The Arbeitsgemeinschaft 2016: Geometric Langlands, Perfectoid Spaces, and the Fargues-Fontaine Curve

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Disclaimer

These notes were live-TEXed by me during the Arbeitsgemeinschaft at Oberwolfach in April 2016. They should not be taken as a faithful transcription of the actual lectures; they represent only my personal perception of the talks (a perception which became increasingly clouded by confusion over the course of the workshop).

I have edited the notes somewhat, but no doubt many typors and errors remain, and that I have introduced some new ones in the process. I have also not attempted to sync the notation used across different talks. Despite these flaws, I hope the notes will be useful to some readers.

Comments and corrections can be sent to me at tonyfeng@stanford.edu.

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Part I

Day One

Chapter 1

Adic Spaces

1.1 Huber rings

The basic building blocks of adic spaces are *Huber rings*.

Definition 1.1.1. A *Huber ring* is a topological ring A, such that there exists an open subring $A_0 \subset A$ and a *finitely generated* ideal $I \subset A_0$ such A has the I-adic topology. We call A_0 the *ring of definition* and I the *ideal of definition*.

Definition 1.1.2. We denote by A^{00} the set of topologically nilpotent elements of A and A^0 the set of power-bounded elements. $A^0 \subset A$ is an open subring and A^{00} is an ideal in A^0.

Definition 1.1.3. A *Tate ring* is a Huber ring A with a topologically nilpotent element ϖ, which we call a *pseudo-uniformizer*. Equivalently, a Tate ring is a Huber ring A such that $A^{00} \cap A^\times \neq \emptyset$.

Remark 1.1.4. If A is a Tate ring and ϖ is a pseudo-uniformizer, then

1. If $\varphi\colon A \to B$ is a continuous homomorphism of Huber rings then B is a Tate ring and $\varphi(\varpi)$ is a pseudo-uniformizing unit of B, so B is automatically a Tate ring.

2. If A is a Tate ring then we may assume that A_0 contains ϖ and $I = \varpi A_0$. It follows that $A = A_0[1/\varpi]$.

Example 1.1.5. Let k be a non-archimedean field, i.e. a topological field whose topology is given by a non-archimedean absolute value of height 1:

$$|\cdot|\colon k \to \mathbb{R}_{\geq 0}.$$

Then k is a Tate ring, with $k^0 = O_k$. We take k^0 to be the ring of definition. Any $\varpi \in k$ with $|\varpi| < 1$ is a pseudo-uniformizer.

Example 1.1.6. The *Tate algebra*

$$A = k\langle X_1, \ldots, X_n\rangle = \{\sum a_I x^I \mid a_I \to 0 \text{ for } I \to \infty\}$$

is a Huber ring, with ring of definition

$$A_0 := A^0 = O_k\langle X_1, \ldots, X_n\rangle.$$

This is a Tate ring, and a uniformizer is again any ϖ with $|\varpi| < 1$.

1.2 Affinoid adic spaces

1.2.1 Underlying set

Definition 1.2.1. A *Huber pair* (affinoid ring) is a pair (A, A^+) where A is a Huber ring and $A^+ \subset A^0$ is an open subring which is integrally closed.

To such a pair we will define an *affinoid adic space*. We begin by describing the underlying set:

Definition 1.2.2. For a Huber pair (A, A^+) we define the *adic spectrum* (just a set for now) to be

$$X = \operatorname{Spa}(A, A^+) := \left\{ |\cdot| \colon A \to \Gamma \cup \{0\} \mid {}^{\text{continuous, multiplicative, non-arch.}}_{|f|\leq 1 \text{ for all } f\in A^+} \right\}.$$

where Γ is a totally ordered abelian group. The meaning of continuity is that for all γ,

$$\{a \in A \mid |a| < \gamma\} \subset A \text{ is open.}$$

Remark 1.2.3. It is equivalent to demand that $\{a \in A \colon |a| \leq \gamma\}$ is open for all γ. Indeed, the non-archimedean property implies that both versions (with strict or non-strict inequalities) are groups, and any group containing an open subset is open.

For $x \in X$ and $f \in A$, we denote

$$|f(x)| := x(f).$$

1.2.2 Topology of rational subsets

Let $T \subset A$ be a finite subset such that $T \cdot A$ generates an open ideal. (If A is a Tate ring then this is equivalent to $TA = A$.)

Definition 1.2.4. We define a *rational open subset* of $X := \operatorname{Spa}(A, A^+)$ to be a subset of the form

$$X\left(\frac{T}{s}\right) = \{x \in X \mid \forall t, |t(x)| \leq |s(x)| \neq 0\}.$$

Theorem 1.2.5. *There is a unique topology on X in which $X\left(\frac{T}{s}\right)$ forms a basis consisting of quasicompact open subsets such that the system of rational subsets stable under finite intersections. With this topology, X is a spectral space. (i.e. homemorphic to* $\operatorname{Spa}(R)$ *for some ring R).*

This gives a functor

$$(\text{Huber pairs}) \to (\text{spectral spaces}).$$

Lemma 1.2.6. $\mathrm{Spa}(\widehat{A}, \widehat{A}^+) \to \mathrm{Spa}(A, A^+)$ *is a homeomorphism preserving rational subsets.*

This means that we can always pass to the completion.

Proposition 1.2.7. *Let (A, A^+) be a Huber pair, and assume A is complete. Then*

1. $\mathrm{Spa}(A, A^+) = \emptyset \iff A = 0$,
2. *$f \in A$ is invertible if and only if $|f(x)| \neq 0$ for all x.*
3. *$f \in A^+ \iff |f(x)| \leq 1$ for all x.*

Remark 1.2.8. Where do we need the fact that A^+ is integrally closed? It is used in the third part of the preceding proposition.

This is everything that we need to say about the topological space underlying an affinoid adic space. Next we describe the structure sheaf.

1.2.3 Structure (pre)sheaf

From now on, we abbreviate Huber pairs (A, A^+) by A.

Theorem 1.2.9 (Localization)**.** *Let T, s be as above. Then there exists a morphism of Huber pairs $A \to A\langle \frac{T}{s} \rangle$ which is universal for morphisms of Huber pairs $\varphi\colon A \to B$ with B complete such that $\varphi(s) \in B^\times$ and for all $t \in T$ we have $\varphi(t)\varphi(s)^{-1} \in B^+$. (This implies that $A\langle \frac{T}{s} \rangle$ is a complete ring.)*

Remark 1.2.10. By the preceding proposition, the property we are asking for is exactly that the induced morphism of adic spectra factors through the open subset $X\left(\frac{T}{s}\right)$.

Lemma 1.2.11. *The natural map*

$$\mathrm{Spa}(A\langle \frac{T}{s} \rangle) \to \mathrm{Spa}(A)$$

is an open embedding, with image $X\left(\frac{T}{s}\right)$.

Let $X = \mathrm{Spa}(A)$. We now define the structure presheaf $(\mathcal{O}_X, \mathcal{O}_X^+)$ by

$$\mathcal{O}_X\left(X\left(\frac{T}{s}\right)\right) = A\langle \frac{T}{s} \rangle$$

and

$$\mathcal{O}_X^+\left(X\left(\frac{T}{s}\right)\right) = A\langle \frac{T}{s} \rangle^+.$$

In particular, $\mathcal{O}_X(X) = \widehat{A}$. This is a presheaf of complete topological rings with basis $X\left(\frac{T}{s}\right)$.

Definition 1.2.12. We call A *sheafy* if $\mathcal{O}_X$ is a sheaf (which automatically implies that $\mathcal{O}_X^+$ is a sheaf).

Any point $x \in X$ is a valuation, and induces a valuation on $\mathcal{O}_{X,x}$ (the usual stalk in the category of ringed spaces). You can check that $\mathfrak{m}_x = v_x^{-1}(0)$ is the unique maximal ideal in $\mathcal{O}_{X,x}$, so the latter is a local ring.

Definition 1.2.13. We define the category $\mathcal{V}$ to have objects tuples $(X, \mathcal{O}_X, \{v_x\}_{x \in X})$ where

- X is a topological space,
- $\mathcal{O}_X$ is a sheaf of complete topological rings such that $\mathcal{O}_{X,x}$ is local, and
- v_x a valuation on $\kappa(x)$.

Morphisms are the natural morphisms of such data.

Proposition 1.2.14. *The functor*

$$\textit{(sheafy Huber pairs)} \to \mathcal{V}$$

sending $A \mapsto \mathrm{Spa}(A)$ *is fully faithful.*

The image of this functor are the *affinoid adic spaces*.

1.3 Adic spaces

Definition 1.3.1. A *adic space* is an object of V which is locally isomorphic to $\mathrm{Spa}(A)$ where A is a sheafy Huber ring.

It is annoying that A is not always sheafy. However, here are some conditions that guarantee the sheafiness.

Theorem 1.3.2. *Let* (A, A^+) *be a complete Huber pair. It is sheafy if any of the following are satisfied:*

1. *A has a Noetherian ring of definition.*
2. *A is a Tate ring and $A\langle X_1, \dots, X_n\rangle$ is noetherian for all $n \geq 0$.*
3. *A is a Tate ring and for every rational subset $U \subset \mathrm{Spa}(A, A^+)$ the ring of power-bounded elements $\mathcal{O}_X(U)^0$ is a ring of definition.*

Example 1.3.3. There is a fully faithful embedding

$$\text{(locally noetherian formal schemes)} \hookrightarrow \text{(adic spaces)}$$

sending

$$\mathrm{Spf}(A) \mapsto \mathrm{Spa}(A, A).$$

In fact, (A, A^+) is also sheafy if A has the discrete topology, so $A \mapsto \mathrm{Spa}(A, A)$ embeds the full category of schemes into the category of adic spaces.

Example 1.3.4. If k is a non-archimedean field, then there is a fully faithful embedding

$$(\text{rigid analytic spaces}/k) \hookrightarrow (\text{adic spaces})$$

sending

$$\mathrm{Spm}(A) \mapsto \mathrm{Spa}(A, A^0).$$

We haven't yet said what perfectoid spaces are, but they form a subcategory of adic spaces.

Example 1.3.5. Let k be a non-archimedean field with absolute value $|\cdot|$ and $k^0 = O_k$. Then we have an embedding

$$\mathrm{Spa}(k, k^0) = \{|\cdot|\} \hookrightarrow \mathrm{Spa}(k^0, k^0).$$

This is not surjective: $\mathrm{Spa}(k^0, k^0)$ has the valuation

$$\begin{cases} (k^0)^\times \mapsto 1 \\ k^{00} \mapsto 0 \end{cases}$$

which obviously does not extend to k.

Suppose O_k is a DVR. Then we have a fully faithful functor

$$(\text{formal schemes l.f.t. } / \, O_k,) \hookrightarrow (\text{adic spaces}/ \, \mathrm{Spa}(O_k, O_k)).$$

The local finite type hypothesis on a formal scheme X means that $X = \mathrm{Spf}(A)$ where there exists a surjection $O_k[[T_1, \ldots, T_n]]\langle X_1, \ldots, X_n\rangle \twoheadrightarrow A$. The theory of Raynaud/Bertholot attaches to such a scheme its generic fiber, which is a rigid analytic space over k. This also embeds fully faithfully into adic spaces over $\mathrm{Spa}(k, O_k)$ via "taking the generic fiber" (or more precisely base chang against $\mathrm{Spa}(k, O_k) \to \mathrm{Spa}(O_k, O_k)$, and we have the following commutative diagram:

$$\begin{array}{ccc} (\text{formal schemes l.f.t. } / \, O_k,) & \hookrightarrow & (\text{adic spaces}/ \, \mathrm{Spa}(O_k, O_k)) \\ \downarrow{\scriptstyle \text{Raynaud-Bertholot}} & & \downarrow \\ (\text{rigid analytic space } / \, k) & \hookrightarrow & (\text{adic spaces}/ \, \mathrm{Spa}(k, O_k)) \end{array}$$

Chapter 2

Geometric Class Field Theory

In the first half we will explain the unramified picture from the geometric point of view, and in the second half we will sketch the generalization to the ramified situation.

2.1 The unramified case

Let p be a prime and $\mathbb{F}_q$ a finite field over $\mathbb{F}_p$. Let ℓ be a prime not equal to p.

The central actor in our story is a smooth projective geometrically connected curve $X/\mathbb{F}_q$. Let $K = K(X)$ be the field of rational functions on X. For $x \in |X|$ (the set of closed points of X), we denote $O_x = \widehat{O}_{X,x}$ and $K_x = \mathrm{Frac}(O_x)$. Let

$$\mathbb{A}_K = \prod{}' K_x.$$

The goal of unramified class field theory is to understand all abelian extensiosn of K which are everywhere unramified.

Theorem 2.1.1 (Unramified CFT)**.** *There is an isomorphism*

$$\left(\mathbb{G}_m(K)\backslash\mathbb{G}_m(\mathbb{A}_K)/\mathbb{G}_m(\prod O_x)\right)^{\wedge} \cong (G_K^{\mathrm{unr}})^{\mathrm{ab}}$$

such that

$$(a_x) \mapsto \prod_x \mathrm{Frob}_x^{\mathrm{ord}_x(a_x)}$$

where the $\wedge$ *means profinite completion.*

2.1.1 First geometric reformulation

We want to understand this statement more geometrically. The right hand side can be interpreted as

$$G_K^{\mathrm{unr}} = \pi_1(X).$$

(We are suppressing the base points.) The left hand side can be interpreted as

$$\mathbb{G}_m(K)\backslash\mathbb{G}_m(\mathbb{A}_K)/\mathbb{G}_m(\prod \mathcal{O}_x) \cong \operatorname{Pic}(X).$$

So here is a geometric reformulation.

Theorem 2.1.2. *We have a natural bijection*

$$\{\textit{characters } \pi_1(X) \to \overline{\mathbb{Z}}_\ell^\times\} \leftrightarrow \{\textit{characters} \operatorname{Pic}(X) \to \overline{\mathbb{Z}}_\ell^\times\}.$$

If we denote it by $\rho \mapsto \chi_\rho$, then

$$\rho(\mathrm{Frob}_x) = \chi_\rho(\mathcal{O}([x])) \textit{ for all } x \in X.$$

2.1.2 Second reformulation: categorification

The goal is to upgrade this statement by categorifying both sides. The point is to obtain a formulation of local nature, so that you can apply things like descent. *Although the initial statement is specific to working over a finite field, the categorical reformulation will not be.*

Let's first categorify the left sidde of Theorem 2.1.2. That's easy: it is the same thing as rank 1 local systems on X, up to isomorphism:

$$\{\text{characters } \pi_1(X) \to \overline{\mathbb{Z}}_\ell^*\} = \pi_0(\mathrm{Loc}_1(X) := \{\text{rank 1 local systems}/X\}).$$

For the right hand side, recall the following fact.

Theorem 2.1.3. *If G is a connected commutative algebraic group over $\mathbb{F}_q$, then the set of characters $G(\mathbb{F}_q) \to \overline{\mathbb{Z}}_\ell^*\}$ is isomorphism classes (i.e. π_0) of the category of "character local systems"*

$$\mathrm{CharLoc}(G) := \{\textit{character local systems } /G\}.$$

This $\mathrm{CharLoc}(G)$ is the category of local systems $L \in \mathrm{Loc}_1(G)$ such that

$$m^*L \cong p_1^*L \otimes p_2^*L \text{ on } G \times G.$$

Alternatively, one can think of the isomorphism being given

$$\psi\colon m^*L \cong p_1^*L \otimes p_2^*L \text{ on } G \times G$$

but then one also has to guarantee a cocycle condition (it is non-trivial to show that there always exists a unique such datum, i.e. that the two definitions presented are equivalent).

Remark 2.1.4. One can also think of a character local system as a *homomorphism* from G to $B\,\mathrm{GL}_1$. (A general rank 1 local system would be any morphism $G \to B\,\mathrm{GL}_1$.)

Proof. If $(M, \psi) \in \mathrm{CharLoc}(G)$, then we get a function $G(\mathbb{F}_q) \to \overline{\mathbb{Z}}_\ell^*$ by

$$g \mapsto \mathrm{Tr}(\mathrm{Frob}_y \mid M_g).$$

This is simply the function-sheaf correspondence.

The converse is trickier; it uses the *Lang isogeny*

$$L_G : G \to G$$

defined by $g \mapsto \mathrm{Frob}(g)g^{-1}$. This is an abelian étale cover of G with Galois group $G(\mathbb{F}_q)$. This construction gives an $N \in \mathrm{Loc}_1(G)$ for any $\chi: G(\mathbb{F}_q) \to \overline{\mathbb{Z}}_\ell^*$.

Exercise 2.1.5. Check that N is in fact a character local system, and that these constructions are inverse.

□

2.1.3 The Abel-Jacobi map

In the geometric formulation there is an obvious choice of bijection. To describe this we need to recall the Abel-Jacobi map

$$AJ : X \to \underline{\mathrm{Pic}}(X)$$

Here $\underline{\mathrm{Pic}}(X)$ denotes the Picard variety (as opposed to the group). For $x \in X$, we have

$$AJ(x) := \mathcal{O}([x]).$$

Theorem 2.1.6. *AJ^* induces an equivalence of categories*

$$\mathrm{CharLoc}(\underline{\mathrm{Pic}}(X)) \cong \mathrm{Loc}_1(X).$$

We denote the inverse by $L \mapsto \mathrm{Aut}_L$.

Remark 2.1.7. Although we stated the theorem for connected abelian G, we seem to be applying it to $\underline{\mathrm{Pic}}(X)$ which is not connected. Fortunately, it is also true for $G = \mathbb{Z}$ (but not necessarily for finite groups!).

Remark 2.1.8. Some observations:

1. This makes sense over any field, even $\mathbb{C}$. (We used the fact that we were over a finite field to arrive at this geometric formulation, but the final statement makes no reference to that.)

2. We have the following compatibility: for $x \in X(\mathbb{F}_q)$,

$$\mathrm{Tr}(\mathrm{Frob}_x \mid L_x) = \mathrm{Tr}(\mathrm{Frob}_{\mathcal{O}([x])} \mid \mathrm{Aut}_{L,\mathcal{O}([x])}).$$

   This is the desired compatibility condition from earlier.

3. There is a Hecke eigensheaf condition: if

$$h\colon X \times \underline{\mathrm{Pic}}(X) \to \underline{\mathrm{Pic}}(X)$$

is the map

$$(x, \mathcal{L}) \mapsto \mathcal{L} \otimes O([x]),$$

then

$$h^* \operatorname{Aut}_L \cong L \boxtimes \operatorname{Aut}_L .$$

Proof. (Deligne) There is a clear map in one direction: given a character local system on $\underline{\mathrm{Pic}}(X)$, we can pull it back to one on X via the Abel-Jacobi map.

In the other direction, we will descend to the space of line bundles of sufficiently large degree, and then use the character sheaf property to extend to all of $\underline{\mathrm{Pic}}(X)$. Fix $d > 2g - 2$, we have

$$X^d \to [X^d/S_d] \to \mathrm{Sym}^d(X) \to \underline{\mathrm{Pic}}^d(X).$$

Some observations:

- The map $\mathrm{Sym}^d(X) \to \underline{\mathrm{Pic}}(X)$ is a projective space bundle for $d \gg 0$.
- The map $[X^d/S_d] \to \mathrm{Sym}^d(X)$ is a coarse moduli space. The only difference between the two spaces is that the stack has nontrivial stabilizers.
- The map $X^d \to [X^d/S_d]$ is a S_d-torsor.

Step 1. Given L, we form $L^{\boxtimes d} \in \mathrm{Loc}_1(X^d)$. This is evidently invariant under S_d, so descends at least to $\widetilde{L}^{(d)} \in \mathrm{Loc}_1([X^d/S_d])$. To descend to the coarse space $\mathrm{Sym}^d X$, you need to check that stabilizers act trivially. That works here *because* we are in the rank 1 situation.

Example 2.1.9. For $d = 2$, we have $(x, x) \in \Delta \subset X \times X$. This has a stabilizer S_2. A local system on $L^{\boxtimes 2}_{(x,x)}$ has stalk $L_x \otimes L_x$, which is also the stalk $\widetilde{L}^{(2)}_{(x,x)}$. The S_2 action here is the switch action. But *because we're in the rank one case*, the switch map is the identity.

Step 2. This implies that $L^{\boxtimes d}$ descends to $L^{(d)}$ on $\mathrm{Sym}^d(X)$ for $d \gg 0$. The last step is easy, thanks to Deligne's observation that if we have a projective space bundle then the source and target have the same fundamental group:

$$\pi_1(\mathrm{Sym}^d(X)) \cong \pi_1(\underline{\mathrm{Pic}}^d(X))$$

so $L^{(d)}$ descends to Pic^d_X.

Step 3. To extend to all of $\underline{\mathrm{Pic}}(X)$, we use the following fact: for $d, e > 2g - 2$ the map

$$+\colon \ \mathrm{Pic}^d_X \times \mathrm{Pic}^e_X \to \mathrm{Pic}^{d+e}_X$$

obtained by tensoring the corresponding line bundles has the following "character sheaf" property. If we call the descended object $\operatorname{Aut}_{L,d} \in \mathrm{Loc}_1(\mathrm{Pic}^d_X)$ then

$$+^* \operatorname{Aut}_{L,d+e} \cong \operatorname{Aut}_{L,d} \boxtimes \operatorname{Aut}_{L,e} .$$

Because of this we can extend to all of $\underline{\mathrm{Pic}}_X$.

□

2.2 The ramified case

2.2.1 Generalized Picard varieties

We want to do something similar on open curves. Fix $S = \{x_1, \ldots, x_n\} \subset |X|$ (we may and do assume that $n \geq 1$.) Let $U = X - S$. We want to understand rank 1 local systems on U, i.e. extensions of K which are unramified outside U.

This involves "generalized Picard varieties". To introduce these, we need some setup. Let $D := [x_1] + \ldots + [x_n]$ and $D_m := mD$. As m varies we get a tower

$$D_1 \subset D_2 \subset \ldots \subset D_\infty := \text{formal completion of } X \text{ along } D.$$

Definition 2.2.1. We define $\underline{\mathrm{Pic}_{D_n}}(X)$ to be the moduli space for pairs $\{L \in \underline{\mathrm{Pic}}(X), \psi \colon L|_{D_n} \cong \mathcal{O}_{D_n}\}$.

We get a tower

$$\underline{\mathrm{Pic}_{D_\infty}}(X) = \lim\left(\ldots \to \mathrm{Pic}_{D_2}(X) \to \mathrm{Pic}_{D_1}(X) \to \mathrm{Pic}(X)\right).$$

The map $\mathrm{Pic}_{D_1}(X) \to \mathrm{Pic}(X)$ is a $\mathbb{G}_m^n$-bundle. Then $\mathrm{Pic}_{D_2}(X) \to \mathrm{Pic}_{D_1}(X)$ is a $\mathbb{G}_a^n$ bundle, and similarly for the rest of the maps. In particular, since the transition maps are affine morphisms the limit makes sense. This $\underline{\mathrm{Pic}_{D_\infty}}(X)$ is a *pro-algebraic group* over the ground field.

2.2.2 The generalized Abel-Jacobi map

We want to do an analog of the previous story in the unramified case. To do that we need an Abel-Jacobi map

$$AJ \colon U \to \underline{\mathrm{Pic}}_{D_\infty}(X)$$

which sends

$$y \mapsto (\mathcal{O}(y), ?)$$

We need to also say how to trivialize this at D_n. But there is a *canonica*l trivialization of $\mathcal{O}(y)$ at every D_n since $y \in U$ is disjoint from D_n; the map AJ can then be described

$$y \mapsto (\mathcal{O}(y), \text{canon.}).$$

Theorem 2.2.2. *The pullback AJ^* induces an isomorphism*

$$\mathrm{CharLoc}(\underline{\mathrm{Pic}}_{D_\infty}(X)) \cong \mathrm{Loc}_1(U).$$

This encodes class field theory because it tells us how to translate local systems into bundle-theoretic data, and you can translate that into an adelic description. In order to do that, we need a sheaf-function correspondence for pro-algebraic groups.

The goal of the rest of the talk is to explain why this theorem amounts to a local statement. First, however, we remark on connections with the more classical versions.

2.2.3 Some remarks

1. There exists a version with bounded ramification. It basically says that if we restrict to $\underline{\mathrm{Pic}}_{D_n}$, then we get Galois extensions such that in the upper numbering of the ramification groups, everything above n acts trivially.

2. We get the classical formulation of CFT via the function-sheaf dictionary.

3. In characteristic 0, we have $\pi_1(\mathbb{A}^n) = 0$ so

$$\mathrm{CharLoc}(\underline{\mathrm{Pic}}_{D_\infty}(X)) = \mathrm{CharLoc}(\underline{\mathrm{Pic}}_D(X)).$$

   So in this case we can proceed as before using the following observation:

   There exists $d \gg 0$ such that

$$\mathrm{Sym}^d\, U \to \underline{\mathrm{Pic}}_D^d(X)$$

   is an affine space bundle.

   You can also do something like in the case of tame ramification, but wild ramification truly presents new difficulties.

4. Serre's classifcal proof (as in *"Algebraic groups and class fields"*) uses the following two results.

   Theorem 2.2.3 (Rosenlicht)**.** *The Abel-Jacobi map $AJ\colon U \to \underline{\mathrm{Pic}}_{D_\infty}(X)$ is the universal map for $U \to G$ for G a commutative smooth algebraic group.*

   This tells us that if we know that local systems are always pulled back from commutative groups then they are even pulled back from $\underline{\mathrm{Pic}}_{D_\infty}(X)$; this turns out to apply here.

   Theorem 2.2.4. *If A is a finite abelian group, then any A-torsor $V \to U$ is pulled back via*

$$\begin{array}{ccc} V & \longrightarrow & G' \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ U & \longrightarrow & G \end{array}$$

   where $\pi\colon G' \to G$ is an isogeny of commutative smooth algebraic groups with kernel A.

   Example 2.2.5. If $A = \mathbb{Z}/p$, you can see this by Artin-Schreier theory.

2.2.4 The descent step

Instead of discussing this classical stuff we want to focus on explaining what happens if you try to imitate the proof in the unramified case.

Proof. Assume $D = [x]$. (This isn't necessary but simplifies the discussion.) Fix $L \in \mathrm{Loc}_1(U)$. We want to use the descent; we get $L^{(d)} \in \mathrm{Loc}_1(\mathrm{Sym}^d(U))$. The hard step is to descend $L^{(d)}$ to $\underline{\mathrm{Pic}}_{D_\infty}(X)$ along $\mathrm{Sym}^d X \to \underline{\mathrm{Pic}}^d_{D_\infty}(X)$.

Consider the cartesian diagram

$$\begin{array}{ccc} \mathrm{Sym}^d(X) & \longleftarrow & T \\ \downarrow & & \downarrow \\ \underline{\mathrm{Pic}}^d(X) & \longleftarrow & \underline{\mathrm{Pic}}^d_{D_\infty}(X) \end{array}$$

The map $\underline{\mathrm{Pic}}^d(X) \leftarrow \underline{\mathrm{Pic}}^d_{D_\infty}(X)$ is a torsor for O_x^*, since the fiber over a point is the space of rigidifications of D_∞ (remember that we are assuming that $D = \{[x]\}$). The fiber product T is the moduli space for the datum of $D' \in \mathrm{Sym}^d(X)$ plus a trivialization for $O(D')$ on the formal completion along x.

Now consider the base change with respect to $\mathrm{Sym}^d(U) \to \mathrm{Sym}^d(X)$.

$$\begin{array}{ccc} \mathrm{Sym}^d(U) & \longleftarrow & ??? \\ \downarrow & & \downarrow \\ \mathrm{Sym}^d(X) & \longleftarrow & T \\ \downarrow & & \downarrow \\ \underline{\mathrm{Pic}}^d(X) & \longleftarrow & \underline{\mathrm{Pic}}^d_{D_\infty}(X) \end{array}$$

Since U is disjoint from $\{x\}$, we get a 0-section

$$\mathrm{Sym}^d(U) \to \mathrm{Sym}^d(U) \times O_x^*.$$

so the fibered product will be a trivial O_x^* torsor over $\mathrm{Sym}^d(U)$:

$$\begin{array}{ccc} \mathrm{Sym}^d(U) & \longleftarrow & \mathrm{Sym}^d(U) \times O_X^* \\ \downarrow & & \downarrow \\ \mathrm{Sym}^d(X) & \longleftarrow & T \\ \downarrow & & \downarrow \\ \underline{\mathrm{Pic}}^d(X) & \longleftarrow & \underline{\mathrm{Pic}}^d_{D_\infty}(X) \end{array}$$

Let's remind ourselves of our goal: we want to descend a local system from $\mathrm{Sym}^d(U)$ to $\underline{\mathrm{Pic}}^d_{D_\infty}(X)$. The map $T \to \underline{\mathrm{Pic}}^d_{D_\infty}(X)$ is a projective space bundle, so we can descend any local system along it; therefore it suffices to descend to T.

For this, the strategy is to find some $M \in \mathrm{CharLoc}(O_x^*)$ such that $L^{(d)} \boxtimes M \in \mathrm{Loc}_1(\mathrm{Sym}^d(U) \times O_x^*)$ extends to T. Since everything is smooth we only have to extend along codimension-one points of the complement. Since the map is base-changed from $\mathrm{Sym}^d(U) \to \mathrm{Sym}^d(X)$ against a 0-dimensional torsor, the situation basically looks the same as for the two maps. What are the codimension-one points of $\mathrm{Sym}^d(X) - \mathrm{Sym}^d(U)$? They correspond to the subset parametrizing divisors where *two points collide*, so in codimension 1 we are reduced to the $d = 1$ case. To do this we use a local analogue of this story, which we explain presently.

2.2.5 Local geometric class field theory

For X the formal disk and $x \in X$ the closed point, U the punctured disk, we get by analogous constructions an O_x^*-torsor $T \to X$ whose fiber over $y \in X$ is the space of trivializations of $O(y)$ at x. This splits canonically over U, so $T|_U \cong U \times O_x^*$.

