf{Z}(1) \to 0$$

provides a surjection

$$H^2_{\text{ét}}(X; \mathbf{Z}/\ell^n\mathbf{Z}(1)) \twoheadrightarrow H^3_{\text{ét}}(X; \mathbf{Z}_\ell(1))[\ell^n].$$

Proposition 2.6. *This is compatible for the pairings* $\langle \cdot, \cdot \rangle_n$ *and* $\langle \cdot, \cdot \rangle_{\mathrm{AT}}$ *in the sense that the following diagram commutes*

$$\begin{array}{ccccc} H^2_{\text{ét}}(X; \mathbf{Z}/\ell^n\mathbf{Z}(1)) & \times & H^2_{\text{ét}}(X; \mathbf{Z}/\ell^n\mathbf{Z}(1)) & \xrightarrow{\langle \cdot,\cdot \rangle_n} & H^5_{\text{ét}}(X; \mathbf{Z}/\ell^n\mathbf{Z}(1)) \\ \downarrow & & \downarrow & & \downarrow \sim \\ H^3_{\text{ét}}(X, \mathbf{Z}_\ell(1))[\ell^n] & \times & H^3_{\text{ét}}(X; \mathbf{Z}_\ell(1))[\ell^n] & \xrightarrow[\langle \cdot,\cdot \rangle_{\mathrm{AT}}]{} & H^5_{\text{ét}}(X; \mathbf{Q}_\ell/\mathbf{Z}_\ell(1))[\ell^n] \end{array}$$

Thanks to Proposition 2.6, to prove Theorem 1.2 it suffices to show that $\langle \cdot, \cdot \rangle_n$ is alternating, which is what we'll do. What makes $\langle \cdot, \cdot \rangle_n$ better to work with than $\langle \cdot, \cdot \rangle_{\mathrm{AT}}$? Philosophically this seems to come down to the fact that $\mathbf{Z}/\ell^n\mathbf{Z}$ enjoys a *ring structure* while $\mathbf{Q}_\ell/\mathbf{Z}_\ell$ does not. The ring structure is (partly) responsible for extra structure on $H^*_{\text{ét}}(X; \mathbf{Z}/\ell^n\mathbf{Z})$ that will play a crucial role in our argument.

To give a sample illustration of the significance of the ring structure, we'll show how it leads immediately to a proof of the skew-symmetry.

Proposition 2.7. *The pairing* $\langle \cdot, \cdot \rangle_n$ *is skew-symmetric.*

Proof. The assertion is equivalent to

$$x \smile \beta y + y \smile \beta x = 0.$$

Since β is a derivation, we have $x \smile \beta y + y \smile \beta x = \beta(x \smile y)$. But the boundary map

$$\beta \colon H^4_{\text{ét}}(X; \mathbf{Z}/\ell^n\mathbf{Z}(2)) \to H^5_{\text{ét}}(X; \mathbf{Z}/\ell^n\mathbf{Z}(2))$$

vanishes because its image is the kernel of

$$[\ell^n] \colon H^5_{\text{ét}}(X; \mathbf{Z}/\ell^n\mathbf{Z}(2)) \to H^5_{\text{ét}}(X; \mathbf{Z}/\ell^{2n}\mathbf{Z}(2))$$

which is identified with the inclusion $\ell^n\mathbf{Z}/\ell^{2n}\mathbf{Z} \hookrightarrow \mathbf{Z}/\ell^{2n}\mathbf{Z}$ by Poincaré duality. □

Corollary 2.8. *The Artin-Tate pairing is skew-symmetric.*

Remark 2.9. Note that the same argument cannot be made directly for $\langle \cdot, \cdot \rangle_{\mathrm{AT}}$, since $\mathbf{Q}_\ell/\mathbf{Z}_\ell$ is not a ring.

Since we are interested in the distinction between skew-symmetric and alternating, we may now set $\ell = 2$. Next I'm going to describe the fuel that powers the argument, namely the *Steenrod squares*. These are operators on $H^*_{\text{ét}}(X; \mathbf{Z}/2\mathbf{Z})$ that are subtler consequences of the ring structure on $\mathbf{Z}/2\mathbf{Z}$.

3. Steenrod squares

3.1. **The Steenrod algebra.** Let us momentarily place ourselves in the context of singular cohomology. If M is a topological space, what algebraic structure does $H^*(M; \mathbf{Z}/2\mathbf{Z})$ have? One gets for free that it is a group. Furthermore, it acquires a (graded) commutative ring structure thanks to the ring structure on $\mathbf{Z}/2\mathbf{Z}$. But that's not all: topologists have known for many years that it has the structure of a *module* over a very important algebra, named the *Steenrod algebra* after its discoverer.

The Steenrod algebra has a very explicit presentation in terms of generators Sq^i, $i \geq 0$, and relations. What this means is that there are canonical cohomology operations

$$\mathrm{Sq}^i \colon H^*(M; \mathbf{Z}/2\mathbf{Z}) \to H^{*+i}(M; \mathbf{Z}/2\mathbf{Z})$$

which are universal in the sense of being functorial with respect to pullback.

