ARTHUR PACKETS

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These are notes for two lectures by Wee Teck Gan on Arthur packets for $\mathrm{SO}(n+1)$, which formed a segment of a course by Wee Teck Gan and Hiraku Atobe.

1. Questions about multiplicity

Let F be a local field and W_F its Weil group.

We'll focus our discussion on $G = \mathrm{SO}(2n+1)$, so $\widehat{G} = \mathrm{Sp}_{2n}(\mathbf{C})$. However, much of what we'll say goes through for classical groups in general.

Let $\psi\colon \mathrm{WD}_F \times \mathrm{SL}_2(\mathbf{C}) \to \widehat{G}$ be an Arthur parameter. Arthur attaches an A-packet

$$\Pi_\psi^{\mathrm{Ar}} = \{\pi_\eta \colon \eta \in \mathrm{Irr}(\overline{A}_\psi)\}.$$

However, various basic questions about Π_ψ are not answered:

(1) Is π_η multiplicity-free?
(2) Is $\oplus_\eta \pi_\eta$ multiplicity-free?

Why would one care about these questions? Local A-packets are local components of summands in $\mathscr{A}_{\mathrm{disc}}(G/k)$. If the answer to question (2) is "yes", then Multiplicity One holds for $\mathscr{A}_{\mathrm{disc}}(G/k)$.

Theorem 1.1 (Moeglin-Renard)**.** *If F is p-adic, then $\bigoplus_\eta \pi_\eta$ is multiplicity-free.*

This is a very difficult theorem, but see the exposition by Bin Xu.

In today's lectures, we will focus on the case $F = \mathbf{R}$ or $\mathbf{C}$.

Theorem 1.2 (Moeglin-Renard)**.**

(1) If $F = \mathbf{C}$, then multiplicity freeness holds.
(2) If $F = \mathbf{R}$, then multiplicity freeness almost holds.

We'll see later what the word "almost" means here.

The proof is not conceptual. They construct A-packets in an independent way, and verify that this agree with Arthur's by checking the endoscopic character relations. From the explicit construction, one verifies by inspection that the packets are multiplicity-free.

We'll restrict our attention to the case where F is archimedean.

2. Initial steps

Let $\psi\colon W_F \times \mathrm{SL}_2(\mathbf{C}) \to \mathrm{Sp}_{2n}(\mathbf{C})$ be an Arthur parameter. We compose with the standard representation $\mathrm{Sp}_{2n}(\mathbf{C}) \hookrightarrow \mathrm{GL}_{2n}(\mathbf{C})$; call the composite parameter ψ^{GL}.

Viewing ψ as a representation of $W_F \times \mathrm{SL}_2(\mathbf{C})$, we decompose it into irreducibles as

$$\psi = \bigoplus_{i \in I} \psi_i.$$

For each $i \in I$, the irreducibility implies that ψ_i has the form

$$\psi_i = \rho_i \otimes S_{d_i}$$

where ρ_i is an irreducible representation of W_F and S_{d_i} is an irreducible d_i-dimensional representation of SL_2. In the Archimedean case, the irreducible representations of W_F are 1-dimensional if $F = \mathbf{C}$, and 1 or 2-dimensional if $F = \mathbf{R}$. The 2-dimensional representations correspond to discrete series of $\mathrm{GL}_2(\mathbf{R})$.

The upshot is that $\dim \rho_i = 1$ or 2, so by the local Langlands correspondence, ρ_i gives rise to an irreducible representation τ_{ρ_i} of $\mathrm{GL}_1(F)$ or $\mathrm{GL}_2(F)$. Furthermore, it is part of the definition of Arthur parameters that $\psi(W_F)$ is bounded. Hence $\psi(W_F)$ corresponds to a tempered representation.

Set τ_{ψ_i} to be the Langlands quotient of

$$\tau_{\rho_i}|\cdot|^{\frac{d_i-1}{2}} \times \tau_{\rho_i}|\cdot|^{\frac{d_i-3}{2}} \times \ldots \times \tau_{\rho_i}|\cdot|^{\frac{1-d_i}{2}}$$

This is called a *Speh representation*. (The $\times$ notation means parabolic induction.)

Finally, define

$$\tau_\psi = \times_{i \in I} \tau_{\psi_i}. \tag{2.1}$$

This is an irreducible unitary representation of $G(F)$.

3. Twisted endoscopic character identities

The Arthur packets are uniquely specified by the *twisted endoscopic character identities*. We will briefly indicate the flavor of what these are.