Theorem 2.2.6 (Local class field theory)**.** *There is an equivalence*

$$\mathrm{Loc}_1(U)/\,\mathrm{Loc}_1(X) \cong \mathrm{CharLoc}(O_x^*).$$

Moreover, $[L] \in \mathrm{Loc}_1(U)/\,\mathrm{Loc}_1(X)$ *corresponds to* $M \in \mathrm{CharLoc}(O_x^*)$ *if and only if* $L \boxtimes M|_{O_x^*}$ *extends to* T.

This informs us how to choose M locally.

□

Chapter 3

The Fargues-Fontaine Curve

3.1 Preliminaries on Fontaine's Rings

3.1.1 Construction of $C^\flat$

We start with some (pre)historical remarks. We denote by C a complete algebraically closed field of characteristic 0; we can imagine $C = \mathbb{C}_p$. We associate to C the set

$$C^\flat = \{(x^{(n)}) \mid (x^{(n+1)})^p = x^{(n)} \text{ for all } n\}.$$

We can define on this set multiplication and addition operations making it into a commutative ring:

$$(x^{(n)})(y^{(n)}) := (x^{(n)}y^{(n)})$$

and

$$(x^{(n)}) + (y^{(n)}) := \left(\lim_{k\to\infty}(x^{(n+k)} + y^{(n+k)})^{p^k}\right).$$

For $x \in C$, we get $x^\flat = (x, x^{1/p}, x^{1/p^2}, \ldots) \in C^\flat$ which is well-defined *up to* $\epsilon^{\mathbb{Z}_p}$ where $\epsilon = (1, \zeta_p, \zeta_{p^2}, \ldots)$. Then we denote

$$x^\sharp := x^{(0)}.$$

Theorem 3.1.1. *$C^\flat$ is an algebraically closed field of characteristic p, complete for the valuation*

$$v_{C^\flat}(x) = v_p(x^\sharp)$$

and we have $k_{C^\flat} = k_C$.

3.1.2 Construction of A_{inf}

Definition 3.1.2. Let $A_{\text{inf}} = W(\mathcal{O}_{C^\flat})$. An element $x \in A_{\text{inf}}$ can be (uniquely) represented as

$$x = \sum_{k\in\mathbb{N}}[x_k]p^k, \quad x_k \in \mathcal{O}_{C^\flat}.$$

We have a Frobenius endomorphism φ on A_{inf} by

$$\varphi\left(\sum [x_k]p^k\right) = \sum [x_k^p]p^k.$$

We also have a map

$$\theta\colon A_{\text{inf}} \twoheadrightarrow \mathcal{O}_C$$

sending

$$\sum [x_k]p^k = \sum x_k^\sharp p^k.$$

Proposition 3.1.3. *θ is a surjective ring homomorphism with kernel generated by $p - [p^\flat]$. We have $\mathcal{O}_C = A_{\text{inf}}/(p - [p^\flat])$.*

3.1.3 Construction of B_{dR} and B_{cris}

Definition 3.1.4. We define

$$B_{\text{dR}}^+ := \varprojlim_k A_{\text{inf}}[1/p]/(p - [p^\flat])^k$$

and the subring

$$A_{\text{cris}} := A_{\text{inf}}\left[\frac{(p-[p^\flat])^k}{k!}, k \in \mathbb{N}\right]^\wedge.$$

The ring A_{cris} has an element $t := \log[\epsilon]$. It is easy to see that $\varphi(t) = pt$. Define $B_{\text{cris}} = A_{\text{cris}}[1/t]$, which has an action of φ. We have $B_{\text{cris}} \subset B_{\text{dR}} := B_{\text{dR}}^+[1/t]$. Finally, we define $B_e = B_{\text{cris}}^{\varphi=1}$.

These rings are related by the "fundamental exact sequence"

$$0 \to \mathbb{Q}_p \to B_e \to B_{\text{dR}}/B_{\text{dR}}^+ \to 0.$$

Note that this implies

$$\operatorname{Gr} B_e = \mathbb{Q}_p + \frac{1}{t}C[1/t].$$

Surprisingly, B_e is a PID. This is the starting point for everything.

3.2 The Fargues-Fontaine curve

3.2.1 Informal description

The p-adic comparison theorems for crystalline/de Rham/étale cohomology lead one to consider the category of pairs (W_e, W_{dR}^+) where W_e is a free B_e-module and W_{dR}^+ is a free B_{dR}^+-module such that

$$B_{\text{dR}} \otimes_{B_e} W_e = B_{\text{dR}} \otimes_{B_{\text{dR}}^+} W_{\text{dR}}^+.$$

(In comparison theorems W_e is the crystalline cohomology, W_{dR}^+ is the de Rham cohomology, and the étale cohomology can be recovered from this setup.)

Fargues and Fontaine were looking for a geometric object that would explain why this category has good properties. Roughly speaking, they constructed a curve X from Spec B_e completed by adding a point at ∞ (corresponding to the valuation given by the grading; in general you should imagine that you can add a "point at ∞" whenever one has a filtered Dedekind domain).

So this curve X has the properties that $B_e = \mathcal{O}(X - \{\infty\})$, and $B_{\mathrm{dR}}^+ = \mathcal{O}_{X,\infty}$. Then the fundamental exact sequence can be interpreted as follows. The fact that

$$B_e \twoheadrightarrow B_{\mathrm{dR}}/B_{\mathrm{dR}}^+$$

is a surjection is saying that we can find find a function which has any particular polar part at ∞. The short exact sequence tells us that the global sections are $\mathbb{Q}_p$, which makes us imagine that the curve is "proper". (Note however that the residue field at ∞ is C, which is weird since it's infinite-dimensional over $\mathbb{Q}_p$.)

In these terms the category of pairs (W_e, W_{dR}^+) corresponds to the category of vector bundles over $X = \operatorname{Spec} B_e \coprod$ (formal neighborhood around ∞) by the Beauville-Laszlo interpretation. The comparison isomorphism is what you need to glue two vector bundles.

What is the meaning of $B_e = (B_{\mathrm{cris}})^{\varphi=1}$? It suggests that our X should be obtained by taking the quotient of some bigger space by φ. Indeed, we have

$$X^{\mathrm{ad}} = Y^{\mathrm{ad}}/\varphi^{\mathbb{Z}}$$

where $Y^{\mathrm{ad}} = \operatorname{Spa}(A_{\mathrm{inf}}) - (p[p^\flat])$.

Remark 3.2.1. One might wonder why we don't build Y using $\operatorname{Spa}(B_{\mathrm{cris}})$, in light of $B_e = (B_{\mathrm{cris}})^{\varphi=1}$. This is bad because φ is not an automorphism of B_{cris}; we should only quotient by automorphisms. If we were to replace B_{cris} by the largest subring on which φ is an isomorphism, then one does indeed arrive at the same Y.

3.2.2 First construction

Definition 3.2.2. Let $[E : \mathbb{Q}_p] < \infty$. Define $A_{\mathrm{inf},E} = \mathcal{O}_E \otimes_{W(k_E)} A_{\mathrm{inf}}$ where ϖ is a uniformizer of E. For $x \in A_{\mathrm{inf},E}$ we can write

$$x = \sum [x_k]\varpi^k$$

which admits an action of $\varphi_E = 1 \otimes \varphi^f$ where $q = |k_E| = p^f$. Then

$$\varphi_E(\sum [x_k]\varpi^k) = \sum [x_k^q]\varpi^k.$$

The expression suggests that $A_{\mathrm{inf},E}$ is similar to $\mathcal{O}_C[[T]]$.

This suggests defining the Newton polygon

$$\mathrm{NP}_x := \text{convex hull } \{(k, v_{C^\flat}(x_k))\}.$$

Remark 3.2.3. The theory of Newton polygons is a little subtler than usual because there are infinitely many coefficients, but it is a theorem that things work out.

Theorem 3.2.4. *If $\lambda < 0$ is a slope of NP_x with multiplicity d then there exist $a_1, \dots, a_d \in O_{C^\flat}$ such that $\frac{v_{C^\flat}(a_i)}{v_p(\varpi)} = -\lambda$ for each i and*

$$(\varpi - [a_1]) \dots (\varpi - [a_k]) \mid x.$$

Remark 3.2.5. This is the result we would have expected if we were working instead with $\mathbb{C}[[t]]$. In the case $O_C[[T]]$ or $O_{C^\flat}[[T]]$ the a_i would be unique, but they are *not* unique here.

Corollary 3.2.6. *The closed prime ideals of $A_{\inf,E}$ are*

- (0)*, with residue field* $\mathrm{Frac}(A_{\inf,E})$*,*
- *maximal ideals, with residue fields $k_{C^\flat} = k_C$.*
- (ϖ)*, with residue field $C^\flat$.*
- $(\varpi - [a])$*, up to some equivalence relation, with residue field K_a which is algebraically closed and complete for v_p, and has $K_a^\flat \cong C^\flat$.*
- $W(\mathfrak{m}_{C^\flat})$*"* $= [\varpi^\flat]$*" with residue field is $\widetilde{E} := E \otimes W(k_{C^\flat})$.*

Now we define the curve $Y_{E,C^\flat} =: Y_E := \mathrm{Spec}' A_{\inf,E} - \{\varpi[\varpi^\flat] = 0\}$, where Spec' means that we take only the *closed* prime ideals.

Proposition 3.2.7. *The points of Y_E correspond to E-untilts of $C^\flat$.*

Proof. For $y \in Y_E$ of the form $y = (\varpi - [a])$ the residue field K_y is an "E-untilt" of $C^\flat$. An E-untilt is a pair $(K \supset E, \iota\colon K^\flat \cong C^\flat)$. Given an E-untilt, we produce a point $(\varpi - [\iota(\varpi^\flat)]) \in Y_E$. This shows: □

3.2.3 Lubin-Tate theory

The description of the points of Y_E above has some problems; for instance there is no easy description for when $(\varpi - [a]) = (\varpi - [b])$. We can get a better parametrization of Y_E via Lubin-Tate theory. Associated to (E, ϖ), for $\alpha \in O_E$ we have $\sigma_\alpha \in \alpha T + T^2 O_E[[T]]$ such that

$$\sigma_\alpha(X \oplus Y) = \sigma_\alpha(X) + \sigma_\alpha(Y)$$

and $\sigma_\varpi \equiv T^q \mod \varpi$. Then we can define

$$\sigma_{\alpha/\varpi^n}(T) = \sigma_\alpha(T^{q^{-n}}).$$

This gives an action of E on $\mathfrak{m}_{C^\flat}$, with α acting by the reduction of σ_α modulo p.

Theorem 3.2.8. *If $x \in \mathfrak{m}_{C^\flat}$, then*

1. $[x]_\varpi := \lim_{n\to+\infty} \sigma_{\varpi^m}([x^{q^{-n}}])$ *is the unique lift of* x *in* $A_{\mathrm{inf},E}$ *such that*
$$\varphi_E([x]_\varpi) = \sigma_\varpi([x]_\varpi).$$

2. *We have* $\sigma_\alpha([x]_\varpi) = [\sigma_a(x)]_\varpi$

3. *The map* $x \mapsto \xi_x := \frac{[x]_\varpi}{[x^{1/q}]_\varpi}$ *gives a bijection between*
$$(\mathfrak{m}_{C^\flat} - \{0\})/\mathcal{O}_E^* \cong Y_E.$$

3.3 The analytic curve

3.3.1 Construction

As we have just seen, Y_E is "the punctured open ball over $C^\flat$ modulo $\mathcal{O}_E^*$". So we would like to say:
$$Y_E = \widetilde{D}^*_{C^\flat}/\mathcal{O}_E^* = \widetilde{D}^*_C/\mathcal{O}_E^*$$
where D is the open unit ball. To make sense of this we need diamonds; indeed, giving rigorous meaning to this expression was one of the motivations for Scholze's theory of diamonds.

The *adic Fargues-Fontaine curve* Y_E^{ad} is defined to be
$$Y_E^{\mathrm{ad}} := \mathrm{Spa}(A_{\mathrm{inf},E}) - \{\varpi[\varpi^\flat] = 0\}.$$

We will eventually define
$$X_E^{\mathrm{ad}} := Y_E^{\mathrm{ad}}/\varphi^{\mathbb{Z}}$$
after we show that this makes sense.

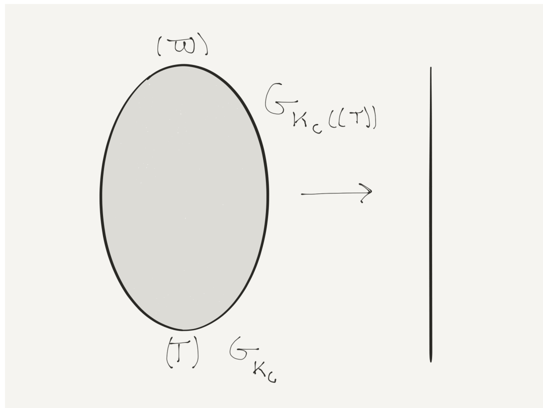
Remark 3.3.1. Proving that these really are adic spaces, i.e. the structure sheaves are sheafy, is quite nontrivial.

3.3.2 Some properties

For any $u \in \mathfrak{m}_{C^\flat} - \{0\}$, we get
$$\mathcal{O}_{\widetilde{E}}[[T^{q^{-\infty}}]] \hookrightarrow A_{\mathrm{inf},E}$$
by sending $T \mapsto [u]$.

Theorem 3.3.2. $A_{\mathrm{inf},E}$ *is the* (ϖ, T)*-completion of the maximal extension of* $\mathcal{O}_{\widetilde{E}}[[T^{q^{-\infty}}]]$ *unramified outside* $T = 0$.

We have $\mathcal{O}_{\widetilde{E}}[[T]] \to \mathcal{O}_{\widetilde{E}}[[T^{q^{-\infty}}]] \to A_{\inf,E}$. Therefore, we can view $\mathrm{Spa}(A_{\inf,E})$ as a cover of $\mathrm{Spa}\,\mathcal{O}_{\widetilde{E}}[[T]]$, which we can think of as a "unit disk".



That is, $\mathrm{Spa}(A_{\inf,E})$ is a profinite covering ramified only over the point (T), with fiber $G_{k_C((T))}$ over all points except (T), where it has fiber G_{k_C}.

There is a map $\delta\colon \mathrm{Spa}\,\mathcal{O}_E[[T]] \to [0,\infty]$ defined by

$$\delta(\overline{x}) := \frac{v_{\overline{x}}(\varpi)}{v_{\overline{x}}([\varpi^\flat])}.$$

(In terms of absolute values, this would be the "radius function" on the unit disk.) Composing this with the map from $\mathrm{Spa}\,A_{\inf,E}$, we obtain a map

$$\mathrm{Spa}\,A_{\inf,E} \to [0,+\infty]$$

which sends

$$v_x \mapsto v_{\overline{x}} \mapsto \delta(\overline{x})$$

and $Y_E^{\mathrm{ad}} = \delta^{-1}((0,\infty)) \subset \mathrm{Spa}\,A_{\inf,E}$.

Definition 3.3.3. We define $Y_I := \delta^{-1}(I)$, with $\mathcal{O}(Y_I)$ being $A_{\inf,E}[1/\varpi, 1/[\varpi^\flat]]$ completed with respect to the family of valuations v_r for $r \in I$, where

$$v_r\left(\sum [x_k]\varpi^k\right) = \begin{cases} \inf(v_{C^\flat}(x_k) + krv_p(\varpi)) & r \geq 1, \\ \inf(\frac{1}{r}v_{C^\flat}(x_k) + kv_p(\varpi)) & r \leq 1. \end{cases}$$

Proposition 3.3.4. *Y_I enjoys the following properties:*

1. *$\mathcal{O}(Y_I)$ is a Fréchet algebra, and is Banach if I is compact.*
2. *If $\min I \neq 0$ then $\mathcal{O}(Y_I)$ is Bézout, and is even a PID if I is compact.*

We have

$$\delta(\varphi(x)) = \frac{1}{q}\delta(x),$$

so φ acts properly on Y_E. This implies that $X_E^{\mathrm{ad}} := Y_E^{\mathrm{ad}}/\varphi^{\mathbb{Z}}$ is compact, since $Y_{[1,q]}$ covers it.

Theorem 3.3.5. $X_E^{\mathrm{ad}} = (X_E)^{\mathrm{ad}}$ *for some scheme* X_E.

If $[E' : E] = +\infty$ then

$$X_{E'} = E' \otimes_E X_E.$$

The content here is that

$$\mathcal{M}(Y_{E'}^{\mathrm{ad}})^{\varphi_{E'}=1} = E' \otimes_E \mathcal{M}(Y_E^{\mathrm{ad}})^{\varphi_E=1}.$$

where $\mathcal{M}$ denotes meromorphic functions.

Theorem 3.3.6. *All finite étale coverings of* X_E *are of this shape, so* $\pi_1(X_E) = \mathrm{Gal}(\overline{E}/E)$.

Chapter 4

Perfectoid Spaces

4.1 Perfectoid Rings

Fix a prime p. We make references throughout to [12] Scholze's IHES paper on perfectoid spaces, [13] Scholze's MSRI lectures.

Recall that a *complete Tate ring A* is a complete topological ring A such that there exists a complete subring $A_0 \subset A$ which is open and a finitely generated ideal $I \subset A_0$ such that $\{I^n : n \in \mathbb{N}\}$ is a neighborhood basis of 0; also, there exists $\varpi \in A^\times$ such that $\varpi^n \to 0$ as $n \to \infty$. Such a ϖ is called a *pseudo-uniformizer*.

Definition 4.1.1. A subset $S \subset A$ is *bounded* if $S \subset \varpi^{-n} A_0$ for some n. We denote by A^0 the ring of *power-bounded elements* of A:

$$A^0 = \{a \in A \mid \{a^n\} \text{ is bounded}\}.$$

Definition 4.1.2. A *perfectoid ring* is a complete Tate ring A such that

- A^0 is bounded,
- there exists a pseudo-uniformizer $\varpi \in A$ such that $p/\varpi^p \in A^0$, and
- the map $\Phi \colon A^0/(\varpi) \mapsto A/(\varpi^p)$ sending $x \mapsto x^p$ is an isomorphism.

Example 4.1.3. Examples of perfectoid rings:

1. $\mathbb{Q}_p^{\text{cyc}} := \mathbb{Q}_p(\zeta_{p^n} \forall n)^\wedge$ where ζ_{p^n} is a primitive p^nth root of unity.
2. $\mathbb{F}_p((T^{1/p^\infty})) := \left(\bigcup_n \mathbb{F}_p((T^{1/p^n}))\right)^\wedge$ with the T-adic topology.

Example 4.1.4. If A is a complete Tate ring $p = 0$ and A^0 bounded, then

$$A \text{ perfectoid} \iff A \text{ perfect}.$$

Reference: [12, Proposition 5.9]

Example 4.1.5. If $A = K$ is a non-archimedean field then K is perfectoid if and only if the valuation is non-discrete, $|p| < 1$, and $\Phi\colon \mathcal{O}_K/(p) \to \mathcal{O}_K/(p)$ is surjective. [13, Proposition 6.1.8, 6.1.9]

We saw earlier that if we have such a Tate ring, then we can form a Huber pair and then take its adic spectrum. However, it is not clear that this gives rise to an adic space because it is not clear that the structure presheaf will be a sheaf.

Theorem 4.1.6. *Let (A, A^+) be a Huber pair (i.e. $A^+ \subset A$ is an open subring which is also integrally closed) with A perfectoid. Then for all rational subdomains $U \subset X = \mathrm{Spa}(A, A^+)$ the ring $\mathcal{O}_X(U)$ is perfectoid, which implies that $\mathcal{O}_X(U)^0$ is bounded for all U, hence (A, A^+) is sheafy.*

Proof. The original proof was [12, Theorem 6.3iii], but the argument there is different. □

4.2 Tilting

Definition 4.2.1. Let A be a perfectoid ring. We define its *tilt* to be

$$A^\flat := \varprojlim_{x \mapsto x^p} A.$$

Its elements will be expressed as

$$(x^{(0)}, x^{(1)}, \ldots) \quad \text{such that } x^{(n)} = (x^{(n+1)})^p.$$

This is a ring under the operations

$$(x^{(n)})(y^{(n)}) = (x^{(n)}y^{(n)})$$

and

$$(x^{(n)}) + (y^{(n)}) = \left(\lim_{k\to\infty} (x^{(n+k)} + y^{(n+k)})^{p^k}\right).$$

Example 4.2.2. We have $(\mathbb{Q}_p^{\mathrm{cyc}})^\flat = \mathbb{F}_p((T^{1/p^\infty}))$ with

$$T = (1, \epsilon, \epsilon_2, \ldots) - (1, 1, \ldots).$$

where ϵ_n is a primitive p^nth root of unity. We take

$$T^{1/p^n} = (\epsilon_n, \epsilon_{n+1}, \ldots) - (1, 1, \ldots).$$

We have $\mathbb{Z}_p^* \xrightarrow{\sim} \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{cyc}}/\mathbb{Q}_p)$ via the cyclotomic character:

$$a \mapsto (\epsilon_n \mapsto \epsilon_n^{a \mod p^n}).$$

Since tilting is supposed to preserve Galois groups, we get as expected an action of $\mathbb{Z}_p^*$ on $\mathbb{F}_p((T^{1/p^\infty}))$ with

$$a \in \mathbb{Z}_p^*\colon (T^{1/p^n} \mapsto (1 + T^{1/p^n})^a - 1).$$

Lemma 4.2.3. *If A is a perfectoid ring, then*

1. *$A^\flat$ is a perfectoid ring with $p = 0$.*
2. *We have*
$$A^{\flat 0} = \varprojlim_{x \mapsto x^p} A^0 = \varprojlim_{x \mapsto x^p} A^0/(p).$$
3. *There exists a pseudo-uniformizer $\varpi \in A$ with $\varpi^{1/p^n} \in A$ for all n. Write*
$$\varpi^\flat = (\varpi, \varpi^{1/p}, \ldots).$$
Then $A^\flat = A^{\flat 0}[1/\varpi^\flat]$.
4. *There is a multiplicative (but not additive) map $A^\flat \to A$ sending $(x^{(n)}) \mapsto x^{(0)}$, denoted $x \mapsto x^\#$. It induces an isomorphism of rings*
$$A^{\flat 0}/\varpi^\flat \xrightarrow{\sim} A^0/\varpi, \quad \text{where } \varpi = \varpi^{\flat \#}.$$
5. *Fixing A and $A^\flat$, the association $A^+ \rightsquigarrow A^{+\flat} = \varprojlim_{x \mapsto x^p} A^+$ gives a bijection between Huber pairs (A, A^+) and $(A^\flat, A^{\flat +})$*

Proof. See [13, Lemmas 6.2.2 and 6.2.4]. □

Theorem 4.2.4. 1. *There is a homeomorphism $X := \mathrm{Spa}(A, A^+) \to X^\flat := \mathrm{Spa}(A^\flat, A^{+\flat})$ sending $x = |\cdot|_x$ to $x^\flat = |\cdot|_{x^\flat}$ with*
$$|f|_{x^\flat} := |f^\#|_x.$$
Furthermore, it preserves rational subsets.

2. *If $U \subset X$ is rational then $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is perfectoid with tilt $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}(U^\flat))$.*

Proof. See [12, Theorem 6.3 i,ii]. □

Idea of proof of Theorem 4.1.6. If $p = 0$ in A, then we can write
$$(A, A^+) = \left(\varinjlim_i (A_i, A_i^+)^{\mathrm{perf}, \wedge} \right)^\wedge$$
because (A_i, A_i^+) reduced of topologically finite type over a perfectoid field. The question of sheafiness for (A, A^+) can in this way be reduced to that for (A_i, A_i^+), and then it follows from a result of Huber (basically by Noetherian approximation).

If $p \neq 0$ in A, then we can use tilting. The theorem having been established in positive characteristic, we can deduce the result for X from that for $X^\flat$ via
$$\mathcal{O}_{X^\flat}^+(U^\flat)/\varpi^\flat \cong \mathcal{O}_X^+(U)/\varpi.$$
Indeed, being a sheaf means that the first Cech cohomology group of this vanishes. The point is basically that if something is true modulo ϖ^n, then it is true for $\varpi^{n/p}$ by inverting Frobenius. Thus the point is that it allows you to automatically improve bounded results to arbitrarily fine results. □

4.3 Perfectoid Spaces

4.3.1 Étale morphisms

Definition 4.3.1. A *perfectoid space* is an adic space covered by $\mathrm{Spa}(A, A^+)$ with A perfectoid.

Theorem 4.2.4 implies that tilting glues to give a functor $X \mapsto X^\flat$.

We want to discuss the "étale site" of a perfectoid space.

Theorem 4.3.2. *Let A be a perfectoid ring with tilt $A^\flat$.*

1. *Any finite étale A-algebra is perfectoid.*
2. *The functor $B \mapsto B^\flat$ is an equivalence between*
    - *perfectoid A-algebras and $A^\flat$-algebras*
    - *finite étale A-algebras and finite étale $A^\flat$-algebras.*

Proof. See [12, Theorem 5.2]. □

Definition 4.3.3. A morphism $f\colon Y \to X$ of perfectoid spaces is called

1. *finite étale* if for all open affinoids $U = \mathrm{Spa}(A, A^+) \subset X$ the pre-image $f^{-1}(U) = \mathrm{Spa}(B, B^+)$ is also affinoid, with B finite étale over A and B^+ the integral closure of A^+ in B.
2. *étale* if for every $y \in Y$, there exists an open neighborhood $U \subset Y$ containing y and $V \subset X$ an open subset with $f(U) \subset V$, and a diagram

$$\begin{array}{ccc} U & \xhookrightarrow{\text{open}} & W \\ & & \big\downarrow{\scriptstyle \text{finite étale}} \\ & & V \end{array}$$

Remark 4.3.4. Why this definition of étale? One issue is that all perfectoid spaces are reduced, so there's no hope of formulating an infinitesimal lifting criterion. Also, things are never of finite type.

Remark 4.3.5. The analogous result for schemes is *false*. (For instance, you can take the étale locus of a branched cover of $\mathbb{P}^1$ of degree at least 2.) However, the analogous result for rigid analytic spaces is true.

Proof. See [12 Lemma 7.3 and Corollary 7.8]. □

4.3.2 The étale site

Proposition 4.3.6. *Étale morphisms of perfectoid spaces enjoy the following properties:*

1. *Finite étale morphisms of perfectoid spaces are stable under compositions and base change. (In particular, there exist fiber products in the category of perfectoid spaces; this is not the case for general adic spaces.)*

2. *étale morphisms of perfectoid spaces are open.*

3. *$f : X \to Y$ is étale if and only if $f^\flat : X^\flat \to Y^\flat$ is étale.*

Definition 4.3.7. The *étale site* $X_{\text{ét}}$ of X is the category of perfectoid spaces étale over X with topological coverings.

If we apply this to a perfectoid *field*, then we get an identification of absolute Galois groups, recovering a "classical" result of Fontaine-Wintenberger.

4.3.3 The philosophy of tilting

The topological properties of a perfectoid space X and its tilt then all *topological* information, such as $|X|, X_{\text{ét}}$ can be recovered from $X^\flat$. However, a perfectoid space over $\mathbb{Q}_p$ has a structure morphism to $\text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ which is "forgotten" by tilting.

Part II

Day Two

Chapter 5

The Pro-étale and v-Topologies

5.1 The pro-étale topology

The pro-étale topology is a topology on the category Perf of perfectoid spaces. An important property of the category Perf which makes this theory possible is that it has all inverse limits with affinoid transition functions. (This is *not* true for the category of adic spaces.)

5.1.1 Pro-étale morphisms

Definition 5.1.1. A morphism $\mathrm{Spa}(A_\infty, A_\infty^+) \to \mathrm{Spa}(A, A^+)$ of perfectoid spaces is called *affinoid pro-étale* if

$$(A_\infty, A_\infty^+) = \left(\varinjlim (A_i, A_i^+)\right)^\wedge$$

for a filtered system of perfectoid (A, A^+)-algebras (A_i, A_i^+) such that $\mathrm{Spa}(A_i, A_i^+) \to \mathrm{Spa}(A, A^+)$ is étale. Here the $\wedge$ means the ϖ-adic completion for some pseudo-uniformizer ϖ of A, which becomes a pseudo-uniformizer for A_∞ as well.

Definition 5.1.2. A morphism $f\colon X \to Y$ of perfectoid spaces is *pro-étale* if it is affinoid pro-étale locally on source and target.

Remark 5.1.3. This definition is reasonable because the property of being affinoid pro-étale is well-behaved under localization, so the property of being pro-étale is indeed local in the analytic topology.

The content of this assertion is that if $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A_\infty, A_\infty^+)$ is a rational subset then $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ is affinoid pro-étale, and similarly if $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A_\infty, A_\infty^+)$ is finite étale.

The key step to proving these results is to show that rational subdomains or finite étale morphisms come from some finite layer.

Remark 5.1.4. Pro-étale morphisms are *not necessarily open*. For example, the inclusion of a point in a profinite set (considered as an affinoid perfectoid space over some perfectoid field) is affinoid pro-étale. Indeed, you can consider the inverse limit over all open neighborhoods of the point.

Proposition 5.1.5. *Pro-étale morphisms are stable under base change and composition.*

5.1.2 The pro-étale topology

Definition 5.1.6. The *pro-étale topology* on Perf is the (pre)topology whose class of covers is generated by:

- all open covers in the analytic topology,
- all affinoid pro-étale maps $\mathrm{Spa}(A_\infty, A_\infty^+) \to \mathrm{Spa}(A, A^+)$ that are surjective (on points).

Warning 5.1.7. A family of pro-étale morphisms $f_i \colon X_i \to X$ that is jointly surjective is not necessarily a covering, for the same reason that a quasicompactness condition is necessary in the fpqc topology. For instance, the map from the disjoint union of singleton points of a profinite set to the profinite set is not a covering in this topology.

Just as with the fpqc topology, what one needs is an additional quasicompactness condition saying that every quasicompact open on the base is the image of some quasicompact open in the source.

Proposition 5.1.8. *The structure sheaf $X \mapsto O_X(X)$ is a sheaf for the pro-étale topology, and moreover $H^i_{\text{pro-étale}}(X, O_X) = 0$ for all $i > 0$ if X is affinoid perfectoid.*

Proof. It is part of the definition of a perfectoid space that O_X is a sheaf for the analytic topology. We need to know that if $\mathrm{Spa}(A_\infty, A_\infty^+) \to \mathrm{Spa}(A, A^+)$ is affinoid pro-étale and surjective, then the complex

$$0 \to A \to A_\infty \to A_\infty \widehat{\otimes}_A A_\infty \to \dots \tag{5.1}$$

is exact.