A reasonable slogan for Sq^i is that it "comes from a derived enhancement of the Frobenius map," which at the prime 2 means the squaring map $x \mapsto x^2$.

Example 3.1. The operation $\mathrm{Sq}^j \colon H^j(X; \mathbf{Z}/2\mathbf{Z}) \to H^{2j}(X; \mathbf{Z}/2\mathbf{Z})$ is the squaring operation $x \mapsto x^2$. Notice that this is a homomorphism because we are in characteristic 2.

The operation $\mathrm{Sq}^j \colon H^i(X; \mathbf{Z}/2\mathbf{Z}) \to H^{i+j}(X; \mathbf{Z}/2\mathbf{Z})$ is 0 if $j > i$.

This operation Sq^1 coincides with the boundary map for

$$0 \to \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/4\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0.$$

Example 3.1 pretty much covers all of the Steenrod squares that have explicit descriptions in terms of other familiar operations. This is a good thing, since it tells us that the Steenrod squares really bring something fundamentally new to the table!

One of the key points in our method is to ask for an analogous understanding on $H^*(M; \mathbf{Z}/2^n\mathbf{Z})$ for all n. We'll define operations

$$\widetilde{\mathrm{Sq}}^i \colon H^*(M; \mathbf{Z}/2^n\mathbf{Z}(a)) \to H^{*+i}(M; \mathbf{Z}/2^n\mathbf{Z}(2a))$$

for all n, which specialize to Sq^i when $n = 1$.

3.2. **Étale homotopy theory.** The theory of Steenrod operations extends to étale cohomology. My favorite way of thinking about this is that one can embed the theory of étale cohomology of algebraic varieties into the theory of singular cohomology of topological spaces via *étale homotopy theory.* ♠♠♠ TONY: [probably don't have time to really talk about this?]

3.3. **Construction of Steenrod squares.** We'll sketch the construction of Steenrod squares.

A general philosophy is that cohomology operations arise when two cohomology classes are equal for "two different reasons". In the case of Steenrod square, the two reasons come from the tautological identity $[u][v] = [u][v]$ and the graded-commutativity of cohomology: $[u][v] = [v][u]$. (Note that there would usually be a sign, which we are free to ignore in characteristic 2!) When we specialize $[u] = [v]$, we get two relations, and this gives birth to a cohomology operation.

Concretely, for $u, v \in Z^i \subset C^i(M)$, the fact that $[uv] = [vu]$ implies that there exists a cochain $\mathrm{cup}_1(u, v)$ such that

$$d\mathrm{cup}_1(u, v) = uv \pm vu.$$

If we take $\mathbf{Z}/2\mathbf{Z}$ coefficients, then we don't have to worry about the sign: $d\mathrm{cup}_1(u, u) = 0$ so that we get a cohomology class $\mathrm{Sq}^{i-1}([u]) := [\mathrm{cup}_1(u, u)] \in H^{2i-1}(M)$.

The point is that although the cup product on cohomology is graded commutative, the cup product on *cochains* $C^*(M) \otimes C^*(M) \to C^*(M)$ is *not* commutative "on the nose", but only up to homotopy. One can "resolve" it to be S_2-equivariant on the nose. As usual with derived procedures, this introduces higher operations.

4. Plan of the proof

Now I'll outline the plan for the proof of Theorem 1.2. By Proposition 2.6, it suffices to show that

$$\langle x, x \rangle_n = 0 \text{ for all } x \in H^2_{\text{ét}}(X; \mathbf{Z}/2^n\mathbf{Z}(1)).$$

Since $\langle , \rangle_n$ is skew-symmetric the assignment $x \mapsto \langle x, x \rangle_n$ is actually a *homomorphism*. For simplicity we'll restrict our attention to $n = 1$. The structure of the argument is the same for all n, although it took a while to phrase it in a way that generalized well; the use of the operations $\widetilde{\mathrm{Sq}}^n$ is actually one of the key innovations.

The first phase of the plan is to rewrite this homomorphism in terms of Steenrod operations, and more precisely to prove the following theorem:

Theorem 4.1. *For $x \in H^2_{\text{ét}}(X; \mathbf{Z}/2\mathbf{Z})$, we have*

$$x \smile (\beta x) = \mathrm{Sq}^2(\beta x).$$

Remark 4.2. Theorem 4.1 applies also to (compact orientable) manifolds, and leads to a criterion for the linking form to be alternating which was previously unknown in topology.

Thanks to Theorem 4.1, we are reduced to showing that

$$0 = \mathrm{Sq}^2(\beta x).$$

Note that $\mathrm{Sq}^2 \colon H^3_{\text{ét}}(X; \mathbf{Z}/2\mathbf{Z}) \to H^5_{\text{ét}}(X; \mathbf{Z}/2\mathbf{Z})$ must, by Poincaré duality, be represented by a class $v_2 \in H^2_{\text{ét}}(X; \mathbf{Z}/2\mathbf{Z})$.