The first condition is:

> (*Stability*) Let $s_\psi = \psi(-\mathrm{Id}_{\mathrm{SL}_2}) \in \mathrm{Sp}_{2n}(\mathbf{C})$. If Θ_{π_η} is the Harish-Chandra character of the representation π_η, then we demand that the sum
>
> $$\Theta_\psi(s_\psi) := \sum_\eta \eta(s_\psi)\Theta_{\pi_\eta}$$
>
> is a stably invariant distribution. (Here we write $\eta(s_\psi)$ for the trace of s_ψ under the representation η of $\overline{A}_\psi$.) The adjective "stably" means invariant under conjugation by $G(\mathbf{C})$, which is stronger than just conjugation by $G(\mathbf{R})$ (which is automatically satisfied).

Before we can explain the second condition, we need some background. Langlands-Shelstad defined a transfer map from

$$\mathrm{Trans} : \mathrm{I}^\theta(\mathrm{GL}_{2n}) \to \mathrm{SI}(G)$$

where

- $\mathrm{I}(G)$ is the space of (regular semisimple) orbital integrals on $G(\mathbf{R})$, which is regarded as a space of distributions on $C_c^\infty(G)$, and $\mathrm{SI}(G)$ is the analogous space of stable orbital integrals.

- θ is the space of twisted orbital integrals with respect to $\theta \in \mathrm{GL}_{2n}$, given by $\theta(g) = w({}^t g^{-1})w^{-1}$ where

$$w = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}$$

Recall the definition of τ_ψ from (2.1). Now we can state another condition.

> Regarding Θ_ψ as a function on $\mathrm{SI}(G)$, we demand that $\mathrm{Trans}^*(\Theta_\psi)$ is the "θ-twisted trace of τ_ψ".

What is the θ-twisted trace? The representation τ_ψ is θ-invariant, so we can extend it to $\mathrm{GL}_{2n}(F) \rtimes \theta$. Then its "$\theta$-twisted trace" is $g \mapsto |\operatorname{Tr}(\pi_\psi(g, \theta)|$.

When s_ψ is trivial, we get the sum of θ_{π_η}. If we can plug in more values of s_ψ, we can isolate each Θ_{π_η}.

For the other values, we need the "usual endoscopic character identities". That is, for all $s \in \overline{A}_\psi$ we consider

$$\Theta_\psi(s) := \sum \eta(s)\Theta_{\pi_\eta}.$$

For $s \neq s_\psi$, $\Theta_\psi(s)$ can be described as "endoscopic transfer of stable characters on endoscopic groups of G". In the case of $G = \mathrm{SO}(2n+1)$, these endoscopic groups are

$$H_{a,b} = \mathrm{SO}(2a+1) \times \mathrm{SO}(2b+1)$$

with $a + b = n, ab \neq 0$.

Why is this any better? We pretend we understand GL_{2n}. The $H_{a,b}$ would be covered by induction. So we have reduced a question about $\mathrm{SO}(2n+1)$ to a question about "simpler" groups.

4. Stratification of parameters

4.1. Reduction to good parity.

Lemma 4.1. *Given $\psi \in \Psi(G)$, we can write*

$$\psi = \rho \oplus \psi_{\mathrm{bp}} \oplus \rho^\vee \colon W_F \times \mathrm{SL}_2 \to \mathrm{GL}_{2n}.$$

Let's explain how we get this. Since ψ is self-dual, any summand ρ which is not self-dual has to have $\rho^\vee$ appear as well. The self-dual representations preserve a bilinear form which could be symplectic or orthogonal. Orthogonal summands cannot occur in isolation, because an invariant summand has to have a symplectic form, and Schur's Lemma prohibits an irreducible summand from having both a symplectic form and an orthogonal form. So such a thing occurs an even number of types, which we group into $\rho \oplus \rho^\vee$ again. Finally, the symplectic irreducible summands are lumped together into ψ_{bp}.

Conclusion: ρ is the sum of non-self-dual summands or self-dual summands of orthogonal type, and ψ_{bp} is the sum of self-dual summands of symplectic type.

The reasoning for writing things in this way is that

$$\overline{A}_\psi = \overline{A}_{\psi_{\mathrm{bp}}}.$$

We let $\Psi_{\mathrm{bp}} \subset \Psi(G)$ be the subset of parameters ψ such that $\psi = \psi_{\mathrm{bp}}$. They will be called the A-parameters of *good parity* ("bon parity").