Write $(A_\infty, A_\infty^+) = \left(\varinjlim (A_i, A_i^+)\right)^\wedge$ as in the definition. Then consider

$$0 \to A^+/\varpi \to A_\infty^+/\varpi \to A_\infty^+/\varpi \otimes_{A^+/\varpi} A_\infty^+/\varpi \to \dots \tag{5.2}$$

This is the filtered direct limit of the complexes

$$0 \to A^+/\varpi \to A_i^+/\varpi \to A_i^+/\varpi \otimes_{A^+/\varpi} A_i^+/\varpi \to \dots \tag{5.3}$$

Since the map $\mathrm{Spa}(A_\infty, A_\infty^+) \to \mathrm{Spa}(A, A^+)$ is surjective, the same holds for $\mathrm{Spa}(A_i, A_i^+) \to \mathrm{Spa}(A, A^+)$.

We need to use the fact that

$$H^i_{\text{ét}}(X, O_X^+/\varpi) \stackrel{a}{=} \begin{cases} A^+/\varpi & i = 0 \\ 0 & i > 0 \end{cases}$$

where $\stackrel{a}{=}$ means an equality at the almost level. This is a classical result of Tate for rigid analytic spaces. For perfectoid spaces, it is proved by using tilting to reduce to the case of characeristic p. Then you can reduce to rigid analytic spaces using noetherian approximation.

The point is that the fact implies that (5.3) is almost-exact. Hence (5.2) is also almost exact. As we saw yesterday, the perfectoid property allows one to upgrade this to an almost integral level, so

$$0 \to A^+ \to A^+_\infty \to A^+_\infty \otimes_{A^+} A^+_\infty \to \dots$$

is almost exact. □

Corollary 5.1.9. *The pro-étale topology is subcanonical (i.e. every representable functor is a sheaf).*

Proof sketch. You first show that you can glue morphisms in the analytic topology, then you show that you can glue morphisms in the pro-étale topology by reducing the preceding proposition. □

Warning 5.1.10. The property of being pro-étale is *not* local in the pro-étale topology.

5.1.3 Diamonds

If we consider a larger class of morphisms which are pro-étale locally in the pro-étale topology, then it doesn't really change our topology.

Definition 5.1.11. Such a morphism is called *locally quasi-profinite*.

This condition can be checked at the level of geometric fibers:

Proposition 5.1.12. *A morphism is locally quasi-profinite if and only if for all geometric points* $\mathrm{Spa}(C, C^+) \to Y$*, the fiber* $X \times_Y \mathrm{Spa}(C, C^+) \to \mathrm{Spa}(C, C^+)$ *is pro-étale, which is equivalent (in this case) to being a profinite set with points having residue field* C.

Definition 5.1.13. A *diamond* is a sheaf $\mathcal{F}$ on the pro-étale toplogy on perfectoid spaces in characteristic p such that there exists a map $h_Y \to \mathcal{F}$ for some representable functor h_Y which is surjective, relatively representable and locally quasi-profinite.

Remark 5.1.14. This relation between this definition to perfectoid spaces is analogous to the relation between algebraic spaces and schemes.

Remark 5.1.15. We could *not* have made this definition with "locally quasi-profinite" replaced by "pro-étale".

5.2 The v-topology

We just saw that a sheaf for pro-étale covers is the same as a sheaf for locally quasi-profinite covers. Note that there is no flatness assumption here: Proposition 5.1.12 implies that quasi-profiniteness is purely a statement about fibers. This suggests that we can define a topology analogous to the fpqc topology without the "finite presentation" assumption; for this reason the topology was originally named "faithful" but was subsequently renamed.

Definition 5.2.1. The v-topology on Perf is the (pre)topology whose covers are generated by

- all open covers in the analytic topology,
- all surjective $\operatorname{Spa}(B, B^+) \to \operatorname{Spa}(A, A^+)$.

Remark 5.2.2. A good analogy is the category of compact Hausdorff spaces with covers being *all* (not necessarily continuous) surjective maps.

Proposition 5.2.3. *The structure sheaf is a sheaf for the v-topology, and moreover if X is affinoid then $H^i_v(X, \mathcal{O}_X) = 0$ for all $i > 0$.*

Proof. Write $\mathcal{F} = \mathcal{O}_X^+/\varpi$. Given a surjective morphism

$$X' := \operatorname{Spa}(B, B^+) \to \operatorname{Spa}(A, A^+) =: X,$$

we need to show that

$$0 \to \mathcal{F}(X) \to \mathcal{F}(X') \to \mathcal{F}(X' \times_X X') \to \dots \tag{5.4}$$

is almost exact. The idea is to split the statement into two cases:

1. $X' \to X$ is a "w-localization" (in the sense of Bhatt-Scholze),
2. X' is arbitrary but X is "w-local".

Even though we haven't defined these notions yet, it hopefully seems plausible that given such a notion we should be able to reduce to these two cases.

Definition 5.2.4. A spectral space X is called *w-local* if every connected component has a unique closed point and the set X^c of closed points is closed in X. (This implies that

$$|X| \hookrightarrow X \to \pi_0(X)$$

is a homeomorphism of profinite sets).

Fact. For every affinoid perfectoid space X there exists a morphism $X^z \to X$ where X^z is affinoid perfectoid and w-local, which is "universal for morphisms from w-local spaces to X". This X^z is called the "w-localization". It is basically some profinite disjoint union of the localizations of all points:

$$X^z \to X = \varprojlim_{\text{finite open cover}} \left(\coprod U_i \to X\right).$$

We now return to the exactness of (5.4).

Case 1. A w-localization is pro-étale, in which case we already know the result.

Case 2. We assume that $X = \mathrm{Spa}(A, A^+)$ which is w-local. We want to prove the exactness of (5.4), which amounts to showing (by the usual story for faithfully flat descent) that B^+/ϖ is faithfully flat over A^+/ϖ. Therefore, a reformulation of the statement we want to show is that if $f\colon \mathrm{Spa}(B, B^+) \to X$ is any morphism, then B^+/ϖ is flat over A^+/ϖ and faithfully flat if f is surjective. (The point is that everything is flat over w-local spaces, as when dealing with valuative spaces.)

Consider the composition

$$h\colon Y \xrightarrow{f} X \xrightarrow{g} T := \pi_0(X)$$

Define the sheaves $\mathcal{A} := g_*\mathcal{O}_X^+/\varpi$ and $\mathcal{M} := h_*\mathcal{O}_Y^+/\varpi$. Then $H^0(T, \mathcal{M})$ is flat over $H^0(T, \mathcal{A})$ if and only if for all $y \in T$, $\mathcal{M}_y$ is flat over $\mathcal{A}_y$. (This is just a general statement about sheaves of rings on profinite sets.)

Now we use the key property of w-locality: y is the same as an inclusion of a closed point $\mathrm{Spa}(K, K^+) \xrightarrow{y} X$ where $K = K(y)$ is a valued field and K^+ is the valuation ring. You then check that $\mathcal{A}_y = K^+/\varpi$ and $\mathcal{M}_y = B_y^+/\varpi$. But flatness over K^+ is the same as torsion-freeness, so B_y^+ is flat over K^+, and so the same holds after modding out by ϖ.

In summary, the trick is that "you can pass to the fibers rather than the stalks" thanks to the w-locality. □

Corollary 5.2.5. *The v-topology is subcanonical.*

We want to discuss gluing vector bundles in the v-topology.

Theorem 5.2.6. *The groupoid of vector bundles is a stack for the v-topology.*

For the analytic topology this was proved by Kedlaya-Liu. What we need to do additionally here is to establish descent of vector bundles for surjective affinoid maps. The trick is to use an approximation argument to reduce to the case of a point.

Chapter 6

Statement of Galois to Automorphic in Geometric Langlands

6.1 The classical case, $G = \mathrm{GL}_n$

6.1.1 Setup

Let $X/\mathbb{F}_q$ be a proper, smooth, geometrically irreducible curve. For each $x \in |X|$ we denote

- $\mathcal{O}_x = \widehat{\mathcal{O}}_{X,x}$,
- $\mathcal{F}_x$ its field of fractions,
- k_x its residue field, and
- $F = \mathbb{F}_q(X)$ its function field.

Finally, we write

$$\mathbb{A}_F := \prod_{x\in|X|}' F_x \supset \mathcal{O}_F := \prod_{x\in|X|} \mathcal{O}_x.$$

6.1.2 Automorphic side

Fix a prime $\ell \neq p$. The main player is the space of *unramified automorphic functions*

$$\mathcal{A} := \mathrm{Funct}(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)/ \mathrm{GL}_n(\mathcal{O}_F), \overline{\mathbb{Q}}_\ell).$$

This admits an action of a *Hecke algebra* for each $x \in |X|$:

$$\mathcal{H}_x := \mathrm{Funct}(\mathrm{GL}_n(\mathcal{O}_x)\backslash \mathrm{GL}_n(F_x)/ \mathrm{GL}_n(\mathcal{O}_x), \overline{\mathbb{Q}}_\ell).$$

The action is by convolution for $T \in \mathcal{H}_x$ and $f \in \mathcal{A}$, we have

$$(T * f)(g) = \int_{h_x\in\mathrm{GL}_n(F_x)} f(gh_x^{-1})T(h_x)\, dh_x$$

with measure normalized so that $\int_{\mathrm{GL}(O_x)} dh_x = 1$.

Proposition 6.1.1. *We have*

$$\mathcal{H}_x \cong \overline{\mathbb{Q}}_\ell[T_x^1, T_x^2, \dots, T_x^n, (T_x^n)^{-1}]$$

where T_x^i is the characteristic function of the double coset

$$\mathrm{GL}_n(O_x)\begin{pmatrix} \varpi_x & \\ & \ddots \end{pmatrix}\mathrm{GL}_n(O_x)$$

6.1.3 Galois side

Let $\sigma\colon \pi_1(X) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ be an ℓ-adic representation; we can think of this equivalently as a local system $E_\sigma \in \mathrm{Loc}_n(X)$.

We define

$$\mathcal{G} := \{\sigma\colon \pi_1(X) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell) \text{ geometrically irreducible}\}/\cong .$$

Recall that for each $x \in |X|$ we have a $\mathrm{Frob}_x \in \pi_1(X)$.

Theorem 6.1.2 (Drinfeld, Lafforgue, Frenkel-Gaitsgory-Vilonen)**.** *To every $\sigma \in \mathcal{G}$ there corresponds a non-zero $f_\sigma \in \mathcal{A}$ such that*

$$T_x^i * f_\sigma = \mathrm{Tr}(\wedge^i\sigma(\mathrm{Frob}_x))f_\sigma.$$

Moreover, f_σ is unique up to scalar and cuspidal.

Here $\wedge^i\sigma$ is the ith exterior power of σ and "cuspidal" means that

$$\int_{U(F)\backslash U(A_F)} f_\sigma(ug)\, du = 0$$

for all $g \in \mathrm{GL}_n(A_F)$ and U the unipotent radical of proper standard parabolic subgroup of GL_n.

6.2 Geometric reformulation

6.2.1 Geometrization of adeles

Let Bun_n be the stack of rank n vector bundles on X.

Theorem 6.2.1 (Weil's uniformization theorem)**.** *We have*

$$\mathrm{Bun}_n(\mathbb{F}_q) = \mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)/ \mathrm{GL}_n(O_F).$$

This allows us to interpret $\mathcal{A} = \mathrm{Funct}(\mathrm{Bun}_n(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$. An obvious categorification of this is sheaves on Bun_n.

6.2.2 Geometrization of Hecke operators

A geometric version of Hecke algebra is the moduli stack of modifications

$$\mathrm{Hecke}^i = \left\{ (x, M, M', \beta) \colon \begin{array}{c} x \in X, \\ M, M' \in \mathrm{Bun}_n \\ \beta \colon M \hookrightarrow M' \\ M'/M \cong k_x^{\oplus i} \end{array} \right\}$$

This admits maps

$$\begin{array}{ccccc} & & \mathrm{Hecke}^i & & \\ & \swarrow^{h_i^{\leftarrow}} & & \searrow^{h_i^{\rightarrow} \times \pi_i} & \\ \mathrm{Bun}_n & & & & \mathrm{Bun}_n \times X \end{array}$$

where

- $h_i^{\leftarrow}(x, M, M', \beta) = M$,
- $h_i^{\rightarrow}(x, M, M', \beta) = M'$, and
- $\pi_i(x, M, M', \beta) = x$.

To relate these Hecke stacks to the classical Hecke algebra, we define a local Hecke stack Hecke_x^i by the cartesian diagram

$$\begin{array}{ccc} \mathrm{Hecke}_x^i & \longrightarrow & \mathrm{Hecke}^i \\ \downarrow & & \downarrow \\ x & \longrightarrow & X \end{array}$$

On rational points, we have a diagram

$$\begin{array}{ccccc} & & \mathrm{Hecke}_x^i(\mathbb{F}_q) & & \\ & \swarrow & & \searrow & \\ \mathrm{Bun}_n(\mathbb{F}_q) & & & & \mathrm{Bun}_n(\mathbb{F}_q) \end{array}$$

Then the classical Hecke operators can be interpreted as

$$T_x^i * f := (h_i^{\rightarrow})_! (h_i^{\leftarrow})^* f$$

for $T \in \mathcal{H}_x$ and $f \in \mathcal{A}$.

6.2.3 Hecke eigensheaves

Based on this we make the following definition.

Definition 6.2.2. (Geometric Hecke operators) We define

$$\mathbb{T}^i \colon D_c^b(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell) \to D_c^b(\mathrm{Bun}_n \times X, \overline{\mathbb{Q}}_\ell).$$

by

$$\mathcal{F} \mapsto (h_i^{\rightarrow} \times \pi_i)_!(h_i^{\leftarrow})^* \mathcal{F}[i(n-i)].$$

Let $\sigma \in \mathcal{G}$. An object $\mathcal{F} \in D_c^b(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ is called a *Hecke eigensheaf* with respect to σ if for all $i = 1, \ldots, n$ we have $\mathbb{T}^i(\mathcal{F}) \cong \mathcal{F} \boxtimes \bigwedge^i E_\sigma$.

Theorem 6.2.3 (Frenkel-Gaitsgory-Vilonen)**.** *For every $\sigma \in \mathcal{G}$, there exists a non-zero Hecke eigensheaf* Aut_σ *which is cuspidal.*

What is the meaning of cuspidality? You can consider the moduli spaces of flags Fl_{n_1,n_2} for $n_1 + n_2 = n$, which parametrize

$$\{0 \to E_1 \to E \to E_2 \to 0\}$$

with rank $E_i = n_i$. This admits maps

$$\begin{array}{ccc} & \mathrm{Fl}_{n_1,n_2} & \\ {\scriptstyle p} \swarrow & & \searrow {\scriptstyle q} \\ \mathrm{Bun}_n & & \mathrm{Bun}_{n_1} \times \mathrm{Bun}_{n_2} \end{array}$$

where $p = E$ and $q = (E_1, E_2)$. A $\mathcal{F} \in D_c^b(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ is *cuspidal* if $q_! p^* \mathcal{F} = 0$ for all n_1, n_2.

Remark 6.2.4. Frenkel-Gaitsgory-Vilonen show that Aut_σ is a perverse sheaf and is irreducible on each connected component Bun_n^d. If we demand that Aut_σ be irreducible, then it is unique.

6.3 Geometric Satake

Let $k = \overline{k}$. Let G be a connected reductive group over k and $\widehat{G}/\overline{\mathbb{Q}}_\ell$ be the dual group over $\overline{\mathbb{Q}}_\ell$.

Definition 6.3.1. The *affine Grassmannian* is $\mathrm{Gr}_G := LG/L^+G$ where the loop group LG is defined by $LG(R) := G(R((t)))$ and $L^+G(R) := G(R[[t]])$.

Definition 6.3.2. We define the category $\mathrm{Sat} := \mathcal{P}_{L^+G}(\mathrm{Gr}_G)$, perverse sheaves equivariant for the "arc group" L^+G.

Properties.

1. There is a bijection
$$L^+G\backslash \mathrm{Gr}_G(k) \leftrightarrow X_*(T)^+$$
with $L^+G\cdot t^\lambda \leftarrow \lambda$ for $t\in T(k((t)))$. Here for $\lambda\colon \mathbb{G}_m\to T$ we get $k((t))^*\to T(k((t)))$ sending $t\mapsto t^\lambda$.

2. Denoting $L^+G\cdot t^\lambda$ by O_λ, we have
$$\overline{O_\lambda}=\bigcup_{\mu\le\lambda} O_\mu.$$

3. We have
$$\mathrm{Gr}_G(R)=\left\{(\mathcal{E},\beta)\colon \begin{array}{c}\mathcal{E}=G-\text{bundle on } D_R\\ \beta\colon \mathcal{E}|_{D_R^0}\cong G\times D_R^0\end{array}\right\}.$$
Here we are using the notation
   - $D=\mathrm{Spec}\ k[[t]]$, $D^0=\mathrm{Spec}\ k((t))$,
   - $D_R=\mathrm{Spec}\ R[[t]]$, $D_R^0=\mathrm{Spec}\ R((t))$.

4. Consider the diagram
$$\begin{array}{ccccc}\mathrm{Gr}_G\times\mathrm{Gr}_G & \xleftarrow{p} & LG\times\mathrm{Gr}_G & \xrightarrow{q} & LG\times^{L^+G}\mathrm{Gr}_G\\ & & & & \downarrow{\scriptstyle m}\\ & & & & \mathrm{Gr}_G\end{array}$$
For $A,A'\in\mathrm{Sat}$, we define $A\widetilde{\boxtimes}A'\in\mathcal{P}(LG\times^{L^+}\mathrm{Gr}_G)$ by the condition
$$p^*(A\boxtimes A')=q^*(A\widetilde{\boxtimes}A')$$
Define the *fusion product*
$$A*A'=m_!(A\widetilde{\boxtimes}A')\in\mathcal{P}_{L^+G}(\mathrm{Gr}_G).$$

Theorem 6.3.3 (Geometric Satake)**.** *We have an equivalence of categories*
$$(\mathrm{Sat},*)\cong(\mathrm{Rep}\,\widehat{G},\otimes)$$
such that for $\lambda\in X^*(\widehat{T})^+\cong X_*(T)^+$ the highest weight representation V_λ corresponds to $IC_\lambda=IC(\overline{O}_\lambda,\overline{\mathbb{Q}}_\ell)$.

Under the fiber functor to vector spaces
$$\begin{array}{ccc}\mathrm{Sat} & \longrightarrow & \mathrm{Rep}(\widehat{G})\\ & \searrow\quad\swarrow & \\ & Vec_k & \end{array}$$
this equivalence corresponds to taking cohomology.

6.4 Statement of global geometric Langlands for general G

6.4.1 The Hecke stacks

Consider the stack

$$\text{Hecke} = \left\{ (x, M, M', \beta) : \begin{array}{r} x \in X, \\ M, M' \in \text{Bun}_n \\ \beta \colon M|_{X-x} \cong M'|_{X-x} \end{array} \right\}$$

This has maps

$$\begin{array}{ccc} & \text{Hecke} & \\ {}^{h^{\leftarrow}}\swarrow & & \searrow^{h^{\rightarrow}\times\pi} \\ \text{Bun}_G & & \text{Bun}_G \times X \end{array}$$

where $h^{\leftarrow}(x, M, M', \beta) = M$ and $h^{\rightarrow}(x, M, M', \beta) = (M', x)$.

There is an evaluation map

$$\text{ev} \colon \text{Hecke} \to \left[\frac{L^+G \backslash LG / L^+G}{\text{Aut}(D)} \right]$$

which is described as follows. After choosing an isomorphism $D_x \cong D$ and a trivialization of $M|_{D_x}$ and $M'|_{D_x}$ the map β describes some transition function $g_\beta \in LG$, which is $\text{ev}(x, M, M', \beta)$.

Using this we can define an operator

$$\text{Hk} : \text{Rep}(\widehat{G}) \times D_c^b(\text{Bun}_{G, \overline{\mathbb{Q}}_\ell}) \xrightarrow{\text{ev}} D_c^b(\text{Bun}_G \times X, \overline{\mathbb{Q}}_\ell)$$

sending

$$(V, \mathcal{F}) \mapsto (h^{\rightarrow} \times \pi)_!(h^{\leftarrow})^*(\mathcal{F} \otimes IC_V^{\text{Hk}})[\text{shift}]$$

where $IC_V^{\text{Hk}} := \text{ev}^*(IC_V)$ is the pullback of the IC sheaf corresponding to the local system V under the Geometric Satake equivalence. Similarly, for $V_1, \ldots, V_d \in \text{Rep}(\widehat{G})$ we define

$$\text{Hk}_{V_1 \boxtimes \ldots \boxtimes V_d}(\mathcal{F}) \in D_c^b(\text{Bun}_G \times X^d, \overline{\mathbb{Q}}_\ell)$$

in an analogous manner. These satisfy

1. $\text{Hk}_{V_1 \boxtimes V_2}(\mathcal{F})|_{\text{Bun}_G \times \Delta(X)} \cong \text{Hk}_{V_1 \otimes V_2}(\mathcal{F})$,

2. For $s \colon X \times X \to X \times X$ the swap function, we have $s^*(\text{Hk}_{V_1 \boxtimes V_2}(\mathcal{F})) \cong \text{Hk}_{V_2 \boxtimes V_1}(\mathcal{F})$.

6.4.2 Statement of Galois to automorphic

Let E be a $\widehat{G}$-local system on X, viewed as a tensor functor

$$E\colon \operatorname{Rep}(\widehat{G}) \to \operatorname{Loc}(X)$$

denoted $V \mapsto E^V$.

Definition 6.4.1. A *Hecke eigensheaf with eigenvalue E* is a perverse sheaf $\mathcal{F} \in \mathcal{P}(\operatorname{Bun}_G, \overline{\mathbb{Q}}_\ell)$ together with isomorphisms

$$\alpha_V\colon \operatorname{Hk}_V(\mathcal{F}) \cong \mathcal{F} \boxtimes E^V \text{ for all } V \in \operatorname{Rep}(\widehat{G})$$

that are compatible with the symmetric tensor structure on $\operatorname{Rep}(\widehat{G})$ and composition of Hecke operators.

Conjecture 6.4.2. *To every irreducible $\widehat{G}$-local system E, there exists a non-zero Hecke eigensheaf* Aut_E *with eigenvalue E.*

Chapter 7

Discussion Session: The Fargues-Fontaine Curve

These are notes from an impromptu discussion session to elaborate upon / clarify aspects of the Fargues-Fontaine curve. Dennis Gaitsgory recalled the basic setup of the Fargues-Fontaine curve and posed some questions, and then Peter Scholze discussed the answers and some complements.

7.1 Basic setup of the Fargues-Fontaine curve

We have $A_{\mathrm{inf}} = W(\mathcal{O}_{\mathbb{C}_p}^\flat)$. The tilt is

$$B^\flat = \varprojlim_{\Phi} B/pB = \varprojlim_{b \mapsto b^p} B.$$

The universal property of Witt vectors is that if R is perfect and B is p-adically complete, then we have

$$\mathrm{Hom}(W(R), B) = \mathrm{Hom}(R, B^\flat).$$

In other words, formation of Witt vectors is left adjoint to tilting. The unit of the adjunction is

$$\theta \colon A_{\mathrm{inf}} \to \mathcal{O}_{\mathbb{C}_p}.$$

There are two possible generators of $\ker \theta$.

1. $p - [p^\flat]$.

2. $\frac{1-[\epsilon]}{1-[\epsilon^\flat]}$ where $\epsilon = (1, \zeta_p, \zeta_p^2, \dots) \in \mathcal{O}_{\mathbb{C}_p}^\flat$. (This is like a Gauss sum.)

We have maps

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{C}_p}^{\flat} = A_{\mathrm{inf}}/(p) & & \\ \uparrow & & \\ A_{\mathrm{inf}} & \xrightarrow{\theta} & \mathcal{O}_{\mathbb{C}_p} \\ \downarrow & & \\ W(\overline{\mathbb{F}}_p) & & \end{array}$$

We consider $Y := \operatorname{Spec} A_{\mathrm{inf}} - (\operatorname{Spec} \mathcal{O}_{\mathbb{C}_p}^{\flat} \cup \operatorname{Spec} W(\overline{\mathbb{F}}_p))$. What points can we write down in here? Take $a \in \mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}^{\flat}} - 0$ and consider the ideal $(p - [a])$. (For example we could take $a = p^{\flat}$.)

Lemma 7.1.1. *We have* $(A_{\mathrm{inf}}/(p - [a]))^{\flat} = \mathcal{O}_{\mathbb{C}_p}^{\flat}$.

This is a preview of the fact we'll see later that the closed points of the the Fargues-Fontaine curve correspond to "untilts".

Proof. Let's try going directly to the definition:

$$(A_{\mathrm{inf}}/(p - [a]))^{\flat} = \varprojlim_{\Phi} (A_{\mathrm{inf}}/(p, p - [a]))^{\flat} = \varprojlim_{\Phi} \mathcal{O}_{\mathbb{C}_p}^{\flat}/a.$$

Why is this the same as $\mathcal{O}_{\mathbb{C}_p}^{\flat}$? We have a (non-canonical) isomorphism

$$\mathcal{O}_{\mathbb{C}_p}^{\flat} = \overline{\mathbb{F}_q[[T]]}^{\wedge}$$

(by which we mean the normalization in the algebraic closure of its fraction field, completed). We can arrange so that $a = T$. Then the result follows from inspection. □

7.2 Questions

Question 1. Is it true that all closed primes in this thing are of this form?

Question 2. How are these primes parametrized via Lubin-Tate theory?

Question 3. What is the map $Y \to (0, \infty)$.

Question 4. Why do we have $\pi_1(X) = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$?

7.3 Answers

7.3.1 Question 1

Not quite: you also have the 0 ideal, but that's the only exception. We're not going to discuss why.

7.3.2 Question 2

Take $a \in 1 + \mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}^\flat}$, which we can regard as a group under multiplication. (The Lubin-Tate group for $\mathbb{Q}_p$ is just the multiplicative group, which is why we only have to consider this basic object.)

Proposition 7.3.1. *All prime ideals as in Question 1 are of the form* $\frac{1-[a]}{1-[a^{1/p}]}$ *where* $a \in (1 + \mathfrak{m} \setminus \{1\})/\mathbb{Z}_p*$.

Remark 7.3.2. We know by Question 1 that this ideal is of the form $p - [b]$ for some (not unique) b, but it is difficult to express thi b in terms of a; the relation would be a horrible formula.

The proof of this proposition is by approximation on the characteristic p side.

7.3.3 Question 3

Notation. Unless otherwise noted, we abbreviate $\operatorname{Spa}(R) := \operatorname{Spa}(R, R^0)$.

Consider $\operatorname{Spa} \mathbb{F}_p((t)) \times_{\operatorname{Spa} \mathbb{F}_p} \operatorname{Spa} \mathbb{F}_p((u))$.

Lemma 7.3.3. *Let K be a complete nonarchimedean field of characteristic p. Then*

$$\operatorname{Spa} \mathbb{F}_p((t)) \times_{\operatorname{Spa} \mathbb{F}_p} \operatorname{Spa} K$$

is the punctured open disk. The result is that

$$\operatorname{Spa} \mathbb{F}_p((t)) \times_{\operatorname{Spa} \mathbb{F}_p} \operatorname{Spa} K = \mathbb{D}_K^* = \text{“}\{x \mid 0 < |x| < 1\}\text{”}.$$

The quotation marks mean that this is true at the level of K-points, and this is universally true with respect to all fields.

Remark 7.3.4. We are used to thinking of $\operatorname{Spa} \mathbb{F}_p((t))$ as a punctured open disk, but it has only one point so this doesn't quite make sense. However, once we base change to a complete non-archimedean field it does make sense.

Proof. You first compute a fibered product at the level of rings of integral elements:

$$\operatorname{Spa} \mathbb{F}_p[[t]] \times_{\operatorname{Spa} \mathbb{F}_p} \operatorname{Spa} \mathcal{O}_K.$$

This is really the fiber product at the level of formal schemes; anyways the result is

$$\operatorname{Spa}\mathbb{F}_p[[t]] \times_{\operatorname{Spa}\mathbb{F}_p} \operatorname{Spa}\mathcal{O}_K = \operatorname{Spa}\mathcal{O}_K[[t]].$$

In here we have $\operatorname{Spa}\mathbb{F}_p((t))\times_{\operatorname{Spa}\mathbb{F}_p}\operatorname{Spa}K$, which is open and correseponds to $\{t\varpi \neq 0\}$ where $\varpi \in K$ is a pseudo-uniformizer.

$$\begin{array}{ccc} \operatorname{Spa}\mathbb{F}_p[[t]] \times_{\operatorname{Spa}\mathbb{F}_p} \operatorname{Spa}\mathcal{O}_K & = & \operatorname{Spa}\mathcal{O}_K[[t]] \\ \uparrow & & \uparrow \\ \operatorname{Spa}\mathbb{F}_p((t)) \times_{\operatorname{Spa}\mathbb{F}_p} \operatorname{Spa}K & = & \{t\varpi \neq 0\} \end{array}$$

Note the similarity with the situation with $\mathbb{A}_{\text{inf}}$.

> *Guideline.* $\mathcal{O}_K[[t]]$ is an analogue of A_{inf} in equal characteristic.

(Indeed, $\mathcal{O}_K[[t]]$ is the starting point for the construction Fargues-Fontaine curve in equal characteristic, just as A_{inf} is the starting point in mixed characteristic.)

Let's consider imposing the conditions one by one. First, the open subset $\{\varpi \neq 0\}$ is the generic fiber (in the sense of Bertholot) of $\operatorname{Spa}\mathcal{O}_K[[t]] \to \operatorname{Spa}\mathcal{O}_K$, and that turns out to be $\mathbb{D}_K$: the open unit disk. (Why? Consider mapping out of $\mathcal{O}_K[[t]]$: a homomorphism over $\mathcal{O}_K$ is determined by the image of t, and the only restriction is that you have to send t to something topologically nilpotent since it is itself topologically nilpotent.)