Tautologically $\mathrm{Sq}^2(\beta x) = v_2(\beta x)$. Hence it suffices to show that $0 = v_2(\beta x)$. Since β is a derivation, we can rewrite this as

$$\beta(v_2 x) - \beta(v_2)x.$$

As discussed in the proof of Proposition 2.7, β is 0 on $H^4_{\text{ét}}(X; \mathbf{Z}/2\mathbf{Z})$, so the obstruction comes down $\beta(v_2)$. By the long exact sequence this is the obstruction to lifting v_2 to an integral class. The proof is then completed by the following theorem.

Theorem 4.3. *The class $v_2 \in H^2_{\text{ét}}(X; \mathbf{Z}/2\mathbf{Z})$ lifts to $\widetilde{v}_2 \in H^2_{\text{ét}}(X; \mathbf{Z}_2(1))$.*

5. Characteristic classes

5.1. **Wu's formula.** To get some ideas for how to compute Steenrod squares, we begin by looking around in the literature on algebraic topology. Let M be a closed smooth manifold of dimension m. By Poincaré duality, the operation

$$\mathrm{Sq}^i \colon H^{m-i}(M; \mathbf{Z}/2\mathbf{Z}) \to H^m(M; \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}$$

must coincide with cupping with a unique class $v_i \in H^i(M; \mathbf{Z}/2\mathbf{Z})$. What can we say about this canonical cohomology class? (Note that this is exactly the situation of Theorem 4.1.)

This question was answered long ago by Wu, who proved the following theorem:

Theorem 5.1 (Wu)**.** *Let M be a closed smooth manifold. Write*

$$\mathrm{Sq} = 1 + \mathrm{Sq}^1 + \mathrm{Sq}^2 + \dots$$
$$v = v_1 + v_2 + \dots$$
$$w = w_1 + w_2 + \dots$$

where the w_i are the Stiefel-Whitney classes of TM. Then we have $\mathrm{Sq}\, v = w$.

Example 5.2. Let's unpack Theorem 5.1 to find a formula for v_2. It says

$$(1 + \mathrm{Sq}^1 + \mathrm{Sq}^2 + \dots)(1 + v_1 + v_2 + \dots) = 1 + w_1 + w_2 + \dots.$$

Now, matching terms by cohomological degree we find that $v_1 = w_1$ and $v_2 + \mathrm{Sq}^1 v_1 = w_2$, so $v_2 = w_2 + w_1^2$.

5.2. **Wu's theorem in étale cohomology.** We'd like to have an analogue of Theorem 5.1 in étale cohomology. For that to begin to make sense, we would need a definition of étale Stiefel-Whitney classes.

The idea is to try to take Thom's formula (**??**) as a *definition* of w_i. To do this, we need to find for a vector bundle $E \to X$ an embedding $X \hookrightarrow Y$ such that E is isomorphic normal bundle of X and Y. A canonical choice exists: take Y to be the total space of E, and the embedding to be the zero section. Then the cycle class $[X] \in H^*_{\text{ét}}(Y; \mathbf{Z}/2\mathbf{Z})$ has the property that $\operatorname{Sq}^i[X] = \pi^*(w_i) \smile [X]$ for some $w_i \in H^*_{\text{ét}}(X; \mathbf{Z}/2\mathbf{Z})$, where $\pi\colon E \to X$ is the projection, and we define $w_i(E)$ to be this w_i.

Theorem 5.3 (F)**.** *Let X be a smooth projective variety over $\mathbf{F}_q$. Then we have* $\operatorname{Sq} v = w$.

The fact that Theorem 5.3 looks exactly like Theorem 5.1 is a bit deceptive, thanks to the failure of absolute cohomology to behave like geometric cohomology in some important aspects. The proof does begin by trying to imitate the proof of the classical theorem. However, there is one technical point which presents a serious new difficulty, which ends up requiring étale homotopy theory to resolve.

5.3. **Relation to Chern classes.** The inspiration for Theorem 4.3 comes from the following fact: for a complex manifold M, the Stiefel-Whitney classes are the reductions mod 2 of the Chern classes (and in particular admit integral lifts). Our more precise heuristic is that a variety over a finite field looks like a complex surface fibration over S^1, and in particular has the same characteristic classes as the complex surface.

Theorem 5.4 (F)**.** *The class $w_2(TX) \in H^2_{\text{ét}}(X; \mathbf{Z}/2\mathbf{Z})$ is the reduction of $c_1(TX) \in H^2_{\text{ét}}(X; \mathbf{Z}_2(1))$ modulo* 2.

If E is a vector bundle over X, then there is an explicit expression for $w_i(E)$ in terms of $\{c_j(E)\}$ by a universal polynomial depending on the parity of $\operatorname{rank} E$. However, the formula is not as simple as one might guess from our earlier heuristic.