We can reduce to understanding Ψ_{bp}:

Theorem 4.2. *Let $\psi \in \Psi(G)$, and write*

$$\psi = \rho \oplus \psi_{\mathrm{bp}} \oplus \rho^\vee.$$

Let the Arthur packet of ψ_{bp} be

$$\Pi_{\psi_{\mathrm{bp}}} = \{\pi_{\mathrm{bp},\eta} \colon \eta \in \mathrm{Irr}(\overline{A}_{\psi_{\mathrm{bp}}})\}.$$

Then:

- *The representation $\pi_\eta := \tau_\rho^{\mathrm{GL}} \rtimes \pi_{\mathrm{bp},\eta}$ of G is irreducible.*
- $\Pi_\psi = \{\pi_\eta \colon \eta \in \mathrm{Irr}(\overline{A}_\psi) = \mathrm{Irr}(\overline{A}_{\psi_{\mathrm{bp}}})\}$.

The proof (of the second part) is based on the principle that "endoscopic transfer commutes with parabolic induction". (The same holds for Jacquet modules, which are left adjoint to parabolic induction.)

Thus we have reduced to the "good parity part".

4.1.1. *Good parameters for $F = \mathbf{C}$.* For $F = \mathbf{C}$, any A-parameter is of the form

$$\psi = \bigoplus_{i \in I} \chi_{k_i,t_i} \otimes S_{d_i}$$

where $\chi_{k,t}$ is the irreducible character of $W_{\mathbf{C}} = \mathbf{C}^\times$ given by

$$z \mapsto \left(\frac{z}{|z|}\right)^k (z\overline{z})^t$$

Then the condition that $\psi = \psi_{\mathrm{bp}}$ forces $\chi_{k_i,t_i} = 1$ and d_i to be even.

Conclusion: in this situation, a "good parity" A-parameter necessarily has the form

$$\psi = \bigoplus_{i \in I} S_{d_i}, \quad d_i \text{ even}.$$

Definition 4.3. Let $F = \mathbf{R}$ or $\mathbf{C}$. Then ψ is *unipotent* if $\psi|_{W_{\mathbf{C}}} = 1$. We regard $W_{\mathbf{C}} \subset W_{\mathbf{R}}$ as

$$1 \to W_{\mathbf{C}} \to W_{\mathbf{R}} \to \mathrm{Gal}(\mathbf{C}/\mathbf{R}) \to 1.$$

If $F = \mathbf{C}$, then $\Psi_{\mathrm{bp}}(G) = \Psi_{\mathrm{unip}}(G)$. For $\psi \in \Psi_{\mathrm{bp}}(G)$, Barbasch-Vogan '85 have defined a packet

$$\Pi_\psi^{\mathrm{BV}}.$$

Let $\psi \colon \mathrm{SL}_2(\mathbf{C}) \to \mathrm{Sp}_{2n}(\mathbf{C}) = \widehat{G}$. Then we get

$$d\psi \colon \mathfrak{sl}_2(\mathbf{C}) \to \widehat{\mathfrak{g}} = \mathrm{Lie}(\widehat{G}).$$

Set $\lambda_\psi := \frac{1}{2} d_\psi \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \in \widehat{\mathfrak{h}} \subset \widehat{\mathfrak{g}}$. We canonically identify $\widehat{\mathfrak{h}} \cong \mathfrak{h}^*$. Really, λ_ψ is really only well-defined modulo W, so we have a canonical element $\lambda_\psi \in \mathfrak{h}^*/W$. We then consider the infinitesimal character $(\lambda_\psi, \lambda_\psi) \in \mathfrak{h}^* \times \mathfrak{h}^*/W \times W$ for $G(\mathbf{C})$. (Some doubling happened because we pass from real to complex groups.)

Now, ψ also gives

$$d\psi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \widehat{\mathfrak{g}}$$

a nilpotent conjugacy class in $\widehat{\mathfrak{g}}$. The even-ness of d_i forces this conjugacy class to be *special* in the sense of Lusztig.

Then there is a bijection (Lusztig-Spaltenstein) between special nilpotent orbits on $\mathfrak{g}$ and special nilpotent orbits on $\widehat{\mathfrak{g}}$, denoted $O_\psi \leftrightarrow \widehat{O}_\psi$.