Then adding in the condition $t \neq 0$ is the punctured disk $\mathbb{D}_K^*$. □

Let us emphasize again: *if $\mathbb{Q}_p$ is replaced by $\mathbb{F}_p((t))$ and K is algebraically closed then the equal characteristic version of A_{inf} is $\mathcal{O}_K[[t]]$.*

Remark 7.3.5. In constructing the curve we should have started with a complete algebraically closed field of characteristic p, instead of 0. (In the notation of Colmez's talk, we should have started with $C^\flat$ rather than C.) Starting with char 0 gives a *pointed curve* because there is a distinguished choice of untilt.

7.3.4 Question 4

Finally, what is the map $\operatorname{Spa}(A_{\text{inf}}) \setminus \{p[p^\flat]\} \to (0, \infty)$?

In equal characteristic, you have

$$\operatorname{Spa}(\mathcal{O}_K[[t]]) \setminus \{t\varpi = 0\} \to (0, \infty).$$

We can understand this from the picture of the punctured disk. We know that

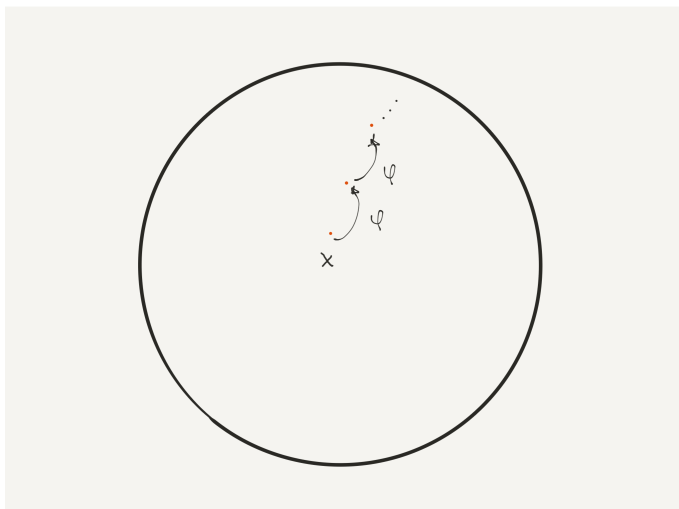
$$\operatorname{Spa}(\mathcal{O}_K[[t]]) = \operatorname{Spa}\mathbb{F}_p((t)) \times_{\operatorname{Spa}\mathbb{F}_p} \operatorname{Spa}K = \mathbb{D}_K^*.$$

The map is a normalization of the "radius". For $x \in \mathbb{D}_K^*(K)$, you have a radius function on the punctured disk which is

$$\kappa(x) = \log_{|\varpi(x)|} |t(x)|.$$

(The "value" of a valuation is not really well-defined, since valuations are only considered up to isomorphism. However, the ratio between two values *is* well-defined.)

There is an action of φ on $O_K[[t]]$ via φ on O_K and $t \mapsto t$. (To remember this, think to the mixed characteristic case, where t is replaced by p. Of course there can be no nontrivial action on p.) This induces an action on $\mathbb{D}_K^*$. In terms of its effect on κ, it decreases κ hence *increases* the "radius" $p^{-\kappa(x)}$.



Warning 7.3.6. As is obvious from the definition, this is *not* an action over K. It is a "geometric" rather than "arithmetic" Frobenius.

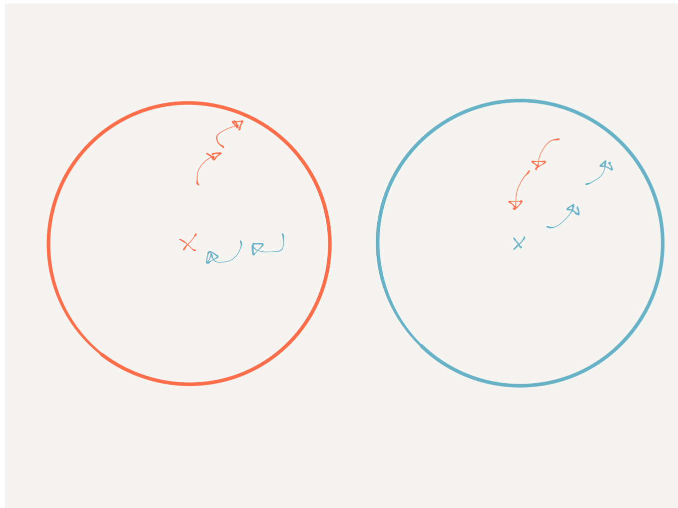
The action of φ on $Y^{\text{ad}} = \mathbb{D}_K^*$ is totally discontinuous and proper. Don't worry about the precise meaning of "proper"; suffice to say that it satisfies the properties that one would want to take a quotient.

Definition 7.3.7. The *adic Fargues-Fontaine curve* is $X^{\text{ad}} = Y^{\text{ad}}/\varphi^{\mathbb{Z}}$. Via κ this is fibered over $(0, \infty)/p^{\mathbb{Z}} = S^1$.

Here's a confusing thing. We have

$$\operatorname{Spa} \mathbb{F}_p((t)) \times_{\operatorname{Spa} \mathbb{F}_p} \operatorname{Spa} \mathbb{F}_p((u)) = \mathbb{D}^*_{\mathbb{F}_p((u))}.$$

Switching factors, we can apply the same reasoning to view this as $\mathbb{D}^*_{\mathbb{F}_p((t))}$. But these two disks have somehow "opposite" coordinates. In particular, if Frobenius is expanding in one picture then it is contracting in the other picture.



The space Y can be compactified by adding two points. For each one the result is easy to understand, but not so for both at once.

In mixed characteristic the story is the same except that $p, [p^\flat]$ replace t, u.

7.4 The scheme-theoretic Fargues-Fontaine curve

7.4.1 Line bundles on X^{ad}

By a *φ-equivariant vector bundle* on Y^{ad} we mean a vector bundle on Y^{ad} equipped with an action of φ over Y^{ad}. For an integer $d \in \mathbb{Z}$, we can form the equivariant line bundle

$$(\mathcal{O}_{Y^{\mathrm{ad}}}, \varphi_d = t^{-d}\varphi).$$

(In mixed characteristic this would be p^{-d} instead.) This descends to the line bundle $\mathcal{O}(d)$ on X^{ad}.

We basically declare $\mathcal{O}(1)$ to be ample. The justification comes from a Theorem of Kedlaya (or Hartl in equal characteristic) that twisting by high enough powers kills cohomology (although that was not the original motivation of Fargues-Fontaine). We can define

$$P := \bigoplus_{d \geq 0} H^0(X^{\mathrm{ad}}, \mathcal{O}(d)).$$

This is a graded $\mathbb{F}_p((t))$-algebra, but it's huge. Letting $P_d := H^0(X^{\mathrm{ad}}, \mathcal{O}(d))$, the fundamental exact sequence reads

$$0 \to \mathbb{F}_p((t)) \to P_1 \to K \to 0.$$

where the map $P_1 \to K$ is evaluation at one fixed point. But K is huge over $\mathbb{F}_p((t))$; for instance it's infinite-dimensional. (There are similar sequences for $d > 1$, which we'll see later.)

Definition 7.4.1. The *schematic Fargues-Fontaine curve* is $X := \operatorname{Proj} P$.

Theorem 7.4.2 (GAGA)**.** *There is a morphism of locally ringed topological spaces $X^{\mathrm{ad}} \to X$, and pullback induces an equivalence of categories*

$$\operatorname{Bun}(X) \xrightarrow{\sim} \operatorname{Bun}(X^{\mathrm{ad}}).$$

7.4.2 The mixed characteristic case

Now we replace $\mathbb{F}_p((t))$ by $\mathbb{Q}_p$. Start with a field C which is complete and algebraically closed of char p. (In the notation of Colmez's talk, this is $C^\flat$.)

We would like to take "$\operatorname{Spa} \mathbb{Q}_p \times \operatorname{Spa} C$".

> *"Fact":* If R is perfect, then one should have
>
> $$\text{“} \operatorname{Spa} \mathbb{Z}_p \times \operatorname{Spa} R\text{”} := \operatorname{Spa} W(R).$$
>
> (The point is that there is no real object to take the fiber product over; for this reason people sometimes write the base as $\mathbb{F}_1$.)

Therefore

$$\text{“} \operatorname{Spa} \mathbb{Z}_p \times \operatorname{Spa} \mathcal{O}_C\text{”} = \operatorname{Spa} W(\mathcal{O}_C) = \operatorname{Spa} A_{\mathrm{inf}}$$

and also

$$\text{“} \operatorname{Spa} \mathbb{Q}_p \times \operatorname{Spa} C\text{”} = \{p[\varpi] \neq 0\}$$

for $\varpi \in C$ a pseudo-uniformizer.

As before, we have a Frobenius φ acting on Y^{ad} and a map

$$\kappa \colon Y^{\mathrm{ad}} \to (0, \infty).$$

Lemma 7.4.3. *The closed non-zero prime ideals of Y^{ad} are in bijection with the set of untilts, which is*

$$\left\{ (C^\#, \iota) \mid \begin{array}{c} C^\# = \text{ complete, algebraically closed extension of } \mathbb{Q}_p \\ \iota \colon (C^\#)^\flat \cong C \end{array} \right\}.$$

Proof. We give one direction. Starting with $C^\#$ and an isomorphism $\iota \colon C \cong (C^\#)^\flat$, we can form

$$\ker(\theta \colon A_{\mathrm{inf}} \to \mathcal{O}_{C^\#}).$$

This is a closed prime ideal. □

This is why fixing a point on Y is fixing an untilt. Alternatively, one can think of it as "giving a $\mathbb{Q}_p$ structure on $C^\flat$." In these terms, the action of φ on Y is through its action on ι.

Remark 7.4.4. In equal characteristic, untilts are just maps

$$\mathbb{F}_p((t)) \to K.$$

For $t \neq 0$, the map $t \mapsto a$ corresponds to the prime ideal $(t - a)$ on the curve.

Again we get line bundles $\mathcal{O}(d)$, and we define

$$P := \bigoplus_{d \geq 0} H^0(X^{\mathrm{ad}}, \mathcal{O}(d))$$

where $P_d = (B^+_{\mathrm{cris}})^{\varphi = p^d}$.

7.4.3 p-adic period rings

What's the connection to Colmez's talk? Fixing $C^\#$, we get a closed point $\infty \in Y^{\mathrm{ad}}$ with residue field $C^\#$ and hence also $\infty \in X^{\mathrm{ad}}$. Since the adic curve maps to the scheme-theoretic curve (a general fact), we also get $\infty \in X$.

In these terms, the p-adic period rings can be described as

- $B^+_{\mathrm{dR}}(C^\#) = \widehat{\mathcal{O}}_{X,\infty}$.
- $B_e = H^0(X - \infty, \mathcal{O}_X) = B^{\varphi=1}_{\mathrm{cris}}$.
- $B_{\mathrm{dR}} = \mathrm{Frac}(B^+_{\mathrm{dR}})$.

There is an element $t = \log[\epsilon] \in (B^+_{\mathrm{cris}})^{\varphi=p} = P_1$. The fundamental short exact sequence is

$$0 \to \mathbb{Q}_p t^d \to P_d \to B^+_{\mathrm{dR}} / \mathrm{Fil}^d \to 0.$$

This is like recording the first d steps of the power series expansion at ∞.

For projective space, one would get finite-dimensional vector spaces over the base field. We have here a *mix* between $\mathbb{Q}_p$ and $C^\#$-vector spaces; this type of object is called a "Banach-Colmez space".

Dividing by t^d and taking the colimit over d gives the fundamental exact sequence

$$0 \to \mathbb{Q}_p \to B_e \to B_{\mathrm{dR}}/B^+_{\mathrm{dR}} \to 0.$$

Chapter 8

Vector Bundles on the Fargues-Fontaine Curve

8.1 Preliminaries on the Fargues-Fontaine curve

Let E be a local ring with residue field $\mathbb{F}_q$. (We can imagine $E = \mathbb{Q}_p$ or $\mathbb{F}_q((t))$.) Let F be an algebraically closed perfectoid extension of $\mathbb{F}_q$. (In terms of the notation of Colmez's talk, $F = C^\flat$.)

We form the adic curve $X^{\mathrm{ad}} := Y^{\mathrm{ad}}/\varphi^{\mathbb{Z}}$. This has a map to the scheme-theoretic Fargues-Fontaine curve $X := \operatorname{Proj} P$, where

$$P = \bigoplus_{d \geq 0} B^{\varphi = \varpi^d}$$

where $B = \mathcal{O}(Y^{\mathrm{ad}})$ is a Fréchet algebra.

Theorem 8.1.1. *X is a regular noetherian scheme of dimension 1.*

If we fixed $\infty \in |X|$ (corresponding to an untilt C) then

$$X \setminus \{\infty\} = \operatorname{Spec}\, B_e$$

where $B_e = B[1/t]^{\varphi=1}$.

Theorem 8.1.2 (Fargues-Fontaine)**.** *The ring B_e is a PID.*

We want to discuss the classification of vector bundles on the Fargues-Fontaine curve. Thanks to the following theorem, we can think interchangeably about the analytic or algebraic curve for this purpose.

Theorem 8.1.3 (Fargues-Fontaine, Hartl-Pink, Kedlaya-Liu)**.** *GAGA for X: the map $X^{\mathrm{ad}} \to X$ induces an equivalence of categories*

$$\mathrm{Bun}_X \cong \mathrm{Bun}_{X^{\mathrm{ad}}}.$$

8.2 Constructing vector bundles

8.2.1 Line bundles

We already know about the line bundles $\mathcal{O}(d)$ for $d \in \mathbb{Z}$. Are these all of them?

The answer is **yes**: $\operatorname{Pic} X \xrightarrow{\sim} \mathbb{Z}$ by $d \mapsto [\mathcal{O}(d)]$. This is saying that the curve has a well-defined notion of degree. This is extremely non-trivial: the usual theory of degree does not apply, because the curve X lives over $\mathbb{Q}_p$ but its residue fields are generally huge (infinite-dimensional over $\mathbb{Q}_p$).

What we are saying here is that if one *redefines* the degree of a point in an appropriate way then it is a theorem that the divisor of any function has degree 0, and that allows us to define a coherent notion of degree.

8.2.2 Higher rank vector bundles

We give an analytic construction of vector bundles on X^{ad}. The key point is that Y^{ad} lives over $\operatorname{Spa} L$ where $L = \breve{E} := \widehat{E^{\mathrm{unr}}}$. So one can pull back φ-equivariant bundles on $\operatorname{Spa} L$ to Y^{ad} to get a functor

$$(\varphi\text{-bundles on } \operatorname{Spa}(L, \mathcal{O}_L)) \to (\varphi\text{-bundles on } Y^{\mathrm{ad}}),$$

which then descend to bundles on X^{ad}. By GAGA (Theorem 8.1.3) this is the same as bundles on X.

This construction gives a functor

$$(\varphi - \text{bundles on } \operatorname{Spa}(L, \mathcal{O}_L)) \to \operatorname{Bun}_X .$$

But of course φ−bundles on $\operatorname{Spa}(L, \mathcal{O}_L)$ are simply classical *L-isocrystals*: finite-dimensional L-vector spaces equipped with bijective semi-linear "Frobenius" endomorphism φ.

This can be made concrete. For $D \in \varphi - \mathrm{Mod}_L$ we get a graded P-module

$$\mathcal{E}(D) := \bigoplus_{d \geq 0} (D \otimes_L B)^{\varphi = \varpi^d}.$$

One then takes the associated quasicoherent sheaf on X. (But it is not clear from this description that this is a vector bundle.)

Warning 8.2.1. $\mathcal{O}(1)$ depends on the choice of ϖ.

8.3 Geometric properties of $\mathcal{E}(D)$

If D is simple of slope $-\lambda$ ($\lambda \in \mathbb{Q}$) then we get a vector bundles $\mathcal{E}(D) =: \mathcal{O}_X(\lambda)$. The *Dieudonné-Manin theorem* gives a classification of irreducible isocrystals in terms of the slope.

What is $\mathcal{O}_X(\lambda)$ concretely? If $\lambda \in \mathbb{Z}$ then it is easy to show that

$$\mathcal{O}_X(\lambda) = \widetilde{P[\lambda]}$$

and this is a line bundle since P is degenerated in degree 1. In general, if $\lambda = \frac{d}{h}$ in reduced form then let E_h/E be the unramified extension of degree h. Then we get a curve $X_{E_h,F} \cong X \otimes_E E_h$ which is a finite étale covering of X with Galois group $\mathbb{Z}/h$. At the level of adic spaces the covering can be described simply as

$$Y^{\mathrm{ad}}/\varphi^{h\mathbb{Z}} \to Y^{\mathrm{ad}}/\varphi^{\mathbb{Z}}.$$

Then $\mathcal{O}_X(\lambda) := \pi_{h*}\mathcal{O}_{X_h}(d)$. Why? This comes from an understanding of the irreducible isocrystals.

Consequences:

1. $\mathcal{O}_X(\lambda) \in \mathrm{Bun}_X$ (since it's the pushforward of a line bundle via a finite étale map).
2. rank $\mathcal{O}_X(\lambda) = h$ and $\deg \mathcal{O}_X(\lambda) = d$, so the slope of $\mathcal{O}_X(\lambda)$ is λ.
3. $\mathcal{O}_X(\lambda)$ is semistable. (This can be checked after pulling back to the finite étale cover Y_{E_h}, where it becomes a direct sum of copies of line bundles of degree d.) In fact it is even stable, by the classification theorem.

8.4 Classification of vector bundles

8.4.1 The classification theorem

Theorem 8.4.1 (Fargues-Fontaine)**.** *The functor*

$$\varphi - \mathrm{Mod}_L \to \mathrm{Bun}_X$$

sending $D \mapsto \mathcal{E}(D)$ is essentially surjective. In other words, any vector bundle is isomorphic to a direct sum of $\mathcal{O}_X(\lambda)$:

$$\mathcal{E} \cong \mathcal{O}_X(\lambda_1) \oplus \ldots \oplus \mathcal{O}_X(\lambda_n) \textit{ for some } \lambda_1 \geq \ldots \geq \lambda_n \in \mathbb{Q}.$$

This expression is unique.

Warning 8.4.2. This is *not* true over non-algebraically-closed fields.

Remark 8.4.3. An important consequence of the classification is that if $\mathcal{E} \in \mathrm{Bun}_X$ is non-zero then

$$\deg \mathcal{E} \geq 0 \implies H^0(\mathcal{E}) \neq 0.$$

This is really hard. To give an example, there is a huge space of extensions

$$0 \to \mathcal{O}(-1) \to \mathcal{E} \to \mathcal{O}(1) \to 0.$$

The classification implies that for any such extension $H^0(\mathcal{E}) \neq 0$. This is difficult; proving it is basically tantamount to proving the theorem.

We also have

$$\mathcal{O}(\lambda)^\vee = \mathcal{O}(-\lambda)$$

and

$$\mathcal{O}(\lambda) \otimes \mathcal{O}(\mu) = \mathcal{O}(\lambda+\mu)^{\oplus ?}$$

which is the best that one can hope for in consideration of the ranks.

8.4.2 Cohomology and consequences

We have

$$\mathrm{Hom}(\mathcal{O}(\lambda), \mathcal{O}(\mu)) = 0 \quad \text{if } \mu < \lambda.$$

On the other hand,

$$\mathrm{Ext}^1(\mathcal{O}(\lambda), \mathcal{O}(\mu)) = 0 \quad \text{if } \mu > 0.$$

These cohomology groups are really big; for instance

$$H^0(\mathcal{O}(1)) = \mathrm{Hom}(\mathcal{O}, \mathcal{O}(1)) = B^{\varphi=\varpi}$$

and

$$H^1(\mathcal{O}(-1)) \cong C/E,$$

which in particular is infinite-dimensional over E.

The "fundamental exact sequence"

$$0 \to E \to B_e \to B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \to 0$$

is equivalent to the statements $H^0(\mathcal{O}) = E$ and $H^1(\mathcal{O}) = 0$.

Corollary 8.4.4. *We have* $\pi_1(X) \cong \mathrm{Gal}(\overline{E}/E)$.

Proof. We have an obvious functor from finite extensions of E to finite étale covers of X, which on fields is $E' \mapsto X \otimes_E E'$. We want to show that this induces an equivalence of categories.

For $f: Y \to X$ with Y connected, we want to show that $f_*\mathcal{O}_Y$ is trivial vector bundle, because then we can try to recover E' as its global sections (it will be a finite-dimensional E-vector space with an E-algebra structure).

Using that that $\mathcal{E}$ has an algebra structure, the classification theorem implies that all slopes of $\mathcal{E}$ are ≤ 0 because if it has some component with positive slope λ, then

$$\mathcal{O}(\lambda) \otimes \mathcal{O}(\lambda) \to \mathcal{E}$$

must be zero since the description of cohomology tells us that a vector bundle on the Fargues-Fontaine curve cannot admit a non-zero map to a vector bundle of smaller slope. Since $\mathcal{E}$ is self-dual we also get that all slopes are non-negative, so all slopes are 0. The classification theorem then implies that the bundle is trivial. □

8.5 Link with p-divisible groups

Let $E = \mathbb{Q}_p$ for simplicity. Fix $\infty \in |X|$ with residue field C (an algebraically closed and complete extension of E). We can then form B_{dR}, etc. (see Colmez's lecture).

8.5.1 Miniscule modifications

For H a p-divisible group over $\overline{\mathbb{F}_p}$, there is an associated covariant Dieudonné isocrystal D, giving a bundle $\mathcal{E} = \mathcal{E}(D) \otimes \mathcal{O}(1)$. (The twisting by 1 is an artifact of the definition of duality for isocrystals, which is normalized to send an isocrystal with slope in $[0, 1]$ to another isocrystal with slope in $[0, 1]$.)

Definition 8.5.1. A *degree* $n \in [0, ht\, H]$ *miniscule modification* of $\mathcal{E}$ is a vector bundle $\mathcal{F}$ fitting into a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{E} \to i_* W \to 0$$

where $i \colon \{\infty\} \hookrightarrow X$ and $\dim_C W = n$.

The key idea is that one can make many modifications trivial using periods of p-divisible groups. The mechanism for this is the *period morphism*, which we now explain.

8.5.2 The period morphism

To H we can attach a *Rapoport-Zink space*, which is rigid space $\mathcal{M}/L$ classifying deformations for p-divisible groups, but where we take deformations not by isomorphisms but by *quasi-isogenies*. There exists an étale period map

$$\mathcal{M} \to Fl$$

where Fl is the flag variety of n-dimensional quotients of D, with $n = \dim H$. The period map is

$$\mathcal{G} \mapsto \mathrm{Lie}\, \mathcal{G}[1/p].$$

The key fact is that $i^*\mathcal{E} \cong D \otimes C$. Granting this fact, the map from $Fl(C)$ to the set of degree n miniscule modifications can be described as

$$x = [D \otimes C \twoheadrightarrow W] \mapsto \mathcal{E}(x) := \ker[\mathcal{E} \to i_* i^* \mathcal{E} \cong i_*(D \otimes C) \twoheadrightarrow i_* W].$$

Theorem 8.5.2. *If x is in the image of the period map, then $\mathcal{E}(x)$ is trivial. (So $\mathcal{E}(x) = V_p(\mathcal{G}) \otimes \mathcal{O}_X$).*

Remark 8.5.3. This is a "sheafy" version of the p-adic comparison theorem for p-divisible groups, whose proof is an easy consequence of the usual version.

There are essentially two cases in which the period map is surjective.

1. If we choose H to be a 1-dimensional height h formal group over $\overline{\mathbb{F}}_p$ then $Fl \cong \mathbb{P}^{h-1}$ (Gross-Hopkins, Laffaille).

2. If we choose H to be a special formal module in the sense of Drinfeld.

Therefore, in these cases p-divisible groups give us many modifications with $\mathcal{F}$ be trivial.

8.6 Sample of ideas in the proof of the classification

The main technical results are that for all $n \geq 1$:

1. all degree 1 modifications of $\mathcal{O}(1/n)$ are trivial (this is a trivial consequence of the theorem statement, but a significant step in the proof).
2. $\mathcal{O}(-1/n)$ is the only degree 1 modification of $\mathcal{O}^{\oplus n}$ without global sections.

Let's sketch how these facts are used in an example. We'll prove that any $\mathcal{E}$ fitting into an extension

$$0 \to \mathcal{O}(-1) \to \mathcal{E} \to \mathcal{O}(1) \to 0$$

has global sections. (This is a special case of Remark 8.4.3.) Suppose otherwise. There is a map $\mathcal{O}(1) \to i_*C$. Consider the composite $\mathcal{E} \to \iota_*C$ and define $\mathcal{F} := \ker(\mathcal{E} \to i_*C)$. So $\mathcal{F}$ is an extension

$$0 \to \mathcal{O}(-1) \to \mathcal{F} \to \mathcal{O} \to 0.$$

By the construction of $\mathcal{F}$ we also have an extension

$$0 \to \mathcal{F} \to \mathcal{E} \to i_*C \to 0.$$

Now for a trick: pick an embedding $\mathcal{O}(-1) \hookrightarrow \mathcal{O}$ and consider the pushout

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(-1) & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \end{array}$$

Then $\mathcal{F}' \cong \mathcal{O}^2$ because $H^1(\mathcal{O}) = 0$. So $\mathcal{F}$ is a degree 1 modification of $\mathcal{O}^2$ but it has no global sections because $\mathcal{E}$ doesn't and $\mathcal{F} \hookrightarrow \mathcal{E}$, so by fact (2) above $\mathcal{F} \cong \mathcal{O}(-1/2)$.

Now dualize the sequence

$$0 \to \mathcal{O}(-1/2) \to \mathcal{E} \to i_*C \to 0$$

to get

$$0 \to \mathcal{E}^\vee \to \mathcal{O}(1/2) \to i_*C \to 0.$$

But this is trivial by (1), so $\mathcal{E}^\vee$ is trivial, hence $\mathcal{E}$ is trivial, contradicting the assumption that $\mathcal{E}$ has no global sections.

Chapter 9

Banach-Colmez Spaces

9.1 Definition of Banach-Colmez Spaces

Definition 9.1.1. A *Banach sheaf* is a contravariant functor $\mathcal{F}$ from Perf_C to topological $\mathbb{Q}_p$-vector spaces such that

1. if $X \in \mathrm{Perf}_C$ is affinoid perfectoid then $\mathcal{F}(X)$ is a Banach space
2. $\mathcal{F}$ is a sheaf on $\mathrm{Perf}_{C,\text{pro-étale}}$.

Morphisms are morphisms of functors (i.e. natural transformations).

A sequence of Banach sheaves is exact if it is is exact as a sequence of sheaves on $\mathrm{Perf}_{C,\text{pro-étale}}$.

Example 9.1.2. Let V be a finite-dimensional $\mathbb{Q}_p$ vector space. Then we have a constant Banach sheaf $\underline{V}$. This is represented by $\mathrm{Spa}(\mathrm{Funct}(V, C), \mathrm{Funct}(V, \mathcal{O}_C))$. The sections can be described explicitly as $\mathcal{F}(X) = \mathrm{Funct}(|X|, V)$.

Example 9.1.3. Let W be a finite-dimensional C-vector space. We can form $\mathcal{F} = W \otimes \mathcal{O}$, which is representable by $W \otimes \mathbb{G}_a$ (but this is not a perfectoid space).

Definition 9.1.4. An *effective Banach-Colmez space* is a Banach sheaf which is an extension

$$0 \to \underline{V} \to \mathcal{F} \to W \otimes \mathcal{O} \to 0.$$

Where W is a C-vector space and V is a $\mathbb{Q}_p$-vector space.

Definition 9.1.5. A *Banach-Colmez space* is a Banach sheaf $\mathcal{F}$ which is a quotient of an effective Banach-Colmez space by a $\mathbb{Q}_p$-vector space:

$$0 \to \underline{V'} \to \underbrace{\mathcal{F}'}_{\text{effective}} \to \mathcal{F} \to 0$$

The category of such is denoted $\mathcal{BC}$. If $\mathcal{F}$ is a BC and V, W, V' are as before we call $\dim_C W$ the *dimension* of the presentation of $\mathcal{F}$ and we call $\dim_{\mathbb{Q}_p} V - \dim_{\mathbb{Q}_p} V'$ the *height* of the presentation.

Example 9.1.6. We have $\dim \underline{\mathbb{Q}_p} = 0$ and $\mathrm{ht}\, \underline{\mathbb{Q}_p} = 1$; while $\dim \mathcal{O} = 1$ and $\mathrm{ht}\, \mathcal{O} = 0$.

Theorem 9.1.7 (Colmez)**.** *$\mathcal{BC}$ is an abelian category. The functor $\mathcal{F} \mapsto \mathcal{F}(C)$ is exact and conservative. Moreover, the dimension and the height do not depend on the presentation. Objects of $\mathcal{BC}$ have a Harder-Narasimhan filtration for the slope function $\mu = -\frac{\mathrm{ht}}{\dim}$.*

9.2 Examples as universal covers of p-divisible groups

9.2.1 Universal cover of a p-divisible group

Let G be a p-divisible group over $\mathcal{O}_C$. We can take $\mathcal{G}$ to be the formal completion of G along its unit section, which is a formal group scheme over $\mathrm{Spf}\, \mathcal{O}_C$. We can then take the generic fiber

$$\mathcal{G}_\eta^{\mathrm{ad}} := \mathcal{G}^{\mathrm{ad}} \times_{\mathrm{Spa}(C, \mathcal{O}_C)} \mathrm{Spa}(\mathcal{O}_C, \mathcal{O}_C).$$

There is an exact sequence of étale sheaves on the big étale site of $\mathrm{Spa}(C)$

$$0 \to \underline{\mathcal{G}_\eta^{\mathrm{ad}}[p^\infty]} \to \mathcal{G}_\eta^{\mathrm{ad}} \xrightarrow{\log} \mathrm{Lie}\, G[1/p] \otimes \mathbb{G}_a \to 0.$$

Here $\mathcal{G}_\eta^{\mathrm{ad}}[p^\infty] = \mathbb{Q}_p/\mathbb{Z}_p \otimes T_p(G)$. Taking the inverse limit of this sequence along the pth power map, we get a short exact sequence

$$0 \to \underline{T_p(G)[p^{-1}]} \to \widetilde{\mathcal{G}_\eta^{\mathrm{ad}}} := \varprojlim_{x \mapsto px} \mathcal{G}_\eta^{\mathrm{ad}} \to \mathrm{Lie}(G)[1/p] \otimes \mathbb{G}_a \to 0.$$

Definition 9.2.1. This $\widetilde{\mathcal{G}_\eta^{\mathrm{ad}}}$ is called the *universal cover* of the p-divisible group G.

Example 9.2.2. For $G = \widehat{\mathbb{G}_m}^d$, if R is an affinoid perfectoid C-algebra then

$$\widetilde{\mathcal{G}}_\eta^{\mathrm{ad}}(R) \cong \varprojlim_p (R^{00})^d \cong (R^{\flat,00})^d.$$

For any G, the universal cover $\widetilde{\mathcal{G}}_\eta^{\mathrm{ad}}$ is an effective BC space.

Example 9.2.3. For $G = \mu_{p^\infty}$ we have $T_p(G) = \mathbb{Z}_p$ (no Galois action since we're over C) and

$$\widetilde{\mu_{p^\infty}}(R) = \varprojlim_p 1 + R^{00} \xrightarrow{\log} R$$

sending $(x_n) \mapsto \log x_0$.

Remark 9.2.4. The sheaf $\widetilde{\mathcal{G}}_\eta^{\mathrm{ad}}$ is always representable by a perfectoid open ball. It is a Banach-Colmez space with height and dimension equal to those of G.

9.2.2 p-divisible groups parametrize all Banach-Colmez spaces

Definition 9.2.5. Define $\mathcal{BC}^+_{p-div}$ to be the full subcategory of $\mathcal{BC}$ consisting of universal covers of p-divisible group. Define $\mathcal{BC}_{p-div}$ to be the full subcategory of $\mathcal{BC}$ obtained as a quotient of an object of $\mathcal{BC}^+_{p-div}$ by a $\mathbb{Q}_p$-vector space.

Proposition 9.2.6. *We have* $\mathcal{BC}_{p-div} \cong \mathcal{BC}$.

Proof. We need to show that every BC space is in $\mathcal{BC}_{p-div}$, i.e. any extension of $W \otimes \mathcal{O}$ by V can be recovered as a quotient of a universal cover of a p-divisible group by a $\mathbb{Q}_p$-vector space.

One important input we need is that $\mathrm{Ext}^1_{\mathcal{BC}}(W \otimes \mathcal{O}, V) \cong \mathrm{Hom}_C(W, V \otimes C)$. What does this even mean? It is saying that any extension

$$0 \to \underline{V} \to \mathcal{F} \to W \otimes \mathcal{O} \to 0$$

fits into a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{V} & \longrightarrow & \mathcal{F} & \longrightarrow & W \otimes \mathcal{O} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow {\scriptstyle f \in \mathrm{Hom}(W, V \otimes C)} & & \\ 0 & \longrightarrow & \underline{V} & \longrightarrow & V \otimes \widetilde{\mu_{p^\infty,\eta}}^{\mathrm{ad}} & \longrightarrow & V \otimes \mathcal{O} & \longrightarrow & 0 \end{array}$$

A result of Fargues, Scholze-Weinstein then says that for all $(W, V, f : W \to V \otimes C)$ there exists a p-divisible group G over $\mathcal{O}_C$ such that $\widetilde{\mathcal{G}}^{\mathrm{ad}}_\eta = \mathcal{F}$. (They take $V = T_p(G)[1/p]$, $W = \mathrm{Lie}(G)[1/p]$, and f is the transpose of the Hodge-Tate map of G^D). Since the fact about $\mathrm{Ext}^1_{\mathcal{BC}}$ tells us that $\mathcal{F}$ is of this form, we are done.

□

Example 9.2.7. Let $G = \mu_{p^\infty}$. Then

$$\widetilde{\mathcal{G}}^{\mathrm{ad}}_\eta(R) = B^+_{\mathrm{cris}}(R^0/p)^{\varphi=p} = B(R)^{\varphi=p}.$$

Corollary 9.2.8. *Banach-Colmez spaces are diamonds over* $C^\flat$.

Now we want to describe more explicitly the objects of $\mathcal{BC}^+ = \mathcal{BC}^+_{p-div}$. Fix a section $\overline{\mathbb{F}}_p \hookrightarrow \mathcal{O}_C/p$. Given a p-divisible group G over $\mathcal{O}_C$, there exists a p-divisible group H over $\overline{\mathbb{F}}_p$ and an isogeny

$$H \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_C/p \cong G \otimes_{\mathcal{O}_C} \mathcal{O}_C/p$$

Theorem 9.2.9 (Scholze-Weinstein)**.** *Let R be a C-perfectoid algebra. Then*

$$\widetilde{\mathcal{G}}^{\mathrm{ad}}_\eta(R) = \widetilde{H}^{\mathrm{ad}}_\eta(R^0/p) = \mathrm{Hom}_{R^0/p}(\mathbb{Q}_p/\mathbb{Z}_p, H)[1/p] = D(H)(R^0/p)[1/p]^{\varphi=p}$$

Evaluating the associated Banach-Colmez space on C gives

$$0 \to \underline{\mathbb{Q}_p} \to (B^+_{\mathrm{cris}})^{\varphi=p} \xrightarrow{\theta} C \to 0.$$

9.3 Banach-Colmez Spaces and the Fargues-Fontaine curve

9.3.1 A t-structure

Let $X = X_{\mathbb{Q}_p, C^\flat}$. We have the abelian category Coh_X on X.

Definition 9.3.1. We define an abelian category.

$$\mathrm{Coh}_X^{0,-} = \left\{ \mathcal{F} \in D^b(X) : \begin{array}{c} H^i(\mathcal{F}) = 0 \forall i \neq -1, 0 \\ \mu(H^0(\mathcal{F})) \geq 0 \\ \mu(H^{-1}(\mathcal{F})) < 0 \end{array} \right\}$$

The point is that we can think of coherent sheaves with positive/negative slopes as a torsion pair, since the classification theorem tells us that $\mathrm{Hom}(\mathcal{E}, \mathcal{E}') = 0$ and $\mathrm{Ext}^1(\mathcal{E}', \mathcal{E}) = 0$ if $\mu(\mathcal{E}) > \mu(\mathcal{E}')$. Therefore, one (admittedly convoluted) way of describing $\mathcal{F} \in \mathrm{Coh}_X$ is as a pair $(\mathcal{F}', \mathcal{F}'')$ with $\mu(\mathcal{F}') < 0$ and $\mu(\mathcal{F}'') \geq 0$ plus an element of $\mathrm{Ext}^1_{\mathrm{Coh}(X)}(\mathcal{F}', \mathcal{F}'') = 0$.

Analogously, we can think to an object of $\mathrm{Coh}_X^{0,-}$ as a pair $(\mathcal{F}', \mathcal{F}'')$ where $\mu(\mathcal{F}') < 0$ and $\mu(\mathcal{F}'') \geq 0$, plus an element of

$$\mathrm{Ext}^1_{\mathrm{Coh}_X^{0,-}}(\mathcal{F}'', \mathcal{F}[1]) = \mathrm{Ext}^2_{\mathrm{Coh}_X}(\mathcal{F}'', \mathcal{F}') = 0.$$

Remark 9.3.2. We can extend additively rank , deg to $D^b(X)$. We can define $\deg^{0,-} = -\operatorname{rank}$ and $\operatorname{rank}^{0,-} = \deg$. If $\mu^{0,-} = \frac{\deg^{0,-}}{\operatorname{rank}^{0,-}}$ then objects of $\mathrm{Coh}_X^{0,-}$ have an HN filtration for this slope function.

Theorem 9.3.3. $\mathrm{Coh}_X^{0,-} \cong \mathcal{BC}$.

9.3.2 Connection to the Fargues-Fontaine curve

Let S be a perfectoid space over $C^\flat$. Then one can define a relative Fargues-Fontaine curve X_S, which you can think of as a family of the usual curves $(X_{k(s)})_{s \in S}$.

Warning 9.3.4. There is no map $X_S \to S$. This is already the case over a field.

- The association $S \rightsquigarrow X_S$ is functorial,
- If $S' \to S$ is pro-étale (surjective) then $X_{S'} \to X_S$ is also.

This allows us to define a morphism of sites

$$\tau\colon (\text{big pro-étale site of } X) \to (\text{sheaves on } \mathrm{Perf}_{C^\flat, \text{pro-étale}})$$

by

$$\mathcal{F} \mapsto \tau_* \mathcal{F}(S) = H^0(X, \mathcal{F}_S := \mathcal{F}|_{X_S}).$$

Proposition 9.3.5. *Let $\mathcal{F} \in \mathrm{Coh}_X$. If $\mu(\mathcal{F}) \geq 0$ then $R^i \tau_* \mathcal{F} = 0$ for all $i \neq 0$. If $\mu(\mathcal{F}) < 0$ then $R^i \tau_* \mathcal{F} = 0$ for all $i \neq 1$.*

Corollary 9.3.6. *We have*

$$\mathrm{Coh}_X^{0,-} \cong \left\{ \mathcal{F} \in D^b(X) \colon \begin{array}{c} H^i(\mathcal{F}) = 0 \text{ if } i \neq -1, 0 \\ R^0\tau_* H^{-1}(\mathcal{F}) = 0, \\ R^1\tau_* H^0(\mathcal{F}) = 0 \end{array} \right\}.$$

In other words, the functor $R^0\tau_* \colon \mathrm{Coh}_X^{0,-} \to \widetilde{\mathrm{Perf}}_{C^\flat,\text{pro-étale}}$ *(where tilde means the category of sheaves) is exact.*

This induces an equivalence $\mathrm{Coh}_X^{0,-} \cong \mathcal{BC}$, implicitly using Scholze to identify sheaves on the proétale sites of C and $C^\flat$.

Example 9.3.7. We have

$$R^0\tau_*\mathcal{O}_X(S) = H^0(X_S, \mathcal{O}_{X_S}) = B^+(R)^{\varphi=1} = \underline{\mathbb{Q}_p}.$$

where $S = \mathrm{Spa}(R)$. Also

$$R^0\tau_*\iota_{\infty*}C = \mathcal{O}.$$

By playing with the sequence

$$\mathcal{O}_X(-1) \to \mathcal{O}_X \to i_{\infty*}C$$

which can be tilted (in the sense of torsion pairs)

$$\mathcal{O}_X \to i_{\infty*}C \to \mathcal{O}_X(-1)[1].$$

we can show that the category depends only on $C^\flat$, and that the curve can be reconstructed from the BC category.

Part III

Day Three

Chapter 10

The Relative Fargues-Fontaine Curve

The goals of this talk are to:

1. Define the relative curves Y_S and $X_S = Y_S/\varphi^{\mathbb{Z}}$ for any $S \in \mathrm{Perf}_{\mathbb{F}_p}$. (To recover Y^{ad} and X^{ad} from before, put $S = \operatorname{Spa} \mathbb{C}_p^\flat$.) These will be adic spaces over $\operatorname{Spa} \mathbb{Q}_p := \operatorname{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$.

2. Describe the relation to untilting and the diamond formula

$$\text{“}Y_S = S \times \operatorname{Spa} \mathbb{Q}_p\text{”}.$$

10.1 Construction of the relative curves Y_S and X_S

10.1.1 The affinoid perfectoid case

Suppose now that $S = \operatorname{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbb{F}_p}$ is affinoid perfectoid of characteristisc p. Fix once and for all a pseudo-uniformizer $\varpi \in R$. We define the ring

$$\mathbb{A}(= \mathbb{A}_{R^+}) = W(R^+) \ni p, [\varpi]$$

with the $(p, [\varpi])$-adic topology.

Definition 10.1.1. We define

$$Y_{(R,R^+)} := \operatorname{Spa}(\mathbb{A}, \mathbb{A}) \setminus V(p[\varpi]).$$

This is, at the moment, a pre-adic space (i.e. the structure presheaf is not yet known to be a sheaf) over $\operatorname{Spa} \mathbb{Q}_p$.

The points of this are continuous valuations

$$\|\cdot\| \colon A \to \Gamma \cup \{0\}$$

such that

- $\|a\| \leq 1$ for all $a \in A$,
- $\|p[\varpi]\| \neq 0$.

Definition 10.1.2. (1) Given $(\|\cdot\|, \Gamma) \in Y_{(R,R^+)}$, its *maximal generalization* is the rank-one point $(\|\cdot\|_{\max}, \mathbb{R}_{\geq 0}) \in Y_{(R,R^+)}$ given by the rule

$$\|a\|_{\max} := p^{-\sup\{r/s \in \mathbb{Q}_{\geq 0} \colon \|[\varpi]\|^r \geq \|a\|^s\}}.$$

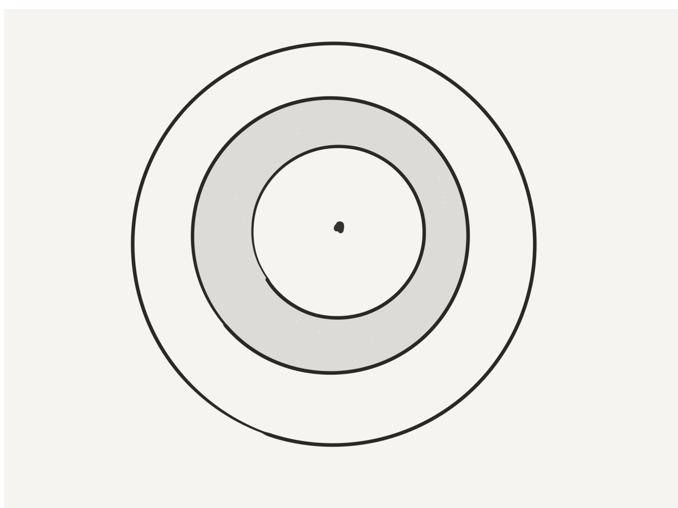
This is the "closest point to $\|a\|$ in the line of Γ generated by $\|[\varpi]\|$".

(2) The *radius* of $(\|\cdot\|, \Gamma)$ is then $\delta(\|\cdot\|) := \|p\|_{\max} \in (0,1)$ (the value in in $[0,1]$ by definition, and cannot be either endpoint because p is topologically nilpotent and not killed). This defines a continuous radius function

$$\delta \colon Y_{(R,R^+)} \to (0,1).$$

(3) Given a closed interval $I \subset (0,1)$, the associated *annulus* is

$$Y^I_{(R,R^+)} := \text{interior of } \delta^{-1}(I) \overset{\text{open}}{\subset} Y_{(R,R^+)}.$$



Lemma 10.1.3. *(i) We have*

$$Y_{(R,R^+)} = \bigcup_{I \subset (0,1)} Y^I_{(R,R^+)}.$$

(ii) If $I = [p^{-r/s}, p^{-r'/s'}]$ with $r, s, r', s' \in \mathbb{N}$, then $Y^I_{(R,R^+)}$ is the rational subdomain

$$\mathrm{Spa}(\mathbb{A}, \mathbb{A})\langle \frac{[\varpi]^r}{p^s}, \frac{p^{s'}}{[\varpi]^{r'}} \rangle \subset Y_{(R,R')}.$$

Proof. With I as in (ii), it follows from the definitions (modulo interior issues) that

$$Y^I_{(R,R^+)} = \{\|\cdot\|\colon \|p\| \in I \subset \Gamma\}$$

which is the claimed rational subdomain. (Note that here we are normalizing $\|\cdot\|$ so that $\|[\varpi]\| = p^{-1}$.) □

Theorem 10.1.4. *$Y_{(R,R^+)}$ is an adic space (i.e. the presheaf is sheafy).*

Idea of proof. Pick some perfectoid field $E \supset \mathbb{Q}_p$ and check that $Y^I_{(R,R^+)} \times_{\operatorname{Spa}\mathbb{Q}_p} \operatorname{Spa} E$ is affinoid perfectoid. (Although we have used finite extensions of $\mathbb{Q}_p$ for our E, references in the literature allow the field E to be perfectoid precisely to deal with this issue.) By definition, this is saying that $Y^I_{(R,R^+)}$ is *pre-perfectoid*. That implies the sheaf property by results of Scholze, or Kedlaya-Liu. □

Remark 10.1.5. Descent of the sheafiness is not hard, but proving that the base change is affinoid perfectoid requires work (to show that the ring of power-bounded elements is bounded), and the original result that affinoid perfectoid spaces are sheafy is hard.

10.1.2 Forming the quotient

We have an action φ on $W(R)$ inducing an action of φ on $Y_{(R,R^+)}$ such that

$$\delta(\varphi(y)) = \delta(y)^{1/p}.$$

Note that in the unit disk picture, this is expanding towards the boundary. Therefore the action of φ on $Y_{(R,R^+)}$ is properly continuous, so we can define

$$\boxed{X_{(R,R^+)} := Y_{(R,R^+)}/\varphi^{\mathbb{Z}}}$$

as an adic space over $\operatorname{Spa}\mathbb{Q}_p$.

Remark 10.1.6. Suppose $I = [a, b] \subset (0, 1)$ such that $b^p < a \le b < a^{1/p}$, so that the interval is translated to a disjoint interval under $x \mapsto x^{1/p}$. Then $Y^I_{(R,R^+)}$ maps isomorphically to an open subset of $X_{(R,R^+)}$.

10.1.3 The map θ

Suppose that $(R, R^+) = (B^\flat, B^{+\flat})$ for some $\operatorname{Spa}(B, B^+) \in \mathrm{Perf}$ over $\operatorname{Spa}\mathbb{Q}_p$ (e.g. $S = \operatorname{Spa}\mathbb{C}_p^\flat$). Then Fontaine's map

$$\theta\colon W(B^{+\flat}) = \mathbb{A} \to B^+$$

induces a closed immersion

$$\theta\colon \operatorname{Spa}(B, B^+) \hookrightarrow Y_{(R,R^+)}. \tag{10.1}$$

Lemma 10.1.7. *The composition*

$$\mathrm{Spa}(B, B^+) \hookrightarrow Y_{(R,R^+)} \twoheadrightarrow X_{(R,R^+)}$$

is a closed immersion.

Proof sketch. Explicitly check that θ has image in an annulus which is small enough so that it maps isomorphically down to the Fargues-Fontaine curve. □

10.1.4 Construction for general $S \in \mathrm{Perf}_{\mathbb{F}_p}$

One checks that the process $(R, R^+) \mapsto Y^I_{(R,R^+)}$ (for any $I \subset (0, 1)$) behaves well under taking rational subdomains. Then it's easy to glue to define Y_S and $X_S = Y_S/\varphi^{\mathbb{Z}}$ (as adic spaces over $\mathrm{Spa}\,\mathbb{Q}_p$) for any $S \in \mathrm{Perf}_{\mathbb{F}_p}$.

One has an obvious analogue of θ if S is a tilt. That is, if $S = (S^\#)^\flat$ then we get a closed embedding

$$\theta\colon S^\# \hookrightarrow Y_S$$

by gluing.

10.2 Diamonds and untilting

10.2.1 Diamonds parametrize untilts

Definition 10.2.1. For X an analytic adic space (i.e. covered by adic spectra of Tate rings) over $\mathrm{Spa}\,\mathbb{Z}_p$, we define

$$X^\diamond\colon \mathrm{Perf}_{\mathbb{F}_p} \to \mathrm{Sets}$$

by

$$\begin{aligned} T &\mapsto \{\text{Untilts over } X \text{ of } T\} \\ &= \left\{ (T^\#, \iota)\colon \begin{matrix} T^\# \in \mathrm{Perf}_X \\ \iota\colon T^{\#\flat} \cong T \end{matrix} \right\}. \end{aligned}$$

Lemma 10.2.2. *If X is perfectoid then* $X^\diamond = \mathrm{Hom}(-, X^\flat)$.

Therefore the formation of diamonds can be thought of as an extension of tilting to adic spaces.

Proof. Let's check that the functors of points agree. For a test space T, $X^\diamond(T)$ is an untilt over X of T. Given such an untilt, we can tilt to obtain a map $T \to X^\flat$.

In the other direction, given a map $T \to X^\flat$, the equivalence between perfectoid spaces over $X^\flat$ and X produces an untilt over X of T. □

In particular, if $X \in \mathrm{Perf}_{\mathbb{F}_p}$ (viewed as an analytic space over $\mathrm{Spa}\,\mathbb{Z}_p$), then $X^\diamond = \mathrm{Hom}(-, X)$, which is just X viewed as a (representable) sheaf.

Lemma 10.2.3. *$X^\diamond$ is a sheaf for the pro-étale topology on* $\mathrm{Perf}_{\mathbb{F}_p}$*, and even a diamond.*

Proof idea. Pick a perfectoid cover of X. To each element in the cover you apply the construction $(-)^\diamond$; this produces diamonds since they are representable. Then you check that anything pro-étale covered by diamonds is itself a diamond. □

Example 10.2.4. $(\mathrm{Spa}\,\mathbb{Q}_p^{\mathrm{cyc}})^\diamond \to \mathrm{Spa}\,\mathbb{Q}_p^\diamond$ is a pro-étale $\mathbb{Z}_p^*$-torsor.

Remark 10.2.5. For many purposes it suffices only to remember that diamonds are a full subcategory of pro-étale sheaves on $\mathrm{Perf}_{\mathbb{F}_p}$.

10.2.2 The diamond equation for the curve

Proposition 10.2.6. *Let $S \in \mathrm{Perf}_{\mathbb{F}_p}$. Then*

$$Y_S^\diamond \cong S^\diamond \times \mathrm{Spa}\,\mathbb{Q}_p^\diamond$$

(in the category of diamonds or $\widetilde{\mathrm{Perf}}_{\mathbb{F}_p,\text{pro-étale}}$).

Remark 10.2.7. Y_S is an analytic adic space because the annuli are Tate algebras.

Proof. Let's compare the functors of points. We have to show that for $T \in \mathrm{Perf}_{\mathbb{F}_p}$ there is a bijection

$$\{\text{untilts } / \, Y_S \text{ of } T\} \leftrightarrow \mathrm{Hom}(T, S) \times \{\text{untilts } / \, \mathrm{Spa}\,\mathbb{Q}_p \text{ of } T\}.$$

Suppose we have a pair $(f, (T^\#, \iota))$ on the right side. We can send this to $(T^\#, \iota)$:

$$(T^\#, \iota) \leftarrow (f, (T^\#, \iota)).$$

At first this seems like it's forgotten f, but that is built into the meaning of $T^\#$, because we need to specify the structure of $T^\#$ as a space over Y_S. This structure is via

$$T^\# \overset{\theta}{\hookrightarrow} Y_{T^{\#\flat}} \overset{\iota}{\cong} Y_T \overset{f}{\to} Y_S.$$

(The embedding θ is (10.1)). In the other direction we send

$$(T^\#, \iota) \mapsto (T \overset{\iota}{\cong} T^{\#\flat} \to S, (T^\#, \iota))$$

where the map $T^{\#\flat} \to S$ is defined is follows. Reduce to the affinoid case $S = \mathrm{Spa}(R, R^+)$. Then we can compose the map $T^\# \to Y_S$ to get $T^\# \to Y_S \to \mathrm{Spa}(W(R^+), W(R^+))$, at which point the universal property of the Witt vectors gives

$$T^{\#\flat} \to \mathrm{Spa}(R, R^+).$$

(At the level of rings formation of Witt vectors is left adjoint to tilting, so at the level of spaces it is right adjoint.) □

Proposition 10.2.8. *The following are in canonical bijection with each other.*

1. *Sections of* $Y_S^\diamond \to S^\diamond$.
2. *Maps* $S^\diamond \to \mathrm{Spa}(\mathbb{Q}_p)^\diamond$.
3. *Untilts over* $\mathrm{Spa}\,\mathbb{Q}_p$ *of* S.
4. *Closed subsets of* Y_S *defined locally by a "degree 1 primitive element" (i.e. a* $\xi \in W(R^+)$ *of the form* $[\varpi] + pu$ *where* $\varpi \in R$ *is a pseudo-uniformizer and* $u \in W(R^+)^*$*).*

Proof. By proposition 10.2.6, (1) is the same as sections of

$$\mathrm{Spa}\,\mathbb{Q}_p^\diamond \to S^\diamond$$

which are the same as maps $S^\diamond \to \mathrm{Spa}\,\mathbb{Q}_p^\diamond$, which is (2).

The set (2) is

$$\mathrm{Hom}_{\widetilde{\mathrm{Perf}}_{\mathbb{F}_p}}(S^\diamond = \mathrm{Hom}(-, S), \mathrm{Spa}\,\mathbb{Q}_p^\diamond)$$

which by Yoneda is $\mathrm{Spa}\,\mathbb{Q}_p^\diamond(S)$, which is (3).

Finally, the identification (3) = (4) is a generalization of the final Lemma from the Peter's discussion yesterday, the idea being that $\ker\theta$ is always generated by a degree 1 primitive element. □

Chapter 11

Beauville-Laszlo Uniformization

11.1 Statement of the results

11.1.1 Setup

Let

- G be a split reductive group over $k = \overline{k}$ an algebraically closed field of characteristic p.
- X be a smooth projective geometrically connected curve over k.
- $x \in |X|$ and $X^0 = X - x$.
- S be a scheme over k.
- $\mathcal{F}$ be a G-bundle over $X \times S$.

11.1.2 Statement of Theorems

Theorem 11.1.1. *There is a surjective étale map $S' \to S$ such that the G-bundle $\mathcal{F} \times_S S' \to X \times_S S'$ has a B-structure.*

Definition 11.1.2. Let $\mathcal{F} \to Y$ be a G-bundle and $B \subset G$ a fixed Borel subgroup. By a *B-structure* of $\mathcal{F}$ we mean a pair $(\mathcal{E}, \eta)$ such that E is a B-bundle and

$$\begin{array}{ccc} \eta\colon G \times^B \mathcal{E} & \xrightarrow{\sim} & \mathcal{F} \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

Remark 11.1.3. There is a natural bijection

$$(B\text{-structures of } \mathcal{F} \to Y) \leftrightarrow (\text{sections } s\colon Y \to B\backslash\mathcal{F}).$$

Here $B\backslash\mathcal{F} = G\backslash B \times^G \mathcal{F}$.

Theorem 11.1.4. *If G is semisimple, then there exists a faithfully flat morphism $S' \to S$ of finite presentation such that $\mathcal{F} \times_S S'|_{X^0 \times_S S'}$ is trivial. In general, if $p \nmid \#\pi_1(G)$ then $S' \to S$ can be chosen to be étale (if $p = 0$, then there are no restrictions).*

Remark 11.1.5. The statements and proofs generalize immediately to a relative curve $X \xrightarrow{\pi} S$, e.g. in Theorem 2 and $D \subset X$ a divisor such that $\pi|_D \colon D \cong S$ then $\mathcal{F}|_{X-D}$ is trivial after base change.

11.1.3 The affine Grassmannian

Recall that $\mathrm{Gr}_G = LG/L^+G$ is an ind-scheme classifying

$$\left\{(\mathcal{F}, i) \colon \begin{array}{rl} \mathcal{F} = & G\text{-bundle}/D = \operatorname{Spec} k[[t]] \\ i = & \text{trivialization of } \mathcal{F}|_{D^\times} \end{array}\right\}$$

Definition 11.1.6. Define $\mathrm{Gr}_{G,x}$ to be the moduli space defined by

$$\mathrm{Gr}_{G,x}(S) = \left\{(\mathcal{F}, i) \colon \begin{array}{c} \mathcal{F} = \ G\text{-bundle}/X \times S \\ i \colon \mathcal{F}|_{X^0 \times S} \xrightarrow{\sim} \mathcal{F}^0|_{X^0 \times S} \end{array}\right\}$$

where F^0 is the trivial bundle.

It is easy to see by a "gluing Lemma" that

$$\mathrm{Gr}_{G,x} \cong \mathrm{Gr}_G .$$

This isomorphism is almost canonical (up to a choice of uniformizer at x).

Why is this relevant? There is a natural map

$$\pi \colon \ \mathrm{Gr}_{G,x} \to \mathrm{Bun}_n(X)$$

sending $(\mathcal{F}, i) \mapsto \mathcal{F}$.

Theorem 11.1.7. *Theorem 11.1.4 says that the map π is surjective in the the faithfully flat topology.*

This statement of the theorem will be generalized to the Fargues-Fontaine curve.

11.2 Proof of Theorem 11.1.1

11.2.1 A simple case

Suppose $S = k = \overline{\mathbb{F}}_p$. For the function field of a curve, Steinberg's Theorem in characteristic 0 or Springer's Theorem in characteristic p tells us that $H^1(k(X), G) = 0$. From this it follows that any G-bundle over the generic point $\eta = \operatorname{Spec} k(X)$ is trivial. Of course the trivial bundle has a B-structure, which by Remark 11.1.3 is equivalent to a section of $B\backslash\mathcal{F}|_\eta$ at the generic point. Such a section spreads out to some open subset $U \subset X$. By the valuative criterion for properness applied to $B\backslash\mathcal{F} \to X$, the section extends (uniquely) to all of X.

11.2.2 Moduli space of B-structures

Remark 11.2.1. We can replace G by G/Z^0, and so assume that G is semisimple.

The idea is to consider the moduli space of all B-structures. We want to show that this has a section after an étale cover; for this it suffices to show that the map from the moduli space to S is smooth and surjective.

Definition 11.2.2. (1) Let $T \subset B$ be the maximal torus and $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ the set of simple roots. For all i and all B-bundles $\mathcal{E} \to X$ we can form a line bundle $\alpha_i(\mathcal{E}) \to X$ via $\alpha_i \colon B \to T \to \mathbb{G}_m$, and we define

$$\deg_i(\mathcal{E}) := \deg \alpha_i(\mathcal{E}).$$

(2) Let $M_{\mathcal{F}}$ be the moduli space of B-structures of F, so

$$M_{\mathcal{F}}(T) = \{B\text{-structures of } \mathcal{F} \times_S T\}.$$

By the way, there are no automorphisms because a B-structure is a section, and sections have no automorphisms.

A section can be identified with a subscheme of the product. By the theory of Hilbert schemes, $M_{\mathcal{F}} \to S$ is a scheme locally of finite presentation (we do not say "locally of finite type" because S may not be Noetherian).