Definition 4.4. We define

$$\Pi_\psi^{\mathrm{BV}} = \{\pi \in \mathrm{Irr}(G) \colon \inf(\pi) = (\lambda_\psi, \lambda_\psi), \mathrm{WF}(\pi) = \overline{O_\psi}\}.$$

Here WF is the "wavefront variety" that gives some measure of π, e.g. if π is generic then $\mathrm{WF}(\pi)$ is regular nilpotent; if π is trivial then it's the 0 orbit.

They showed:

- Every $\pi \in \Pi_\psi^{\mathrm{BV}}$ is unitary.
- There is a natural bijection $\Pi_\psi^{\mathrm{BV}} \leftrightarrow \mathrm{Irr}(\overline{A}_\psi^L)$, Lustzig's quotient of A_ψ. (Really they work on the group side, and use that there is an identification $\mathrm{Irr}(\overline{A}_\psi^L) = \mathrm{Irr}(\overline{A(O_\psi)}_\psi^L)$.
- They verified the endoscopic relations. The input was writing down the characters of Π_ψ^{BV} as parabolic inductions.

Theorem 4.5 (Moeglin-Renard)**.** $\Pi_\psi^{\mathrm{BV}} = \Pi_\psi^{\mathrm{Ar}}$.

Barbasch [2016, Howe's 70th birthday proceedings] showed: Π_ψ^{BV} can be constructed by "iterated theta liftings".

4.2. $F = \mathbf{R}$. Next we turn to the case $F = \mathbf{R}$. We know that irreducible representations of $W_{\mathbf{R}}$ are 1 or 2-dimensional. We break into these parts:

$$\psi = \left(\bigoplus_{i \in I} \underbrace{\rho_{k_i}}_{2-\mathrm{dim}} \otimes S_{d_i} \right) \oplus \left(\bigoplus_{j \in J} \underbrace{\chi_j}_{1-\mathrm{dim}} \otimes S_{b_j} \right)$$

For the 1-dimensional part, we need b_i even, $\chi_j^2 = 1$. The 2-dimensional part is more complicated – the parity of k_i determines whether ρ_{k_i} is orthogonal or symplectic, and then d_i needs to have the opposite parity.

We will define consider chains of containments

$$\Psi(G) \supset \Psi_{\mathrm{bp}}(G) \supset \Psi_{\mathrm{vreg}}(G) \supset \Psi_{\mathrm{unip}}(G).$$

What is $\Psi_{\mathrm{vreg}}(G)$? It means $k_1 \gg k_2 \gg k_3 \gg \ldots \gg k_r \gg 0$.

By definition, $\psi \in \Psi_{\mathrm{unip}}(G) \iff \psi|_{W_{\mathbf{C}}} = 1$, which happens iff $I = \emptyset$.

We also have $\Psi_{\mathrm{AJ}}(G) \subset \Psi_{\mathrm{vreg}}(G)$, for "Adams-Johnson", which is cut out by the condition that $|J| \leq 1$. (Morally J is as small as possible.)

We've seen that the study of $\Psi(G)$ reduces to that of $\Psi_{\mathrm{bp}}(G)$ by parabolic induction. We'd like to similarly reduce to the lower echelons.

4.2.1. *Unipotent parameters.* This case was explained by Moeglin. She showed that for $\psi \in \Psi_{\mathrm{unip}}(G)$, Π_ψ^{Ar} can be constructed by "iterated theta lifting". Roughly it goes like this. Peel off the largest piece of the parameter, thus getting a parameter for a smaller group. By induction, you get a packet for the smaller group. Moeglin shows that you can get the packet for the larger group by theta lifting. Thus you reduce to the case of 1-dimensional representations, which are easy.

How do you check the endoscopic character identities? In principle things would be OK if the endoscopic character identities behave well under the theta correspondence. How then can you check the endoscopic character relations? Answer: use global methods. We can globalize the parameter just by finding a global quadratic character that restricts to the right local quadratic character. For these very degenerate global parameters, Moeglin established (independently of the trace formula) the global correspondence. Knowing the compatibility with the global correspondence amounts to the endoscopic character relations.

The 1-dimensional characters lie inside Ψ_{unip} and Ψ_{AJ}.