We said that we would like the map $M_{\mathcal{F}} \to S$ to be smooth and surjective. Actually it is surjective but not smooth. To rectify this, we look at a certain subspace of it.

(3) For every geometric point $y \in M_{\mathcal{F}}$ (corresponding to a B-bundle $\mathcal{E}_y \to X$) we can consider $d_i(y) := \deg \alpha_i(\mathcal{E}_y) \in \mathbb{Z}$. Then $d_i \colon M_{\mathcal{F}} \to \mathbb{Z}$ is locally constant. Define $M^+_{\mathcal{F}} \subset M_{\mathcal{F}}$ to be the set of $y \in M_{\mathcal{F}}$ such that $d_i(Y) < \min\{1, 2-2g\}$ for all i. This is a union of connected components.

Then Theorem 11.1.1 follows from the two propositions.

Proposition 11.2.3. *The map $M^+_{\mathcal{F}} \to S$ is smooth.*

Proposition 11.2.4. *The map $M^+_{\mathcal{F}} \to S$ is surjective.*

11.2.3 Proof of Proposition 11.2.3

The first proposition is standard deformation theory. Indeed, a geometric point $y \in M^+_{\mathcal{F}}$ corresponds to a section $\sigma \colon X \to B\backslash\mathcal{F}$. A deformation of this B-structure is controlled by $H^0(X, \sigma^* T_{(B\backslash\mathcal{F})/X})$. One checks that $\sigma^* T_{(B\backslash\mathcal{F})/X} = (\operatorname{Lie} G/\operatorname{Lie} B) \times^B \mathcal{E}_y$ where $\mathcal{E}_y$ is the B-bundle corresponding to y. By deformation theory it is enough that the obstruction space

$$H^1(X, (\operatorname{Lie} G/\operatorname{Lie} B) \times^B \mathcal{E}_y) = 0 \text{ for all geometric points } y \in M^+_F.$$

The reason is that

$$\operatorname{Lie} G / \operatorname{Lie} B = \bigoplus_{\alpha<0} \mathfrak{g}_\alpha.$$

In particular, it is enough to show that $H^1(X, \mathfrak{g}_\alpha) = 0$ for all $\alpha < 0$. But we assumed that $\deg \mathfrak{g}_{\alpha_i} < 2 - 2g$ for each simple root α_i, so each simple negative weight space has degree at least $2g - 2$. By Riemann-Roch, $H^1(X, \mathfrak{g}_\alpha) = 0$.

Remark 11.2.5. There was some confusion about why we need the assumption $d_i(Y) < 1$. The answer is that otherwise if $g = 0$ then we could have $d_i(Y) = 1$. For $\alpha = \alpha_i + \alpha_j$ we would then have $\deg \alpha = 2$, so $H^1(X, \mathfrak{g}_{-\alpha})$ would have non-vanishing cohomology.

11.2.4 Proof of Proposition 11.2.4

We can check Proposition 2 at the level of geometric points. It follows from a more precise result:

Proposition 11.2.6. *Let $\mathcal{F} \to X$ be a G-bundle. Then for all N there exists a B-structure $\mathcal{E}$ of $\mathcal{F}$ such that* $\deg_i \mathcal{E} < -N$ *for all i.*

Example 11.2.7. Let $G = \mathrm{SL}_2$ and $\mathcal{F} \to X$ a rank 2 bundle. The proposition is saying that there is a line sub-bundle of degree as small as desired; this is an easy consequence of Riemann-Roch.

Proof. We proceed with several reductions.

Step 1: we may assume that $\mathcal{F}$ is the trivial bundle $\mathcal{F}_0$. The reason is that we know that $\mathcal{F}|_\eta$ is trivial (by Steinberg's or Springer's theorems), so there is an isomorphism

$$\mathcal{F}|_{X-D} \cong \mathcal{F}^0|_{X-D}$$

for some divisor $D \subset X$. But then every B-structure of $\mathcal{F}$ gives one for $\mathcal{F}^0$, by the valuative criterion. If the isomorphism $v|_{X-D} \cong \mathcal{F}^0|_{X-D}$ has "relative position h" then there exists $c(h)$ such that for every B-structure $\mathcal{E}$ of $\mathcal{F}$ the corresponding B-structure $\mathcal{E}^0$ of $\mathcal{F}^0$ satisfies

$$-c(h) < \deg_i \mathcal{E} - \deg_i(\mathcal{E}^0) < c(h).$$

Step 2: we may assume that $X = \mathbb{P}^1$ and G is simply connected. Indeed, take any map $X \to \mathbb{P}^1$. The pullback of a B-structure for the trivial bundle on $\mathbb{P}^1$ will be a B-structure for the trivial bundle on X.

Step 3: Let $\mathrm{Bun}_B^{<-N}$ be the space of B-bundles $\mathcal{E}$ with $\deg_i \mathcal{E} < -N$ for all i. Then we claim that $\mathrm{Bun}_B^{-N} \neq \emptyset$ for all N. Indeed, a T-bundle induces a B-bundle, and a T-bundle is just a direct sum of line bundles, which we can arrange to have any degree we want.

Step 4. We claim that the map $\mathrm{Bun}_B^{-N} \to \mathrm{Bun}_G$ is smooth by Proposition 11.2.3 if $N \gg 0$. Indeed the fibers of this map over a G-bundle are the B-structures on it, and Proposition 11.2.3 shows that this is smooth for $N > 2g - 2$.

Step 5. Since $X = \mathbb{P}^1$, $\mathrm{Bun}_G^{\mathrm{triv}} \subset \mathrm{Bun}_G$ is an open substack. To see this it suffices to calculate that the map $B(G = \mathrm{Aut}(\mathrm{triv})) \to \mathrm{Bun}_G$ is étale, which can be done using tangent spaces since both are smooth. The dimension of BG is $-\dim G$. To calculate the dimension of Bun_G, we use that its tangent complex is $\mathrm{Lie}(G)[1]$, so the dimension of the tangent space is

$$h^0(X, \mathrm{Lie}(G)) - h^1(X, \mathrm{Lie}(G)).$$

This is a bundle of rank $\dim G$ and degree 0 (since it's self-dual by the Killing form). Then Riemann-Roch shows that

$$\chi(X, \mathrm{Lie}(G)) = 0 + (\dim G)(1 - g).$$

The dimension of Bun_G is $(g-1)\dim G$ in general.

Step 6. Finally, it is a general fact that if G is simply-connected then $\mathrm{Bun}_G(X)$ is irreducible. The trivial bundle is open in Bun_G, and the map $\mathrm{Bun}_B^{-N} \to G$ has open image because it is smooth, so its image intersects the trivial bundle. □

11.3 Proof of Theorem 11.1.4

Step 1. We may assume that F comes from a T-torsor $\mathcal{E}_T$.

Proof. By Proposition 11.2.3, we may assume that F has a B-structure $\mathcal{E} \to X \times S$. We have

$$B \twoheadrightarrow T \hookrightarrow B.$$

This gives a map

$$\mathrm{Bun}_B \to \mathrm{Bun}_T \to \mathrm{Bun}_B\,.$$

In particular from $\mathcal{E} \in \mathrm{Bun}_B$ we get $\mathcal{E}' \in \mathrm{Bun}_B$.

We may assume that S is affine since we are proving a local assertion. We want to show that

$$G \times^B \mathcal{E}|_{X^0 \times S} \cong G \times^B \mathcal{E}'|_{X^0 \times S}\,.$$

You'll see the idea if we just do the proof for GL_2. In that case a GL_2-bundle is a rank 2 bundle $\mathcal{F}/X$. A B-structure $\mathcal{E}$ corresponds to a line sub-bundle $\mathcal{F}_0 \hookrightarrow \mathcal{F}$. In terms of the notation above, the G-bundle obtained from $\mathcal{E}'$ is $\mathcal{F}_0 \oplus \mathcal{F}/\mathcal{F}_0$. The claim then boils down to the assertion that

$$\mathcal{F} \cong \mathcal{F}_0 \oplus \mathcal{F}/\mathcal{F}_0.$$

The result then follows from $X^0 \times S$ is affine, so all the extension groups $\mathrm{Ext}^1(\dots)$ vanish. □

Step 2: Reduce to G being simply-connected.

Step 3. Reduce to the GL_2 case. The point is that if G is simply-connected then all T-bundles are controlled by coroots. One can then reduce to showing that two T-bundles differing by a single coroot are isomorphic locally on S, which moves us into the rank 2 case.

Step 4. Doing the case of GL_2. This isn't semisimple, so one has to find an appropriate formulation. The statement becomes:

Let $\mathcal{F}, \mathcal{F}' \to S$ *be two rank* 2 *bundles such that* $\det \mathcal{F} \cong \det \mathcal{F}'$. *Then we have*

$$\mathcal{F}|_{X^0 \times S} \cong \mathcal{F}^0|_{X^0 \times S}$$

after Zariski localization on S.

The proof is that after localizing on S we have a filtration

$$0 \to \mathcal{O} \to \mathcal{F}|_{X^0 \times S} \to \det \mathcal{F}|_{X^0 \times S} \to 0$$

since any bundle has "enough" sections after localizing on S and puncturing X. Then the result follows form the fact that extension groups will vanish after localizing (e.g. so that S is affine).

Part IV

Day Four

Chapter 12

The Classification of G-bundles

We now want to generalize the preceding classification of vector bundles to G-bundles (recovering the old results when $G = \mathrm{GL}_n$).

12.1 Background

12.1.1 Notation

We fix the notation for this talk: let

- E be a local field (of characteristic 0 or p),
- ϖ_E the uniformizer,
- $\mathbb{F}_q$ the residue field,
- $\breve{E}$ the completion of the maximal unramified extension, and
- F an algebraically closed perfectoid field of characteristic p.

12.1.2 Classical G-bundles

Definition 12.1.1. Let G be a connected linear algebraic group over E. A *connected G-bundle* on X can be defined in either of the following two ways:

1. ("INTERNAL") A principal homogeneous space $\mathcal{T}$ under G on X which is locally trivial for the (étale or fppf) topology.
2. ("EXTERNAL") An exact faithful E-linear $\otimes$−functor $\mathrm{Rep}_E\, G \to \mathrm{Vect}_X$.

Example 12.1.2. Why are the two definitions equivalent? We sketch one direction. Given a G-torsor $\mathcal{T}$, we can define the functor

$$V_{\mathcal{T}}((V,\rho)) = \mathcal{T} \times^{G,\rho} V.$$

Definition 12.1.3. We denote by $|\operatorname{Bun}_G|$ the set of isomorphism classes of connected G-bundles on X.

Example 12.1.4. If $G = \mathrm{GL}_n$ then $|\operatorname{Bun}_G| = \operatorname{Vect}_{X,n}$.

12.1.3 The classification of Vect_X

We have a functor

$$\mathcal{E}\colon \varphi - \operatorname{Mod}_{\breve{E}} \to \operatorname{Vect}_X$$

sending

$$(V, \varphi) \mapsto \bigoplus_{d \geq 0} \left(B^+_{E,F} \otimes_{\breve{E}} V\right)^{\varphi = \varpi_E^d}.$$

Theorem 12.1.5. *This $\mathcal{E}$ is a faithful exact E-linear $\otimes$-functor, which is essentially surjective (but not fully faithful, see Warning 12.1.6).*

It also induces an equivalence of categories

$$\textit{(isoclinic } \varphi\textit{-isocrystals)} \leftrightarrow \textit{(semi-stable vector bundles)}$$

and a bijection of objects

$$|\varphi - \operatorname{Mod}_{\breve{E}}| = |\operatorname{Vect}_X|.$$

Warning 12.1.6. The functor is *not* fully faithful because $\operatorname{End}(\operatorname{Triv} \oplus \operatorname{Triv}(1))$ is $E \oplus E$ in the category of isocrystals but a "Banach-Colmez-like object" $\begin{pmatrix} E & BC \\ & E \end{pmatrix}$ in the category of vector bundles.

This theorem is what we want to generalize, from vector bundles to G-bundles.

12.2 G-isocrystals (following Kottwitz)

12.2.1 The definition

Definition 12.2.1. Let G be a connected linear algebraic group over E. A *G-isocrystal* can be defined in either of the following two ways:

1. (EXTERNAL) An exact faithful E-linear $\otimes$-functor

$$N\colon \operatorname{Rep}_E G \to \varphi - \operatorname{Mod}_{\breve{E}}.$$

2. (INTERNAL) An element $b \in G(\breve{E})$. These form a category via

$$\operatorname{Hom}(b, b') = \{g \in G(\breve{E}) \mid gb\sigma(g)^{-1} = b'\}.$$

We denote by $B(G)$ the set of G-isocrystals up to isomorphism.

Example 12.2.2. Why are the internal and external versions equivalent? Given $b \in G(\breve{E})$, we can associate the functor N_b defined by

$$N_b(V, \rho) = (V \otimes_E \breve{E}, \rho(b) \circ (\mathrm{Id} \otimes \sigma))$$

Example 12.2.3. For $G = \mathrm{GL}_n$, the classical isocrystal description of an element $b \in G(\breve{E})$ is $(\breve{E}^{\oplus n}, b \circ \sigma)$.

12.2.2 The Newton and Kottwitz invariants

Let G be reductive. We construct two invariants associated to G-bundles.

The Newton Invariant. Let $b \in G(\breve{E})$. Then we can associate a homomorphism

$$\nu_b \colon \mathbb{D}_{\breve{E}} \to G_{\breve{E}}$$

where $\mathbb{D}$ is the split torus over E with $X^*(\mathbb{D}) = \mathbb{Q}$. This homomorphism ν_b is characterized by the property that for all (V, ρ), the morphism

$$\rho \circ \nu_b \colon \mathbb{D}_{\breve{E}} \to \mathrm{GL}(V_{\breve{E}})$$

has induced $\mathbb{Q}$-grading on $V_{\breve{E}}$ equal to the slope filtration of $(V_{\breve{E}}, b\sigma)$.

The cocharacter group $X_*(G)$ has an action of G, and we set

$$X_*(G)_{\mathbb{Q}}/G = \mathrm{Hom}_{\breve{E}}(\mathbb{D}_{\breve{E}}, G_{\breve{E}})/G(\breve{E}).$$

There is an action of σ on $X_*(G)_{\mathbb{Q}}$, and one can show that $\nu_b \in (X_*(G)_{\mathbb{Q}}/G)^\sigma$ only depends on $[b]$, thus inducing a well-defined map

$$\nu \colon B(G) \to (X_*(G)_{\mathbb{Q}}/G)^\sigma. \tag{12.1}$$

This is the *Newton invariant*.

Example 12.2.4. If G is quasi-split, say with Borel B, maximal torus $T \subset B$, and maximal split torus $A \subset T \subset B$ then the right side of (12.1) can be identified with $X_*(A)^+_{\mathbb{Q}}$.

Remark 12.2.5. There is also an internal definition of the Newton invariant. Given b, there exists b' with $b \sim b'$ such that $s \gg 0$ such that

$$(b'\sigma)^s = s \cdot \nu_{b'}(\varpi_E) \cdot \sigma^s$$

with the equality taking place in $G(\breve{E}) \rtimes \langle\sigma\rangle$. This characterizes $\nu_{[b]} = \nu_{[b']}$ (since ν is supposed to be defined on isomorphism classes).

The Kottwitz invariant. Consider

$$\pi_1(G) = X_*(T)/X_*(T_{sc}).$$

This is canonically and functorially associated to G, and admits an action of Γ. The Kottwitz invariant is described in terms of this fundamental group, as a map

$$\kappa\colon B(G) \to \pi_1(G)_\Gamma.$$

This is not so easy to define, but we will try to give some feeling for it. Roughly $B(G)$ is similar to $\pi_0(LG)$ (but not quite on the nose) and $\pi_0(LG) = \pi_1(G)_\Gamma$.

Theorem 12.2.6. *The map $B(G) \to (X_*(G)_{\mathbb{Q}}/G)^\sigma \times \pi_1(G)_\Gamma$ is injective.*

The description of the image is not easy in general, but in the quasi-split case it is fairly easy to describe it.

Example 12.2.7. Let $G = \mathrm{GL}_n$. Then $X_*(A)^+_{\mathbb{Q}} = (\mathbb{Q}^n)_+$ and $\pi_1(G)_\Gamma = \mathbb{Z}$. In this case the first component of the map gives the slopes of the Newton polygon, and the second component gives the endpoint of the Newton polygon. So in this case the 1st component determines the second, since the endpoint can be determined from the slopes via the formula

$$(\lambda_i) \in (\mathbb{Q}^n)_+ \mapsto \sum \lambda_i.$$

Therefore, the image can be characterized as the tuples whose break points are integers.

Example 12.2.8. Let $G = T$. Then $X(A)^+_{\mathbb{Q}} = X_*(T)_\Gamma \otimes \mathbb{Q}$. (There are no positivity conditions because there are no roots.) The second component is $\pi_1(T)_\Gamma = X_*(T)_\Gamma$. In this case the second component determines the first, via

$$\gamma \in X_*(T)_\Gamma \to X_*(T)_\Gamma \otimes \mathbb{Q}.$$

(And the first component determines the second up to torsion.)

12.2.3 More structure to $B(G)$

First, there is an analogue of the semistable/isoclinic set.

Definition 12.2.9. Let $B(G)_{\text{basic}} = \{[b] \mid \nu_b = \text{ central homomorphism}\}$.

Example 12.2.10. For GL_n, this means isoclinic.

Inside $B(G)_{\text{basic}}$ there is the subset $B(G)^0_{\text{basic}} = \{[b] \mid \nu_b = \text{ trivial}\}$. This is the analogue of the unit root isocrystals.

These form a section to the Kottwitz invariant. In other words, κ induces *bijections*

$$B(G)_{\text{basic}} \to \pi_1(G)_\Gamma$$

and

$$B(G)^0_{\text{basic}} \to \pi_1(G)_{\Gamma,\text{tors}} \cong H^1(E, G).$$

In this sense $B(G)$ is a generalization of Galois cohomology.

12.2.4 The automorphism group

Another piece of structure is the automorphism group. For $b \in G(\breve{E})$, we can associate a a group

$$J_b(R) := \{g \in G(\breve{E} \otimes R) \mid gb\sigma(g)^{-1} = b\}.$$

Then $J_b(E) = \mathrm{Aut}(b)$. This turns out to always be a reductive group over E.

Remark 12.2.11. The $J_b(E)$ are Levis if G is quasiplit.

Some facts.

- An element $b \in G(\breve{E})$ is basic if and only if J_b is an inner form of G.
- If $Z(G)$ is connected then *every* inner form comes from some basic b.
- If G is quasisplit, then $B(G)$ can be described in terms $B(M)_{\mathrm{basic}}$ for standard Levi subgroups $M \subset G$.
- $B(G)$ is a partially ordered set and its basic elements are the minimal ones.

12.3 G-bundles on the Fargues-Fontaine Curve

12.3.1 Semistable G-bundles

We want to define a functor

$$\mathcal{E}_G \colon G - \text{isocrystals} \to \mathrm{Bun}_G\,.$$

There are again two definitions.

1. (EXTERNAL) Given a G-isocrystal $\mathrm{Rep}_E \to \varphi - \mathrm{Mod}_{\breve{E}}$ in the external sense, composing with $\mathcal{E}$ gives
$$\mathrm{Rep}_E \to \varphi - \mathrm{Mod}_{\breve{E}} \xrightarrow{\mathcal{E}} \mathrm{Vect}_X\,.$$
This is a G-bundle in the external sense.
2. (INTERNAL) Given $b \in G(\breve{E})$, form $G_{\breve{E}} \times_{\breve{E}} Y_E/\varphi^{\mathbb{Z}}$ with φ acting diagonally by φ on Y_E and by $g \mapsto b\sigma(g)$ on $G_{\breve{E}}$.

Theorem 12.3.1. *Assume that* ch $E = 0$*. Then this functor* $\mathcal{E}_G$ *is faithful and induces a bijection*

$$B(G) \to |\,\mathrm{Bun}_G\,|.$$

Furthermore, $\mathcal{E}_G$ *induces an equivalence of categories between* $B(G)_{\mathrm{basic}}$ *and the category of semi-stable G-bundles.*

Definition 12.3.2. A G-bundle $\mathcal{T}$ is *semi-stable* if

1. (HALF-EXTERNAL) $\mathcal{T}(\mathrm{Lie}\, G, \mathrm{Ad})$ is a semi-stable vector bundle.
2. (EXTERNAL) $\mathcal{T}(V, \rho)$ is a semi-stable vector bundle if ρ is homogeneous. (Remark: we are using here that tensor of semistable is semistable, which follows from the classification of vector bundles.)
3. (INTERNAL) Let $P \subset G$ be a power-bounded subgroup. Let A_P be the split part of the center of P. We have dually A'_P the split part of the cocenter of P. Then the map

$$A_P \to A'_P$$

is an isogeny, identifying the rational cocharacter groups. Let $\mathcal{T}$ be a G-bundle and suppose $\mathcal{T}_P$ is a P-structure on $\mathcal{T}$. Then we define the *slope* cocharacter $\mu(\mathcal{T}_P) \in X_*(A_P)_{\mathbb{Q}}$ which is characterized by the property

$$\langle \mu(\mathcal{T}_P), \lambda \rangle = \deg \lambda_*(\mathcal{T}_P) \quad \text{for all } \lambda \in X^*(A'_P).$$

Finally, we define $\mathcal{T}$ to be semi-stable if and only if

$$\langle \mu(\mathcal{T}_P), \alpha \rangle \leq 0 \forall \alpha \in \mathrm{Lie}\, N_P.$$

12.3.2 The two invariants

How are the two invariants expressed in terms of the corresponding G-bundles?

Newton invariant. First assume that G is quasi-split, with $A \subset T \subset B$ as before. Let $\mathcal{T}$ be a G-bundle. The *Harder-Narasimhan reduction theorem* says that there exists a unique pair $(P, \mathcal{T}_P)$ with P a standard parabolic subgroup and $\mathcal{T}_P$ a P-bundle such that

1. $\mathcal{T}_P \times^P M_P$ is a semistable M_P-bundle, and
2. $\mu(\mathcal{T}_P) \in X_*(A_P)^{++}_{\mathbb{Q}}$.

Now the maximal split subtorus $A_P \subset A$ gives a map from $X_*(A_P)^{++}_{\mathbb{Q}} \to X_*(A)^{+}_{\mathbb{Q}}$, sending $\mu(\mathcal{T}_P) \mapsto \nu_{\mathcal{T}} \in X_*(A)^{+}_{\mathbb{Q}}$.

Proposition 12.3.3. *We have* $[\nu_b] = -\nu_{\mathcal{T}(b)}$.

Why the minus sign? It came up already in Dospinescu's talk: a minus sign was taken to get compatibility of endomorphisms.

Kottwitz invariant. We know that

$$|\mathrm{Bun}_G| = H^1_{\text{ét}}(X, G).$$

Fargues defines a G-equivariant Chern class

$$c_1^G \colon H^1_{\text{ét}}(X, G) \to \pi_1(G)_\Gamma.$$

Proposition 12.3.4. *We have*

$$\kappa(b) = c_1^G(\mathcal{E}_G(b)).$$

12.3.3 What's wrong in characteristic p?

In the book of Fargues and Fontaine, they construct various categories of φ-isocrystals which give vector bundles. One of these functors is not exact. When you want to apply the external definition of G-bundles you need an exact functor; this uses the fact that in characteristic 0 the representation theory is semisimple.

Chapter 13

Proof of Geometric Langlands for GL(2), I

13.1 Some recollections

13.1.1 Notation

Let

- $X/\mathbb{F}_q =: k$ be a smooth projective geometrically connected curve.
- $F = k(X)$,
- for all $x \in |X|$ we denote $\mathcal{O}_x, F_x$ to be the completed local ring and its fraction field, respectively.
- $G = \mathrm{GL}_n$, $G_x := G(F_x)$,
- $K_x := G(\mathcal{O}_X)$,
- $\mathcal{H}_x$ the spherical Hecke algebra at x,
- $\mathcal{O}_F = \prod \mathcal{O}_x$.

13.1.2 Goal

Given an everywhere unramified $\sigma\colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, i.e. a local system E on X, we want to

1. Construct an unramified automorphic form

$$f_\sigma\colon G(F)\backslash G(\mathbb{A})/G(\mathcal{O}) \to \overline{\mathbb{Q}}_\ell$$

such that for all $x \in |X|$, the action $\mathcal{H}_x$ on $\overline{\mathbb{Q}}_\ell \cdot f_\sigma$, the eigencharacter for the action of is

$$\mathcal{H}_x \overset{\text{Sat}}{\cong} R(G^\vee) \xrightarrow{\chi_{\gamma_x}} \overline{\mathbb{Q}}_\ell$$

given by

$$V \mapsto \operatorname{Tr}(\gamma_x \mid_V)$$

where $\gamma_x = [\sigma(\text{Frob}_x)^{ss}]$.

2. Upgrade f_σ to a (perverse Hecke eigen-)sheaf Aut_E on Bun_n, recalling that

$$|\operatorname{Bun}_n(\mathbb{F}_q)| = G(F)\backslash G(\mathbb{A})/G(O_F).$$

In this talk we'll get as far as we can along the construction of a sheaf (not quite the Aut_E) on a Bun'_n which maps to Bun_n. Roughly speaking, Bun'_n is the moduli space of pairs $(L, \Omega^{\otimes(n-1)} \hookrightarrow L)$ where L is a GL_n-bundle, so the map to Bun_n is the forgetful map. In Heinloth's talk, this sheaf will be shown to descend.

13.2 Classical motivation

This section will be about how, given a Galois representation σ, we could make a guess of Aut_E. By analogy, suppose you had an elliptic curve over $\mathbb{Q}$ and you wanted to show that it was modular. A naïve strategy might be to write down the Fourier expansion of the modular form from the local data. Then you have to check some invariance properties. This is hard to carry out in that setting, but it's basically what we'll try to do here.

13.2.1 Fourier expansion of cusp forms on GL_n

For $n = 2$, let

$$\varphi\colon \text{GL}_2(F)\backslash \text{GL}_2(\mathbb{A}) \to \overline{\mathbb{Q}}_\ell$$

be a cusp form. Fix $g \in \text{GL}_2(\mathbb{A})$. Then the function

$$x \mapsto \varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right)$$

is periodic, i.e. descends to a function on $F\backslash\mathbb{A}_F$. This gives a Fourier expansion in the characters $\widehat{F\backslash\mathbb{A}}$:

$$\varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = \sum_{\widehat{F\backslash\mathbb{A}}} \cdots$$

Fixing a nontrivial character Ψ of F, we can identify $\widehat{F\backslash\mathbb{A}} \cong F$ by

$$\gamma \in F \mapsto (x \mapsto \Psi(\gamma x)).$$

Then the Fourier expansion of Ψ is

$$\varphi\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}g\right)=\sum_{\gamma\in F}\left(\int_{F\backslash\mathbb{A}}\varphi\left(\begin{pmatrix}1 & y\\ & 1\end{pmatrix}g\right)\Psi_\gamma(y)^{-1}dy\right)\cdot\Psi_\gamma(x).$$

For $\gamma=0$ the integral vanishes by cuspidality. For $\gamma\neq 0$, a change of variables g

$$\int_{F\backslash\mathbb{A}}\varphi\left(\begin{pmatrix}1 & y\\ & 1\end{pmatrix}\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}g\right)\Psi(y)^{-1}\,dy.$$

The conclusion (taking $x=0$) is that

$$\varphi(g)=\sum_{\gamma\in F^*}W_{\varphi,\psi}\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}g\right)$$

where

$$\begin{aligned}W_{\varphi,\psi}(g)&=\int_{F\backslash\mathbb{A}}\varphi\left(\begin{pmatrix}1 & y\\ & 1\end{pmatrix}g\right)\Psi(y)^{-1}\,dy\\ &=\int_{N(F)\backslash N(\mathbb{A})}\varphi(ng)\Psi(n)^{-1}\,dn.\end{aligned}$$

More generally, for GL_n we get the Fourier expansion

$$\varphi(g)=\sum_{N_{n-1}(F)\backslash\mathrm{GL}_{n-1}(F)}W_{\varphi,\psi}\left(\begin{pmatrix}\boxed{\gamma} & \\ & 1\end{pmatrix}g\right). \tag{13.1}$$

The Whittaker property is

$$W_{\varphi,\psi}(ng)=\Psi(n)W_{\varphi,\psi}(g)$$

for all $n\in N(\mathbb{A})$. Here $\Psi(n)$ is defined by

$$n=\begin{pmatrix}1 & u_{12} & & & \\ & \ddots & \ddots & & \\ & & 1 & u_{n-1,n} \\ & & & 1\end{pmatrix}\mapsto\Psi\left(\sum u_{i,i+1}\right).$$

More precisely, this expansion yields a $G(\mathbb{A})$-equivariant isomorphism

$$C^\infty(G(\mathbb{A}))^{(N(\mathbb{A}),\psi)}\overset{\mathbb{G}(A)}{\cong}C^\infty(P_1(F)\backslash G(\mathbb{A}))_{\mathrm{cusp}}$$

where P_1 is the mirabolic subgroup

$$P_1=\begin{pmatrix}\boxed{*} & \boxed{*}\\ & 1\end{pmatrix}\subset P=\begin{pmatrix}\boxed{*} & \boxed{*}\\ & *\end{pmatrix}$$

the isomorphism being given by $\varphi\mapsto W_{\varphi,\psi}$, and the inverse being the Whittaker expansion (13.1).

The strategy for producing f_σ is to use the local theory write down an element W_σ on the left side, and then take the Fourier expansion to get an element of the right side. The hard work is the descent on the right hand side.

13.2.2 Building W_σ

Let γ_x be the semisimpole conjugacy class of $\sigma(\mathrm{Frob}_x)$. It is a fact that for all G and all $x \in |X|$, there exists a unique $W_{\gamma_x} : G(F_x) \to \overline{\mathbb{Q}}_\ell$ satisfying the conditions:

1. (NORMALIZATION) $W_{\gamma_x}(1) = 1$,
2. (SPHERICAL WHITTAKER CONDITION) for all $n \in N(F_x), g \in G(F_x), k \in G(\mathcal{O}_x)$

$$W_{\gamma_x}(ngk) = \psi_x(n) W_{\gamma_x}(g)$$

3. (HECKE EIGENVALUES) for all $h \in \mathcal{H}_x$,

$$h \cdot W_{\gamma_x} = \chi_{\gamma_x}(h) W_{\gamma_x}.$$

This builds local Whittaker functions. To build the global ones, we take their product.

Definition 13.2.1. Define $W_\sigma : G(\mathbb{A}) \to \overline{\mathbb{Q}}_\ell$ by

$$W_\sigma((g_x)) = \prod W_{\gamma_x}(g_x).$$

We then define f'_σ to be the Fourier expansion of W_σ, as in (13.1). This is a priori only left invariant under the mirabolic, so

$$f'_\sigma \in C^\infty(P_1(F)\backslash G(\mathbb{A})/G(\mathcal{O}))$$

and f'_σ has the correct Hecke eigenvalues.

Remark 13.2.2. For general G the local Whittaker functions exist, but not global (what is a generalization of the mirabolic?).

Conjecture 13.2.3. *The function f'_σ is (left) $\mathrm{GL}_n(F)$-invariant.*

13.3 Geometrization

The aim of the rest of the talk is to geometrize f'_σ on a subset of its domain, corresponding to $|\mathrm{Bun}'_n(\mathbb{F}_q)| \subset P_1(F)\backslash G(\mathbb{A})/G(\mathcal{O})$. We won't elaborate on this yet, but we emphasize that this applies to a particular *subset* of the domain.

13.3.1 Setup

We now replace GL_n by a group *scheme* over X, denoted GL_n^J, whose functor points is

$$\mathrm{GL}_n^J(R) = \{\text{invertible } n \times n \text{ matrices } (A) \in \Gamma(\mathrm{Spec}\ R, \Omega^{j-i})\}.$$

Example 13.3.1. For GL_2,

$$\mathrm{GL}_2^J = \begin{pmatrix} \mathcal{O} & \Omega^1 \\ \Omega^{-1} & \mathcal{O} \end{pmatrix}.$$

Likewise, we define $N^J, P_1^J, B^J, \dots$

We can then construct a (more) canonical character

$$\Psi\colon N(F)\backslash N(\mathbb{A})/N(\mathcal{O}) \to \overline{\mathbb{Q}}_\ell^*$$

which depends only on the choice of character of the residue field $\psi\colon k \to \overline{\mathbb{Q}}_\ell^*$, setting

$$\Psi = \prod \Psi_x$$

where

$$\Psi_x\colon \begin{pmatrix} 1 & u_{12} & & \\ & \ddots & \ddots & \\ & & 1 & u_{n-1,n} \\ & & & 1 \end{pmatrix} \mapsto \prod_{i=1}^{n-1} \psi(\mathrm{Tr}_{k(x)/k}\,\mathrm{Res}_x(u_{i,i+1} \in \Omega^1)).$$

This is invariant by $N(\mathcal{O})$ and $N(F)$ by the residue theorem.

Now construct W_σ and f'_σ on $\mathrm{GL}_n^J(\mathbb{A})$ in the same way as before.

13.3.2 Difficulties with geometrization

Now we've arrived at the proper work, which has to do with trying to geometrize this. There are difficulties in doing this. Perhaps the two main ones are:

1. Local: If we were to try to geometrize the local Whittaker functions, then we would run into the problem that the orbits of $N(F_x)$ on the affine Grassmanin $G(F_x)/G(\mathcal{O}_X)$ on ∞-dimensional over k.

2. Global: $P_1(F)\backslash G(\mathbb{A})/G(\mathcal{O})$ are n-dimensional bundles L plus "generic embeddings" $\Omega^{n-1} \hookrightarrow L$. There can be poles with no control. Thus this is not an object of classical algebraic geometry.

A lot of the work in geometric Langlands in recent years is about developing a sensible theory of such things. However, in this case there is a hack to get around the obstacles: there are two things which come together to save us.

1. Local: we have the *Shintani (for* GL(n)*) and Casselman-Shalika (for general* G*)* formula, which tells us about the support of the spherical Whittaker function.

2. Global: $G(\mathbb{A})^+/G(\mathcal{O})$ is the set of $\mathbb{F}_q$-points of a scheme, where (using $G = \mathrm{GL}_n$) $G(F_x)^+ := \mathrm{GL}_n(F_x) \cap M_n(\mathcal{O}_X)$.

It turns out that modulo center, the interesting part of the Whittaker function is supported on the scheme underlying $G(\mathbb{A})^+/G(\mathcal{O})$.

13.3.3 Casselman-Shalika formula

We have the NAK decomposition. We know how the Whittaker function transforms under left translation by N and right translation by K, so we need to figure out what happens on A. The answer is that for all $x \in |X|$ and γ_x a semisimple conjugacy class (which gives rise to the Whittaker function W_{γ_x}):

1. $W_{\gamma_x}(\lambda(\varpi_x)) = 0$ for $\lambda \in X_*(T) - X_*(T)_+$ (*non*-dominant cocharacters).

2. For $\lambda \in X_*(T)_+$,
$$W_{\gamma_x}(\lambda(\varpi_x)) = q_x^{\mathrm{scalar}(\lambda)} \operatorname{Tr}(\gamma_x \mid_{V(\lambda)}).$$
the right hand side being "trace of γ on the highest weight representation with weight λ of $G^\vee$".

Remark 13.3.2. The first part is an easy exercise.

13.3.4 Consequence

We're going to draw a picture of where various things live.

Define
$$\widetilde{Q} := N(F)\backslash N(\mathbb{A}) \times_{N(O)} G(\mathbb{A})^+/G(O).$$
The Whittaker sheaf W_σ is supported mod center in $G(\mathbb{A})^+/G(O)$. The map $(u, g) \mapsto ug$ presents $\widetilde{Q}$ over a space
$$B(F_x)^+ = N(F_x)(T(F_x)^+ := T(F_x) \cap \mathrm{Mat}_n(O_x)).$$
This is where the Whittaker function lives. It in admits a map down to
$$|\operatorname{Bun}'_n(\mathbb{F}_q)| \subset P_1(F)\backslash G(\mathbb{A})/G(O)$$
where our fake automorphic form lives, which then maps down to where the actual automorphic form lives
$$\begin{array}{ccc}
\widetilde{Q} := N(F)\backslash N(\mathbb{A}) \times_{N(O)} G(\mathbb{A})^+/G(O) & & \\
\downarrow & & \\
Q := N(F)\backslash B(\mathbb{A})^+/B(O) & \hookrightarrow & N(F)\backslash G(\mathbb{A})/G(O) \\
\downarrow & & \\
|\operatorname{Bun}_n(\mathbb{F}_q)| := \mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A})/\mathrm{GL}_n(O). & &
\end{array}$$

We start with W_σ on the top and descend it to f'_σ on the second layer. To descend further down the ladder, we want to geometrize, but we can only do so on the subsets $\widetilde{Q}$ and Q.

Theorem 13.3.3. *There is a sheaf $\mathcal{F}_E$ on an algebraic stack $\widetilde{Q} \xrightarrow{\widetilde{\nu}} \mathrm{Bun}'_n$ such that on $\mathbb{F}_q$-points*

$$|\widetilde{Q}(\mathbb{F}_q)| = \widetilde{Q}$$

and admitting a map

$$\widetilde{\nu}\colon \widetilde{Q} \to |\,\mathrm{Bun}'_n(\mathbb{F}_q)|$$

such that

$$\mathrm{Tr}(\widetilde{\nu}_! \mathcal{F}_E) = f'_\sigma|_{\mathrm{Bun}'_n(\mathbb{F}_q)}.$$

In the last few minutes we'll try to tell you as much as possible about the construction of $\mathcal{F}_E$.

13.4 Laumon's construction

The most significant part of $\mathcal{F}_E$ is that which geometrizes the Casselman-Shalika formula. This is a remarkable construction due to Laumon.

Laumon defined $\mathcal{L}_E$ on a stack $\mathrm{Coh}^n \leftarrow \widetilde{Q}$. Here Coh is the algebraic stack/$\mathbb{F}_q$ parametrizing torsion coherent sheaves of finite length on X. That is, $\mathrm{Hom}(S, \mathrm{Coh})$ is the groupoid whose objects are coherent sheaves $\mathcal{T}$ on $X \times S$ that are finite flat over S.

We can then define a substack Coh^n of Coh which is the open substack of those $\mathcal{T}$ such that at each closed point of S, $\mathcal{T}|_{\mathrm{pt}}$ is a sum of at most n indecomposable summands:

$$\mathcal{T} = \bigoplus_i \mathcal{O}_X/\mathcal{O}_X(-D_i).$$

This breaks up into a union of components by degree:

$$\mathrm{Coh}^n = \prod_{m>0} \mathrm{Coh}^{n,m}$$

where $\mathrm{Coh}^{n,m}$ is the "degree m part" of Coh^n. For a local system E on X, we get a local system $E^{\boxtimes m}$ on X^m, which then admits an obvious map π to $X^{(m)}$. We descend

$$E^{\boxtimes m} \rightsquigarrow (\pi_* E^{\boxtimes m})^{S_m} =: E^{(m)}.$$

Then $E^{(m)}|_{X^{(m),rss}}$ is a local system. We have a map

$$X^{(m),rss} \to \mathrm{Coh}^{n,m,rss}$$

and $E^{(m)}|_{X^{(m)},rss}$ descends to a local system $\mathcal{L}^{rss}_{E,m}$ on $\mathrm{Coh}^{n,m,rss}$. Finally, we define $\mathcal{L}_{E,m}$ as a middle extension sheaf for $j\colon\ \mathrm{Coh}^{n,m,rss} \hookrightarrow \mathrm{Coh}^n$:

$$\mathcal{L}_{E,m} := j_{!*}\mathcal{L}^{rss}_{E,m}.$$

Definition 13.4.1. *Laumon's sheaf* $\mathcal{L}_E$ is (up to shift) the perverse sheaf on Coh^n whose restriction to each $\mathrm{Coh}^{n,m}$ is $\mathcal{L}_{E,m}$.

The relation to Casselman-Shalika is described in (the second part of) the following theorem:

Theorem 13.4.2 (Laumon)**.** *The function* $\mathrm{Tr}(\mathcal{L}_E)\colon \mathrm{Coh}^n(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell$ *is given by*

1. $\mathrm{Tr}(\mathcal{L}_{E,m})(\mathcal{T}) = \prod_{x\in|X|} \mathrm{Tr}\,\mathcal{L}_{E,m_x,x}(\mathcal{T}_x)$ *where* $\mathrm{Coh}^{n,m}(x) \to \mathrm{Coh}^{n,m}$ *is defined in the same way but for torsion sheaves supported at x,* $\mathcal{L}_{E,m_x,x}$ *is the pullback of* $\mathcal{L}_{E,m}$*, and* $\mathcal{T}_x$ *is the restriction of* $\mathcal{T}$ *to a neighborhood of x where it is only supported at x.*

2. *We have*
$$\mathcal{L}_{E,m,x} \cong \bigoplus_{\lambda\in X_*(T)_{++,m}} IC(\mathrm{Coh}^{n,m,\lambda}(x)) \otimes E_x(\lambda)$$
(where indexing is means $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n \geq 0$ *and* $m = \sum \lambda_i$*). Here* $Coh^{n,m,\lambda}$ *are st rata of* $\mathrm{Coh}^{n,m}(x)$ *and* $E_x(\lambda)$ *is obtained by composing* E_x *with the representation of heighest weight* λ*. Then* Frob_x *acts by* $\mathrm{Tr}(\sigma(\mathrm{Frob}_x)|_{V(\lambda)})$ *(i.e. the Casselman-Shalika formula).*

Chapter 14

Discussion Session: p-Divisible Groups

These are notes from a discussion session of p-divisible groups. Some questions were posed by Dennis Gaitsgory, and then their answers were discussed by Jared Weinstein.

14.1 Questions

14.1.1 Question 1

We have a p-divisible group over $\mathcal{O}_{\mathbb{C}_p}$. There was a "universal cover" $\widetilde{\mathcal{G}}$. What is this? Also, please explain the short exact sequence

$$0 \to T_p(\widetilde{\mathcal{G}}^{\mathrm{an}})[1/p] \to \widetilde{\mathcal{G}}^{\mathrm{an}} \to \mathrm{Lie}(G) \otimes \mathbb{G}_a \to 0.$$

14.1.2 Question 2

How do you associate isocrystals to p-divisible groups? What is the period map?

14.1.3 Question 3

How do you describe modifications of bundles on X in terms of p-divisible groups?

14.1.4 Question 4

What is the analogue of this stuff in equal characteristic?

14.2 Discussion of Question 1

Let me first write down the sequence without taking universal covers. The exact sequence is basically coming from the logarithm:

$$0 \to G[p^\infty](\mathcal{O}_C) \to G(\mathcal{O}_C) \xrightarrow{\log} \mathrm{Lie}(G) \otimes C \to 0. \tag{14.1}$$

14.2.1 Basics on p-divisible groups

There are often implicit identifications made in talking about p-divisible groups. If G a p-divisible group, then it is "represented" by a formal scheme which usually is also denoted G. What do we mean by this? (What is the associated formal scheme?) By definition a p-divisible group G is an inductive system

$$G = \varinjlim G_n$$

where G_n are group schemes. We could view G as a sheaf on the category Nilp_p of rings in which p is nilpotent. Then it turns out to be representable by a formal scheme.

Example 14.2.1. For the p-divisible group

$$\mu_{p^\infty} = \varprojlim \mu_{p^n}$$

the formal scheme is $\widehat{\mathbb{G}}_m$. In general it can be difficult to describe.

So if G is a p-divisible group, we denote by $G(\mathcal{O}_C)$ the points of the formal scheme. This is a $\mathbb{Z}_p$-module. We have a logarithm map

$$G(\mathcal{O}_C) \xrightarrow{\log} \mathrm{Lie}\, G.$$

Before discussing what this is technically, we give some examples.

Example 14.2.2. For $G = \mathbb{Q}_p/\mathbb{Z}_p$ the constant group scheme, this map is

$$\mathbb{Q}_p/\mathbb{Z}_p \to 0$$

so in this case the sequence (14.1) is

$$0 \to \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0 \to 0.$$

Example 14.2.3. For $G = \mu_{p^\infty}$, we have $G(\mathcal{O}_C) = 1 + \mathfrak{m}_C$ (considered as a multiplicative group). Why? You might think at first that $G(\mathcal{O}_C)$ should be the p-power roots of unity, but we cannot evaluate it directly on $\mathcal{O}_C$ because $\mathcal{O}_C$ is not an object of Nilp_p.

Instead, we have to view $\mathcal{O}_C$ as an inverse limit of $\mathcal{O}_C/p^n$. Then

$$\mu_{p^\infty}(\mathcal{O}_C) = \varprojlim_n \mu_{p^\infty}(\mathcal{O}_C/p^n)$$

Now on $\mathcal{O}_C/p^n$, which really is an object of Nilp_p, we can apply the definition of μ_{p^∞} literally. But in this ring there will be many p-power roots of unity - anything close to 1 works. So

$$\mu_{p^\infty}(\mathcal{O}_C) = \varprojlim_n \mu_{p^\infty}(\mathcal{O}_C/p^n) = \varprojlim_n 1 + \mathfrak{m}_C/p^n = 1 + \mathfrak{m}_C.$$

Alright, let's finally start building the short exact sequence. Again, the right map is the p-adic logarithm:

$$1 + \mathfrak{m}_C \xrightarrow{\log} C$$

The kernel of the logarithm is the torsion subgroup, so in this case (14.1) is

$$0 \to (1 + \mathfrak{m}_{C_p})[p^\infty] \to 1 + \mathfrak{m}_C \xrightarrow{\log_p} C \to 0.$$

The two examples just discussed, $\mathbb{Q}_p/\mathbb{Z}_p$ and μ_{p^∞}, can be thought of as the "building blocks" of p-divisible groups. Everything else "looks like" a mix between them. For instance, if G is a general p-divisible group then $G(\mathcal{O}_C)$ will look like a product of disks (as in $1 + \mathfrak{m}_{\mathcal{O}_{C_p}}$ for μ_{p^∞}) times a product of $\mathbb{Q}_p/\mathbb{Z}_p$ factors.

14.2.2 Analytification

Let's now return to the discussion of the short exact sequence

$$0 \to G[p^\infty](\mathcal{O}_C) \to G(\mathcal{O}_C) \to \mathrm{Lie}(G) \otimes C \to 0.$$

We now construct the analogous sequence at the level of analytic spaces. To a p-divisible group G there is an associated adic space G^{an} over C. The construction passes through the associated formal scheme over $\mathcal{O}_C$. We know that there is a fully faithful embedding from formal schemes over $\mathcal{O}_C$ to adic spaces over $\mathrm{Spa}(C, \mathcal{O}_C)$; then we form the generic fiber over $\mathrm{Spa}(C)$. (This was denoted G^{ad}_η in Arthur's talk.)

The claim is that there is a short exact sequence

$$0 \to G[p^\infty] \to G^{\mathrm{an}} \to \mathrm{Lie}(G) \otimes \mathbb{G}_a^{\mathrm{an}} \to 0.$$

What is $\mathbb{G}_a^{\mathrm{an}}$ and why did we tensor with it? We tensored with it because we want an exact sequence of objects in the category of adic spaces, so we have to turn the vector space $\mathrm{Lie}(G)$ into an "adic vector space". (We are viewing $G[p^\infty]$ as a discrete adic space. Strictly speaking, maybe we should underline it) Now, $\mathbb{G}_a$ is what you expect in terms of the functor of points:

$$\mathbb{G}_a^{\mathrm{an}}(R, R^+) = R.$$

However, it is slightly subtle to present this as an adic space. This is *not* represented by an affinoid adic space. It is something like the whole affine line, which is not quasi-compact as an analytic space: it should be presented as a rising union of infinitely many disks of increasing radius.

Now what is the logarithm actually? Let's go to the very basics. If G is a formal group of dimension 1, then by definition there is some power series

$$X +_G Y = X + Y + \dots.$$

What's the Lie algebra? You just choose a coordinate X, and the addition law is given by this power series. Multiplication by p should be finite, so we should have

$$[p]_G(X) = uX^{p^h} + \dots \quad u \in \mathcal{O}_C^*.$$

Now we need to give a map

$$G(\mathcal{O}_C) \to C.$$

As a set $G(\mathcal{O}_C)$ is $\mathfrak{m}_C$, but with group law given by that power series. So

$$\log_G(X) = \lim_{n\to\infty} \frac{[p^n]_G(X)}{p^n}.$$

Exercise 14.2.4. Do this explicitly for $\widehat{\mathbb{G}}_m$.

Why is this valued in Lie(G)? The Lie algebra is dual to differentials. So if ω is an invariant differential on G, there should be a natural way to evaluate

$$\langle \log(x), \omega \rangle = \text{``}\int_0^x \omega\text{''} \in C.$$

What does this mean precisely? We can write the Kähler differential as $\omega = df$, and there is a unique normalization of f so that $f(0) = 0$. We set

$$\int_0^x \omega := f(x)$$

for this f.

14.2.3 Passing to the universal cover

Now, what is the universal cover? We could describe as a formal scheme whose functor of points is

$$\widetilde{G}(R) = \varprojlim_p G(R).$$

This should be a $\mathbb{Q}_p$-vector space. Indeed, applying $\varprojlim_p$ to any $\mathbb{Z}_p$-module gives a $\mathbb{Q}_p$-vector space.

Example 14.2.5. If $G = \mathbb{Q}_p/Z_p$ then $\widetilde{G} = \mathbb{Q}_p$.

Example 14.2.6. If $G = \mu_{p^\infty}$ then

$$\widetilde{G}(R) = \varprojlim 1 + R^{00}$$

where R^{00} is the ideal of topologically nilpotent elements. Why? Again we need to express R as a limit of rings in Nilp_p in order to compute:

$$\mu_{p^\infty}(R) = \varprojlim_n \mu_{p^\infty}(R/p^n) = \varprojlim_n 1 + R^{00}/p^n$$

since at the finite levels anything topologically nilpotent is a p^nth root of unity.

Note that this limit looks like $R^\flat$ if R is perfectoid. In fact, recall that there are two parallel constructions of the tilt: one in characteristic 0, and one after modding out by p. Indeed we have here

$$\begin{aligned}\widetilde{G}(R) &= \mu_{p^\infty}(R)\\ &= \varprojlim_n 1 + R^{00}/p^n\\ &= \varprojlim_{x\mapsto x^p} 1 + R^{00}/p\\ &= \widetilde{G}(R/p)\end{aligned}$$

The preceding example reflects the general phenomenon that

$$\widetilde{G}(R) \to \widetilde{G}(R/p)$$

is always an isomorphism. We might say that $\widetilde{G}$ is a "crystalline" construction because it is insensitive to infinitesimal extensions.

Now what about the exact sequence? There is a map

$$\begin{array}{c}\widetilde{G}(\mathcal{O}_C)\\ \downarrow\\ G(\mathcal{O}_C)\end{array}$$

which is projection onto the 0th coordinate. Let's compare the logarithm maps for G and its analytification.

$$\begin{array}{ccccccccc} & & \widetilde{G}(\mathcal{O}_C) & \longrightarrow & \mathrm{Lie}(G)\otimes C & \longrightarrow & 0 & & \\ & & \downarrow & & \| & & & & \\ 0 & \longrightarrow & G[p^\infty](\mathcal{O}_C) & \longrightarrow & G(\mathcal{O}_C) & \longrightarrow & \mathrm{Lie}(G)\otimes C & \longrightarrow & 0\end{array}$$

The map $\widetilde{G}(\mathcal{O}_C) \to \mathrm{Lie}(G)\otimes C$ can therefore be thought of as the composition with projection and the logarithm map for $G(\mathcal{O}_C)$. In particular, the kernel consists of elements whose 0th part is killed by the classical logarithm, i.e. consists of elements whose 0th part is torsion.

$$\begin{array}{ccccccccc} & & T_p(G) & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & T_pG \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & \longrightarrow & \widetilde{G}(O_C) & \longrightarrow & \mathrm{Lie}(G) \otimes C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & G[p^\infty](O_C) & \longrightarrow & G(O_C) & \longrightarrow & \mathrm{Lie}(G) \otimes C & \longrightarrow & 0 \end{array}$$

Theorem 14.2.7. *$\widetilde{G}$ is a perfectoid space.*

Example 14.2.8. For $G = \mu_{p^\infty}$, what is $\widetilde{G}$ as a perfectoid space? It turns out to be the "perfectoid open ball of radius 1" (since these are precisely the topologically nilpotent elements for multiplication). This is easiest to describe at the level of points:

$$\widetilde{\mu}_{p^\infty}^{\mathrm{an}}(R) = \varprojlim_{x \mapsto x^p} 1 + R^{00}.$$

How do we describe this as a perfectoid space in terms of affinoid charts? Again, the description is a little complicated: it is certainly not affinoid since the open ball is not quasicompact. We can exhaust it from inside by closed balls.

Note that $\mathrm{Spa}(C\langle x^{1/p^\infty}\rangle)$ is a "perfectoid closed ball of radius 1". First of all, what does $C\langle x^{1/p^\infty}\rangle$ even mean? $C\langle X\rangle$ is the Tate algebra, with elements being convergent power series. Then $C\langle x^{1/p^\infty}\rangle$ is obtained by adjoining all p-power roots of X and completing. Now to exhaust the open ball from within, we need to taking a rising union of rescaled closed disks $|X| \leq |p^\epsilon|$ as $\epsilon \to 0$:

$$\varinjlim_{\epsilon \to 0} \mathrm{Spa}(C\langle \left(\frac{X}{p^\epsilon}\right)^{1/p^\infty} \rangle).$$

Perhaps a slicker way to describe this is as $(\mathrm{Spf}\, O_C[[X^{1/p^\infty}]])^{\mathrm{an}}$.

14.3 Discussion of Question 2

14.3.1 Dieudonné modules

Let k be a perfect field of characteristic p. There is an equivalence of categories

$$M\colon \{p\text{-div groups}/k\} \xrightarrow{\sim} \{\text{Dieudonné modules}/W(k)\}. \tag{14.2}$$

What are Dieudonné modules?

Definition 14.3.1. A *Dieudonné module* is a finite free $W(k)$-modules M, together with maps

$$F, V\colon M \to M$$

where F is σ-linear and V is σ^{-1}-linear and $FV = p$.

Now, what we actually discussed were not Dieudonné modules but *isocrystals*, which looked similar but were defined over fields. We can get that from a Dieudonné module by inverting p. But then what happens to the equivalence (14.2)?

On the left, we get p-divisible groups up to isogeny. On the right, we don't need to specify the V because it is determined by F once p is invertible, but there is still a condition because V had to preserve a lattice. So the right side becomes the category of isocrystals over k (which by definition modules over $W(k)[1/p]$) with slopes in $[0, 1]$.

$$\{p\text{-div groups}/k\}/\text{isogeny} \xrightarrow{\sim} \{\text{isocrystals}/k \text{ with slopes in } [0,1]\}.$$

Example 14.3.2. $M(\mathbb{Q}_p/\mathbb{Z}_p) = W(k)$ with $F = p\sigma$.

Example 14.3.3. $M(\mu_{p^\infty}) = W(k)$ with $F = \sigma$.

In general, we have

$$\begin{aligned} \mathrm{ht}(G) &= \mathrm{rank}\ M(G) \\ \dim G &= \dim_k M(G)/VM(G) \end{aligned}$$

Note that the latter is a module over $W(k)/p$ since V divides p.

So what is this equivalence M? Given G/k, lift to $G'/W(k)$ arbitrarily. Then it is a fact G' has a *universal vector extension*. What does this mean? A vector extension is an extension of G' by a sheaf of $W(k)$-algebras isomorphic to $\mathbb{G}_a^n$.

$$0 \to V \cong \mathbb{G}_a^n \to EG' \to G' \to 0.$$

They form a category with morphisms required to be linear over $W(k)$ on the vector parts; the universal vector extension is the initial object. This turns out not to depend on G'. The reason is basically that the difference between different lifts is divisible by p, so the logarithm converges.

A good analogy to keep in mind is the following. Given a curve, its Picard scheme depends on the complex structure, but the universal vector extension is the stack of local systems on the curve, which is *independent of complex structure*. Then the Dieudonné module is

$$M(G) := \mathrm{Lie}\, EG'.$$

Why is this actually a Dieduonné module? Our original G has a Frobenius morphism

$$F\colon G \to G^{(p)}$$

inducing

$$V\colon M(G) \to M(G^{(p)}) = M(G) \otimes_{W(k),\sigma} W(k).$$

because of our conventions (note that this is σ^{-1}-linear). Since $F\colon G \to G^{(p)}$ divides p, $V\colon M(G) \to M(G^{(p)})$ also divides p, so we can define F as well.

Remark 14.3.4. There is also a contravariant version of the Dieudonné module in which "F actually induces F".

14.3.2 The period map

Now what's the period map? It's usually attributed to Gross-Hopkins or Grothendieck-Messing. Fix a p-divisible group G. The target of the period morphism is $\mathrm{Gr}(d, M(G))^{\mathrm{an}}$ (d-dimensional quotients of the Dieudonné module) where $d = \dim G$ and $n = \mathrm{ht}\, G$. The source is a deformation space for p-divisible groups, denoted $\mathcal{M}_G$. This is an adic space. What are its points? Roughly speaking, an (R, R^+)-point is a deformation of G to R^+. (Here (R, R^+) is an affinoid algebra over $(W(k)[1/p], W(k))$.)

$$\mathcal{M}_G(R, R^+) \approx \left\{ \begin{array}{c} G'/R = \ p\text{-div group} \\ \iota = \text{quasi-isogeny}\colon G \otimes_k R^+/p \to G' \otimes_{R^+} R^+/p \end{array} \right\}$$

(Caveat: we have to sheafify, and this only applies to bounded rings. If R is not bounded, then we need to first express it as a rising union of bounded subrings.)

Now we can finally define the period morphism. Given (G', ι) we have

$$0 \to V \to EG' \to G' \to 0.$$

We have a rigidification

$$\mathrm{Lie}\, EG'[1/p] \overset{\iota}{\cong} M(G) \otimes_{W(k)} R.$$

This map is induced by the "crystalline property" of the Dieudonné module because ι is only defined modulo p. Then the map $\mathrm{Lie}\, EG'[1/p] \to \mathrm{Lie}\, G'[1/p]$ defines a point in the classifying space of d-dimensional quotients of $M(G)$, which is $\mathrm{Gr}(d, M(G))$.

14.4 Discussion of Question 3

Let $G/\mathcal{O}_C$ be a p-divisible group. Then we have an exact sequence

$$0 \to V \to EG \to G \to 0.$$

The universal cover fits as

$$\begin{array}{ccccccccc} & & & & & & \widetilde{G} & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & V & \longrightarrow & EG & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

In fact the covering map factors through EG.

$$\begin{array}{ccccccccc} & & & & & & \widetilde{G} & & \\ & & & & & \swarrow & \downarrow & & \\ 0 & \longrightarrow & V & \longrightarrow & EG & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

Why? Informally speaking, $\widetilde{G} = \{(x_0, x_1, \dots)\}$ so

$$\lim_{i\to\infty} p^i \widetilde{x_i} \in EG$$

defines a lift. The point is that choices were made in lifting x_i to $\widetilde{x_i}$ but they will be killed in the limit, because the ambiguity is measured by V and this is multiplied by higher and higher powers of p.

So we get a map

$$\widetilde{G}(\mathcal{O}_C) \to EG(\mathcal{O}_C) \xrightarrow{\log_{EG}} \operatorname{Lie} EG \otimes C.$$

The composition is called the *quasi-logarithm* $q\log_G$. We can geometrize it to a morphism

$$q\log_G \colon \widetilde{G}^{\mathrm{an}} \to M(G) \otimes \mathbb{G}_a.$$

Both source and target only depend on G modulo p; it is a theorem that the map itself also only depends on G modulo p.