4.2.2. *Bootstrapping from unipotent to very regular.* We take a hint from the special case of Adams-Johnson parameters of the form

$$\psi = \bigoplus_{i \in I} \rho_{k_i} \otimes S_{d_i} \oplus \mathrm{sign}^\delta \otimes S_b.$$

Then

$$\psi|_{W_{\mathbf{C}} \times \mathrm{SL}_2} = \bigoplus ((\chi_{k_i} + \chi_{k_i}^{-1}) \otimes S_{d_i}) \oplus (1 \otimes S_{2b}).$$

This factors through the Levi

$$\prod_{i \in I} \mathrm{GL}(d_i, \mathbf{C}) \times \mathrm{Sp}(2b, \mathbf{C}).$$

and sends SL_2 into the principal SL_2 of each Levi, and $W_{\mathbf{C}}$ to the center of each Levi.

We're missing information about $\psi(j)$, where $j \in W_{\mathbf{R}}/W_{\mathbf{C}}$. It's an element of order 4 whose image normalizes this Levi. On each GL it acts as an outer automorphism, and does nothing on $\mathrm{Sp}(2b, \mathbf{C})$.

Now, $\widehat{L} \rtimes \langle j \rangle$ is the L-group of some twisted Levi $L \subset G$. Indeed, the Langlands parameter lands in the normalizer by an outer automorphism of $\mathrm{GL}(d_i)$, so it looks like the L-group of a unitary group. Conclusion: the twisted Levi has the form $\prod_{i \in I} U(d_i) \times SO(2b+1)$.

The parameter ψ factors through

$$\psi \colon W_{\mathbf{R}} \times \mathrm{SL}_2 \to \widehat{L} \rtimes \langle j \rangle \subset \widehat{G}.$$

Hence we can think of ψ as an A-parameter for L, such that $\psi|_{\mathrm{SL}_2}$ is the principal SL_2 of $\widehat{L}$. This is the A-parameter of a 1-dimensional character, so Π_ψ^L is a set of 1-dimensional characters χ_ψ of L.

4.2.3. *Adams-Johnson theory.* For each $w \in W(G_{\mathbf{R}}, T_{\mathbf{R}})\backslash W(G_{\mathbf{C}}, T_{\mathbf{C}})/W(L_{\mathbf{C}}, T_{\mathbf{C}})$ we have a twisted Levi $L_w \subset G$ of the form $\prod_{i\in I} U(p_i, q_i) \times \mathrm{SO}(2b+1)$. On each $L_w(\mathbf{R})$, we have $\chi_{w,\psi}\colon L_w(\mathbf{R}) \to \mathbf{C}^\times$.

If these were actually Levis instead of twisted Levis, we do parabolic induction. As a replacement, we use "cohomological induction". Set $\pi_w(\psi) := A_{q_w}(\chi_{w,\psi})$ where q_w is a θ-stable parabolic algebra of $\mathfrak{g}_{\mathbf{C}}$. Here θ is the Cartan involution.

(Think of this as the analogue of Deligne-Lusztig induction, which goes from twisted Levis e.g. anisotropic tori. Cohomological induction is a real analogue of Deligne-Lusztig induction. Morally, J.K. Yu's construction of p-adic supercuspidals is a p-adic "cohomological induction".)

Set

$$\Pi_\psi^{\mathrm{AJ}} = \{\pi_w(\psi)\colon w \in \dots W(G_{\mathbf{R}}, T_{\mathbf{R}})\backslash W(G_{\mathbf{C}}, T_{\mathbf{C}})/W(L_{\mathbf{C}}, T_{\mathbf{C}})\}.$$

They defined $\Pi_\psi^{\mathrm{AJ}} \to \mathrm{Irr}(\overline{A}_\psi)$ and verified the usual endoscopic character relations.

Theorem 4.1 (Adams-Moeglin-Renard)**.** $\Pi_\psi^{\mathrm{AJ}} = \Pi_\psi^{\mathrm{Ar}}$.

What remains is to verify the twisted endoscopic relations. The proof is based on a principle. What is it? We're building the packets from very simple packets by cohomological induction. So the heart of the proof comes down to showing that endoscopic transfer commutes with cohomological induction.

In a more general case where $|J| > 1$, we have 1-dimensional characters on the GL-part and unipotent parameters on the classical part. We do cohomological induction.

Why the restriction to "very regular"? The issue is with the construction $\pi_w(\psi) := A_{q_w}(\chi_{w,\psi})$. The regularity lets you control this cohomological induction, e.g. it's irreducible.

To go from "very regular" to "good parity", you use "Zuckerman's translation functor". The principle at use is "translation functor commutes with endoscopic transfer". But the translation functor is not an equivalence, so it collapses some things. This is in why we only "almost" know multiplicity-freeness: at the step of applying Zuckerman's translation functor, some representations could be collapsed.