Theorem 14.4.1. *Let (R, R^+) be a perfectoid C-algebra. There exists an isomorphism*

$$\widetilde{G}(R^+) \to (M(G) \otimes B^+_{\mathrm{cris}}(R^+/p))^{\varphi=1} = H^0(X_{(R,R^+)}, \mathcal{E}_{M(G)}).$$

Proof sketch. We have

$$\widetilde{G}(R^+) = \widetilde{G}(R^+/p) = \operatorname{Hom}_{R^+/p}(\underline{\mathbb{Q}_p}, G).$$

Since we're over R^+/p we know that some power of p dies in G, so this is a *forward* limit

$$\lim_{n\to\infty} \operatorname{Hom}(\mathbb{Q}_p/p^n\mathbb{Z}_p, G) = \operatorname{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, G)[1/p].$$

Now passing to Dieudonné modules, we conclude that

$$\widetilde{G}(R^+) = \operatorname{Hom}_R(\underline{M}(\mathbb{Q}_p/\mathbb{Z}_p), \underline{M}(G))$$

in the category of Dieudonné crystals. This means that whenever you have a PD thickening of R^+/p, we can evaluate this on that thickening. We choose to evaluate it on $A_{\mathrm{cris}}(R^+/p) \to R^+/p$, which is the *universal* PD thickening. Then we get

$$\widetilde{G}(R^+) = \operatorname{Hom}(B^+_{\mathrm{cris}}, M(G) \otimes B^+_{\mathrm{cris}})$$

with the crystal structure on B^+_{cris} being $V = 1$, so

$$\widetilde{G}(R^+) = \operatorname{Hom}(B^+_{\mathrm{cris}}, M(G) \otimes B^+_{\mathrm{cris}}) = (M(G) \otimes B^+_{\mathrm{cris}})^{V=1,\text{ i.e. } F=p}$$

as desired. □

Go back to the exact sequence

$$0 \to T_p(G) \otimes \mathbb{Q}_p \to \widetilde{G} \to \mathrm{Lie}(G) \otimes \mathbb{G}_a \to 0.$$

We now see how to interpret the (R, R^+)-points of the middle term as global sections of a vector bundle $\mathcal{E}_{M(G)}$. Geometrizing to the actual vector bundles, we can interpret our sequence as a modification

$$0 \to T_p(G) \otimes \mathcal{O}_X \to \mathcal{E}_{M(G)} \to ? \to 0$$

Call $i \colon \infty \to X$ the inclusion of the point with residue field C. We can view $\mathrm{Lie}(G) \otimes C$ as $i_* \mathrm{Lie}(G) \otimes C$, getting

$$0 \to T_p(G) \otimes \mathcal{O}_X \to \mathcal{E}_{M(G)} \to i_* \mathrm{Lie}(G) \otimes C \to 0$$

The theorem also goes in the other direction: given any modification with trivial kernel, there is a corresponding p-divisible over $\mathcal{O}_C$ which induces it. Thus, there is a bijection

$$\{p\text{-divisible groups}/\mathcal{O}_C\}/\text{isogeny} \cong \{\text{modifications } 0 \to T \to \mathcal{E} \to i_\infty W \to 0\}$$

where T is trivial and W is miniscule, which means that it is a module over B_{dR}^+/t (i.e. killed by the uniformizer).

Chapter 15

Beauville-Laszlo Uniformization for the Fargues-Fontaine Curve

15.1 The classical affine Grasmannian

Let me first recall the classical story. Let X be a smooth projective connected curve over an algebraically closed field $k = \overline{k}$, $x \in X(k)$, and G/k a semisimple group.

We consider the *affine Grassmannian* Gr_G, which is an ind-(projective scheme). Recall that Gr_G parametrizes G-torsors $\mathcal{F}/X$ plus a trivialization over the punctured curve:

$$\mathcal{F}|_{X \setminus \{x\}} \cong G \times (X \setminus \{x\}).$$

Forgetting the trivialization induces a map

$$\mathrm{Gr}_G \to \mathrm{Bun}_G$$

where Bun_G is the (Artin) moduli stack of G-bundles. We have only defined the map on objects, but we all know how to relativize it in this case. In the case of the Fargues-Fontaine curve, it will be more subtle.

Theorem 15.1.1 (Drinfeld-Simpson)**.** *This map is surjective in the fppf topology.*

Remark 15.1.2. If $X = \mathbb{P}^1$ then we can replace "semisimple" by "reductive". This may be useful for understanding the behavior for the Fargues-Fontaine curve, which behaves like a mix between genus 0 and genus 1 curves.

15.2 The B_{dR}^+-affine Grassmannian

15.2.1 The ring B_{dR}^+

Let R be any perfectoid algebra. Fix $R^+ \subset R$ and a pseudo-uniformizer $\varpi^\flat$. (Ultimately everything will be independent of these choices.) Then we have the map

$$\theta \colon W(R^{\flat +}) \to R^+$$

with $\ker\theta = (\xi)$.

Definition 15.2.1. We define $B^+_{\mathrm{dR}}(R)$ to be the ξ-adic completion of $W(R^{\flat+})[\frac{1}{[\varpi^\flat]}]$. We think of this as "the completion of $\operatorname{Spa} R \times_? \operatorname{Spa} \mathbb{Z}_p$ along the graph map

$$\Gamma_{\operatorname{Spa} R \to \operatorname{Spa} \mathbb{Z}_p} \colon \operatorname{Spa} R \hookrightarrow \operatorname{Spa} R \times \operatorname{Spa} \mathbb{Z}_p."$$

We also define $\operatorname{Fil}^n B^+_{\mathrm{dR}}(R) = \xi^n B^+_{\mathrm{dR}}(R)$ and $B_{\mathrm{dR}}(R) = B^+_{\mathrm{dR}}(R)[\xi^{-1}]$.

Proposition 15.2.2. *The ring $B^+_{\mathrm{dR}}(R)$ enjoys the following properties:*

1. *$B^+_{\mathrm{dR}}(R)$ is ξ-adically complete, ξ-torsion free, and $B^+_{\mathrm{dR}}(R)/\xi = R$. (It looks like the completion along something of codimension one.)*

2. *Assume $p = 0$ in R. Then one can take $\xi = p$, obtaining $B^+_{\mathrm{dR}}(R) = W(R)$. (Thus the characteristic 0 version can be thought of as a deformation of $W(R)$.)*

Remark 15.2.3. If $R = \mathbb{C}_p$ then we get Fontaine's ring $B^+_{\mathrm{dR}} = B^+_{\mathrm{dR}}(\mathbb{C}_p)$ of p-adic periods. This B^+_{dR} is a complete DVR with uniformizer ξ and residue field $\mathbb{C}_p$. That means that it is abstractly isomorphic to $\mathbb{C}_p[[\xi]]$. However, the topology and Galois structures are not compatible.

We would like to play the game of affine Grassmannians in this situation. (Whenever you have a DVR you can take think of the affine Grassmannian as the space of lattices in its fraction field.)

15.2.2 The B^+_{dR}-affine Grassmannian

Let $G/\mathbb{Q}_p$ be a reductive group.

Definition 15.2.4. We define $\mathrm{Gr}_G^{B^+_{\mathrm{dR}}}$ to be the (pre)sheaf (which will be a sheaf for all our topologies) on $\mathrm{Perf}_{\mathbb{F}_p}$ with the following functor of points: if $\operatorname{Spa}(R, R^+) = S$ then

$$\mathrm{Gr}_G^{B^+_{\mathrm{dR}}}(S) = \left\{ \begin{matrix} R^\# = \text{ untilt of } R/\mathbb{Q}_p \\ \mathcal{F} = G\text{-bundle / Spec } B^+_{\mathrm{dR}}(R^\#) \\ \iota \colon \mathcal{F}|_{\operatorname{Spec} B_{\mathrm{dR}}(R^\#)} \cong G \times \operatorname{Spec} B_{\mathrm{dR}}(R^\#) \end{matrix} \right\}$$

Remark 15.2.5. There is a map

$$\mathrm{Gr}_G^{B^+_{\mathrm{dR}}} \to \operatorname{Spa} \mathbb{Q}_p^\diamond$$

which in terms of the functor of points is

$$(R^\#, \mathcal{F}) \mapsto R^\#.$$

Therefore, we can consider $\mathrm{Gr}_G^{B^+_{\mathrm{dR}}}$ as a (pre)sheaf on $\mathrm{Perf}_{\mathbb{F}_p} / \operatorname{Spa} \mathbb{Q}_p^\diamond$. But we have seen that this slice category is precisely $\mathrm{Perf}_{\mathbb{Q}_p}$. Under this identification $\mathrm{Gr}_G^{B^+_{\mathrm{dR}}}$ has the functor of points

$$B \in \mathrm{Perf}_{\mathbb{Q}_p} \mapsto \left\{ \begin{matrix} \mathcal{F} = G\text{-bundle / Spec } B^+_{\mathrm{dR}}(B) \\ + \text{ trivialiation on Spec } B_{\mathrm{dR}}(B) \end{matrix} \right\} \tag{15.1}$$

This is maybe the more natural definition, but we have chosen to give a definition that already lives in the worlds of diamonds.

Example 15.2.6. If $G = GL_n$, then the right side of (15.1) is simply the set of finite projective $B_{\mathrm{dR}}^+(B)$-modules M plus an isomorphism $M[1/\xi] \cong B_{\mathrm{dR}}(R)^n$. In general, we can think in these terms using the Tannakian philosophy.

15.2.3 Schubert cells

Let μ be a conjugacy class of cocharacters $\mathbb{G}_m \to G$. This may not be defined until an extension of $\mathbb{Q}_p$, but let's assume it's defined over $\mathbb{Q}_p$ for simplicity. Then we have a closed Schubert cell

$$\mathrm{Gr}_{G,\mu}^{B_{\mathrm{dR}}^+} \subset \mathrm{Gr}_G^{B_{\mathrm{dR}}^+}$$

parametrizing bundles such that at all geometric points, the relative position is bounded by μ.

If $R = C$ is algebraically closed and complete, and we choose $T \subset G_C$, then we have a Cartan decomposition

$$G(B_{\mathrm{dR}}^+(C))\backslash G(B_{\mathrm{dR}}(C))/G(B_{\mathrm{dR}}^+(C)) = X_*(T)_+.$$

For a proof, choose an isomorphism with $\mathbb{C}_p[[\xi]]$ (see Remark 15.2.3).

Remark 15.2.7. We can think of $\mathrm{Gr}_G^{B_{\mathrm{dR}}^+}$ as the sheafification of

$$R \mapsto G(B_{\mathrm{dR}}(R))/G(B_{\mathrm{dR}}(R^+)).$$

Theorem 15.2.8 (Scholze)**.** $\mathrm{Gr}_{G,\mu}^{B_{\mathrm{dR}}^+}$ *is a diamond.*

Example 15.2.9. (Caraiani-Scholze) If μ is miniscule and $P_\mu \subset G$ is the parabolic subgroup corresponding to μ, then

$$\mathrm{Gr}_{G,\mu}^{B_{\mathrm{dR}}^+} \cong (\underbrace{G/P_\mu}_{\text{rigid space}/\mathbb{Q}_p})^\diamond$$

This is an analogue of the result that for the classical affine Grassmannian, the Schubert cells are the usual flag varieties.

Remark 15.2.10. There is a fully faithful embedding

$$\{\text{seminormal rigid spaces}/\mathbb{Q}_p\} \hookrightarrow \{\text{diamonds}/\operatorname{Spa}\mathbb{Q}_p^\diamond\}$$

sending $X \mapsto X^\diamond$. Seminormality has to do with the topological difference between the curve and its normalization. (A node is seminormal; a cusp is not.) The point is that if $X \to Y$ is a universal homeomorphism, then $X^\diamond \cong Y^\diamond$. I like to think of diamonds as only remembering topological information. So this fully faithful embedding is saying that up to this defect, diamonds remember everything.

Example 15.2.11. For $G = \mathrm{GL}_2$ and $\mu = (n, 0)$ for $n \geq 2$,

$$\mathrm{Gr}_\mu := \mathrm{Gr}_{G,\mu}^{B_{\mathrm{dR}}^+} = \left\{ \begin{matrix} M = B_{\mathrm{dR}}^+ - \text{lattice} \subset B_{\mathrm{dR}}^2 : \\ \xi^n (B_{\mathrm{dR}}^+)^2 \overset{n}{\subseteq} M \overset{n}{\subseteq} (B_{\mathrm{dR}}^+)^2 \end{matrix} \right\}$$

There is a Bott-Samuelson resolution

$$\widetilde{\mathrm{Gr}_\mu} = \left\{ \begin{matrix} M \in \mathrm{Gr}_\mu + \text{flag} \\ M = M_n \overset{1}{\subseteq} M_{n-1} \overset{1}{\subseteq} \ldots \overset{1}{\subseteq} M_0 = (B_{\mathrm{dR}}^+)^2 \\ \text{each } M_i / M_{i-1} \text{ is a line bundle over } R \end{matrix} \right\}$$

Then $\widetilde{\mathrm{Gr}_\mu}$ is a succesion of $\mathbb{P}^1$-fibrations over $\mathbb{P}^1$. You might think that because it is inductively built from classical rigid spaces that it is itself a classical rigid space, but actually it is *not* a rigid space. (However, we would still like to think of it as being "smooth", whatever that means.) Why?

Locally (say $n = 2$) it looks like an extension

$$0 \to \mathbb{A}^1 \to B_{\mathrm{dR}}^+ / \mathrm{Fil}^2 \to B_{\mathrm{dR}}^+ / \mathrm{Fil}^1 = \mathbb{A}^1 \to 0$$

(the left $\mathbb{A}^1$ may be twisted over $\mathbb{Q}_p$, but the twist goes away over $\mathbb{Q}_p^{\mathrm{cyc}}$). The middle space $B_{\mathrm{dR}}^+ / \mathrm{Fil}^2$ is an example of a Banach-Colmez space. This is *not split* étale locally, so it cannot be a rigid space. (To split it we need to make a pro-étale extension adjoining all p-power roots of something.)

15.3 Bun_G

15.3.1 Construction of Bun_G

Recall that the Fargues-Fontaine curve lives over $\mathbb{Q}_p$. We know what its vector bundles are, but it is not clear what parametrizes families of vector bundles over X. The naïve guess is rigid spaces over $\mathbb{Q}_p$, but that's wrong. Instead, *we need to use the relative Fargues-Fontaine curve over $S \in \mathrm{Perf}_{\mathbb{F}_p}$.*

Definition 15.3.1. Let $S \in \mathrm{Perf}_{\mathbb{F}_p}$. Then we have a relative curve $X_S = Y_S / \varphi^{\mathbb{Z}}$, which is an adic space over $\mathrm{Spa}\,\mathbb{Q}_p$. A *$G$-bundle on X_S* is an exact faithful $\mathbb{Q}_p$-linear $\otimes$-functor

$$\mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Bun}_{X_S}.$$

Let Bun_G be the (pre)stack (which again will turn out to be a stack for all possible topologies) on $\mathrm{Perf}_{\mathbb{F}_p}$ which sends

$$S \mapsto \{G\text{-bundles}/X_S\}.$$

Remark 15.3.2. A theorem of Kedlaya-Liu implies that Bun_{X_S} is well-defined. Basically it says that for any analytic adic space, the category of bundles behaves as one would expect (with respect to gluing, etc.).

Remark 15.3.3. Say $S/\overline{\mathbb{F}}_p$ and $b \in G(\breve{\mathbb{Q}}_p)$. Then we can form $\mathcal{E}_b$ over X_S. If we wrote the internal definition we would say that this is "the trivial G-torsor on Y_S, descended via b to X_S".

Proposition 15.3.4. Bun_G *is a stack for the v-topology.*

This uses that vector bundles form a stack for the v-topology, which was discussed in Eugen Hellman's talk.

Remark 15.3.5. In the algebraic case one only gets a stack for the fppf topology. Thus, the proposition is stronger than you might have expected from reasoning by analogy with schemes. But for *perfect* schemes one also gets it for the v-topology, so it's the perfection that makes this possible.

15.3.2 "Smooth Artin stacks" in the category of diamonds

One of the main ideas is that Bun_G is a "smooth Artin stack" (i.e. admits a "smooth" cover by a "smooth" perfectoid space). Unfortunately, we haven't yet figured out what "smooth" should mean. We have some basic examples of things that should be smooth.

Example 15.3.6. If $X \to Y$ is a smooth rigid space over $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, then $X^\diamond \to Y^\diamond$ should be "smooth". (In these cases taking the diamond is like taking the perfection.) Why? We are in the process of developing a six-functor sheaf formalism. Smooth maps should imply that $f^! = f^*$ up to shift. Because all étale information is preserved by taking the diamond, if this is satisfied for $X \to Y$ then it should also be satisfied at the level of diamonds.

Example 15.3.7. If you believe this then you run into funny phenomena. For instance, considering the classifying stack $B\underline{\mathbb{Q}_p}$ for $\underline{\mathbb{Q}_p}$-torsors. Then we claim that $\mathrm{Spa}\,\mathbb{Q}_p^{\mathrm{cyc},\diamond} \times B\underline{\mathbb{Q}_p}$ is smooth.

Under the equivalence of categories of

$$\mathrm{Perf}_{\mathbb{F}_p} / \mathrm{Spa}\,\mathbb{Q}_p^{\mathrm{cyc}\,\diamond} \cong \mathrm{Perf}_{\mathbb{Q}_p^{\mathrm{cyc}}}$$

the two stacks correspond:

$$\mathrm{Spa}\,\mathbb{Q}_p^{\mathrm{cyc}\,\diamond} \times B\underline{\mathbb{Q}_p} \leftrightarrow B\underline{\mathbb{Q}_p}$$

There is an exact sequence (in the category of pro-étale sheaves on $\mathrm{Perf}_{\mathbb{Q}_p^{\mathrm{cyc}}}$):

$$0 \to \underline{\mathbb{Q}_p} \to \widetilde{\mu_{p^\infty}}^{\mathrm{an}} \to \mathbb{G}_a \to 0$$

which induces a map $\mathbb{G}_a \twoheadrightarrow B\underline{\mathbb{Q}_p}$ with fiber $\widetilde{\mu_{p^\infty}}^{\mathrm{an}}$ (the surjectivity is because the map from a point to $B\underline{\mathbb{Q}_p}$ is always surjective; this just says that every torsor is locally trivial). We've declared $\mathbb{G}_a$ to be smooth, since it comes from a smooth rigid analytic space, but also $\widetilde{\mu_{p^\infty}}^{\mathrm{an}}$ is smooth because it is the perfection of the open unit disk. Therefore we are forced to believe that $B\underline{\mathbb{Q}_p}$ is smooth.

Theorem 15.3.8 (Kedlaya-Liu, Fargues)**.** *The semistable locus* $\mathrm{Bun}_G^{ss} \subset \mathrm{Bun}_G$ *is open, and*

$$\mathrm{Bun}_{G^{ss},\overline{\mathbb{F}}_p} \cong \coprod_{b \in B(G)_{\mathrm{basic}} \overset{\kappa}{\cong} \pi_1(G)_\Gamma} \underline{BJ_b(\mathbb{Q}_p)}$$

(If G is a locally finite group then $\underline{G}$ *is the sheaf* $\underline{G}(S) = Map_{cont}(|S|, G)$*.)*

Remark 15.3.9. This is not what you get in the algebraic case (where the semistable locus is open). That may be surprising; it's because we took a different notion of family.

Note that the automorphisms of the trivial G-torsor are a locally profinite group $G(\mathbb{Q}_p)$, and *not* the algebraic group G. That's the reason p-adic groups appear. In the usual case we take the classifying space for a smooth group so it makes sense that we get an Artin stack, but here we are taking the classifying space for a p-adic group and we're not sure what we should get.

15.3.3 Uniformization of G-bundles

We have seen that if $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbb{Q}_p}$, then we get a relative Cartier divisor

$$S \hookrightarrow X_{S^\flat}.$$

As discussed in Definition 15.2.1, we can think of $B_{\mathrm{dR}}^+(R)$ as the completion of $X_{S^\flat}$ along S.

Lemma 15.3.10. *There is a functor*

$$\{B_{\mathrm{dR}}^+ - \textit{lattices in } B_{\mathrm{dR}}(R)^{\oplus n}\} \to \mathrm{Bun}(X_{S^\flat})$$

given by modifying the trivial vector bundle.

This is the Beauville-Laszlo Lemma in this setting. It was also proved by Kedlaya-Liu.

By the Tannakian formalism, for any G we get a map

$$\mathrm{Gr}_G^{B_{\mathrm{dR}}^+}(R, R^+) \to \mathrm{Bun}_G(R^\flat, R^{\flat+}).$$

Theorem 15.3.11 (Fargues)**.** *Assume G is quasisplit. Then the map*

$$\mathrm{Gr}_G^{B_{\mathrm{dR}}^+} \to \mathrm{Bun}_G$$

is surjective. More precisely, for $C/\mathbb{Q}_p$ *we get a point* $\infty \in X := X_{C^\flat}$*, and any G-bundle on X is trivial on* $X \setminus \{\infty\}$*.*

This follows easily from the classification of G-bundles

We claim that one can use this map and its surjectivity to give a smooth cover of G from a smooth space.

Example 15.3.12. Let $G = \mathrm{GL}_2$ and $\mu = (1, 0)$. We have a Schubert cell $\mathrm{Gr}_{G,\mu}^{B_{\mathrm{dR}}^+} \subset \mathrm{Gr}_G^{B_{\mathrm{dR}}^+}$. What does it look like under the uniformization map?

$$\begin{array}{ccc} \mathbb{P}^1 = \mathrm{Gr}_{G,\mu}^{B_{\mathrm{dR}}^+} \subset & \mathrm{Gr}_G^{B_{\mathrm{dR}}^+} \\ & \big\downarrow \\ & \mathrm{Bun}_{\mathrm{GL}_2} \end{array}$$

Inside $\mathbb{P}^1$ we have Drinfeld's upper half space $\Omega^2 \subset \mathbb{P}^1$ and its complement $\mathbb{P}^1(\mathbb{Q}_p) \subset \mathbb{P}^1$. The former maps to $\mathcal{O}(1/2)$ and $\mathbb{P}^1(\mathbb{Q}_p)$ maps to $\mathcal{O} \oplus \mathcal{O}(1)$.

$$\begin{array}{ccc} \Omega^2 & & \mathbb{P}^1(\mathbb{Q}_p) \\ \big\downarrow & & \big\downarrow \\ \mathcal{O}(1/2) & & \mathcal{O} \oplus \mathcal{O}(1) \end{array}$$

So we see that the stratifications on flag varieties are highly non-algebraic!

What is the image of the Schubert cell $\mathrm{Gr}_{G,\mu}^{B_{\mathrm{dR}}^+}$? It is a subset of $B(G)$ called $B(G)(\mu)$, familiar from the theory of Shimura varieties.

The map is $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant. If you quotient by the $\mathrm{GL}_2(\mathbb{Q}_p)$-action then the map is surjective in some smooth topology.

The semistable locus has dimension 0, while its complement has negative dimension. For example, the bundle $[\mathcal{O} \oplus \mathcal{O}(1)] \in \mathrm{Bun}_{\mathrm{GL}_2}$ has automorphism scheme

$$B\begin{pmatrix} \underline{\mathbb{Q}_p^*} & \widetilde{\mu_{p^\infty}}^{\mathrm{an}} \\ & \underline{\mathbb{Q}_p^*} \end{pmatrix}.$$

Chapter 16

Relation to Classical Local Langlands

The goal is to recall the local Langlands correspondence and its refined form for quasi-split groups, and then move towards non-quasisplit groups. Finally, we'll explain the connection to the upcoming conjecture.

16.1 The quasisplit case

16.1.1 The basic conjecture

Let E be a local field of characteristic 0 and G a connected reductive group over E. The basic problem is to understand irreducible admissible representations of $G(E)$. The Langlands correspondence reduces us to understanding the tempered representations $\Pi_{\mathrm{temp}}(G)$.

Definition 16.1.1. Let $\Gamma = \mathrm{Gal}(\overline{E}/E)$. We denote by L_E the *local Langlands group of E*, which is

$$L_E := \begin{cases} W_E & \text{archimedean} \\ W_E \times \mathrm{SU}_2(\mathbb{R}) & \text{non-arch} \end{cases}$$

Definition 16.1.2. The *L-group* of G is is

$$^L G := \widehat{G} \rtimes \Gamma$$

with Weil form $\widehat{G} \rtimes W_E$.

Example 16.1.3. For $G = \mathrm{GL}_n$, $\widehat{G} = \mathrm{GL}_n$. For $G = \mathrm{SL}_n$, $\widehat{G} = \mathrm{PGL}_n$.

Example 16.1.4. For split groups the L-group splits as a direct product. Inner forms of the same group have the same L-group.

Definition 16.1.5. We define $\Phi_{\mathrm{temp}}(G)$ to be the set of *tempered L-homomorphisms*

$$\phi \colon L_E \to {}^L G.$$

which are homomorphisms ϕ as above such that

- ϕ commutes with projection to the Weil group,

$$\begin{array}{ccc} L_E & \xrightarrow{\phi} & {}^L G \\ & \searrow \quad \swarrow & \\ & W_E & \end{array}$$

- (*tempered*) ϕ has bounded image in $\widehat{G}$, and
- (*admissible*) ϕ maps the Weil group to semisimple elements in $\widehat{G}$.

Conjecture 16.1.6 (Conjecture A)**.** *There exists a map*

$$LL\colon \Pi_{\text{temp}}(G) \to \Phi_{\text{temp}}(G)$$

with finite fibers $\Pi_\phi(G) := LL^{-1}(\phi)$*, which are called L-packets for the tempered parameter* ϕ*.*

Example 16.1.7. For GL_n this is a bijection (each Π_ϕ is a singleton). But SL_n already has two elements in its discrete series L-packets.

Remark 16.1.8. The map has nice properties.

- We understand the image (it should be those ϕ factoring through parabolic subgroups relevant to G).
- In the unramified case the correspondence is controlled by the Satake isomorphism.
- We understand how this behaves with respect to parabolic induction.

Main question: How can we address representations in $\Pi_\phi(G)$ individually?

16.1.2 Refined Langlands conjectures

Langlands realized the importance of the group

$$S_\phi = \{g \in \widehat{G} \mid g\phi(L_E)g^{-1} = \phi\}$$

i.e. the centralizer of the L-parameter. Kottwitz showed that S_ϕ^0 is a reductive group (this uses the admissibility condition). We have $Z(\widehat{G})^\Gamma \subset S_\phi$, and we define

$$\overline{S}_\phi = S_\phi / Z(\widehat{G})^\Gamma.$$

We now assume that G is quasi-split, which allows us to choose a *Whittaker datum* $\omega = (B, \psi)$ where $\psi\colon U \to \mathbb{C}^*$ is a non-degenerate character. The parametrization of the L-packet depends on the choice of Whittaker datum.

Conjecture 16.1.9 (Conjecture B)**.** *There is an injective map* $\iota_\omega \colon \Pi_\phi(G) \hookrightarrow \mathrm{Irr}(\pi_0(\overline{S}_\phi))$ *which is bijective if E is p-adic.*

The importance of the centralizer S_ϕ comes up in connection to global computations, using the trace formula. That the Whittaker datum is necessary comes from a conjecture of Shahidi that there is a unique generic constituent of the L-packet and it should correspond under i_ω to the trivial representation.

Conjecture 16.1.10 (Conjecture C)**.** *There is a unique generic constituent of* $\Pi_\phi(G)$ *corresponding to the trivial representation under* ι_ω*.*

There is another part of the conjecture that we're not going to say anything about. Part of the motivation for why we want to access member of the L-packet individually is to make sense of some calculations coming out of the trace formula, namely stabilization and what happens on the spectral side. Because of that, there should be some character relations that are encoded in this map.

Conjecture 16.1.11 (Conjecture D)**.** *If E is non-archimedean, then the bijection*

$$\Pi_\phi(G) \cong \mathrm{Irr}(\pi_0(\overline{S_\phi}))$$

can be re-interpreted as a "perfect pairing" (one has to be careful about what this means in the non-abelian case)

$$\langle \cdot, \cdot \rangle \colon \Pi_\phi(G) \times \pi_0(\overline{S_\phi}) \to \mathbb{C}$$

which defines a virtual character attached to an L-homomorphism ϕ *and* $s \in \pi_0(\overline{S_\phi})$*:*

$$\Theta_\phi^s = \sum_{\pi \in \Pi_\phi(G)} \langle \pi, s \rangle \Theta_\pi$$

where Θ_π *is the Harish-Chandra character. This pairing should satisfy certain endoscopic relations.*

We won't say more other than that these endoscopic relations come from global motivations.

16.2 The non-quasisplit case

16.2.1 Problems for non-quasisplit groups

Now suppose G is not quasisplit. Obviously Conjecture C doesn't make sense because it depends on having a Whittaker datum. Although it is not clear from our brief discussion, Conjecture D also cannot be formulated because the endoscopy relations depend on transfer factors which require the quasi-splitness to normalize. Even if you had a way of looking at transfer factors "up to scalars", it turns out that the map giving the right character relations doesn't exist, because Conjecture 16.1.9 is false.

Example 16.2.1. Let E be a p-adic field and F/E a quadratic extension. Let $G = \mathrm{SL}_2/E$ and $G' = (D^\times)^{\mathrm{Nm}=1}$, where D is a quaternion algebra over E (so G' is an inner form of SL_2). (F is a maximal commutative subalgebra of D, since any quadratic extension embeds into any quaternion algebra.) Let $\sigma \in \mathrm{Gal}(F/E)$ be the non-trivial element. Choose a character

$$\theta\colon F^\times \to \mathbb{C}^\times$$

such that $\theta^{-1} \cdot (\theta \circ \sigma)$ is *non-trivial* of order 2. We define an L-parameter

$$\phi\colon W_{F/E} \to \mathrm{PGL}_2(\mathbb{C})$$

by

$$f \mapsto \begin{pmatrix} \theta(f) & \\ & \theta(\sigma(f)) \end{pmatrix}$$

Then it turns out that

$$\overline{S_\phi} = S_\phi = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

Langlands-Labesse found $\#\Pi_\phi(G) = 4$ but $\#\Pi_\phi(G') = 1$. For the $\pi \in \Pi_\phi(G')$ the hypothetical character relations would imply that $\langle \pi, 1 \rangle = 2$ and $\langle \pi, s \rangle = 0$ for $s \in S_\phi \setminus \{1\}$, which is not a character even up to scalars.

16.2.2 Inner twists

The fundamental idea is that you shouldn't work with a single inner form of a quasi-split group, but rather *treat all of them together at once*. Numerical computations for unitary groups that suggest that this is reasonable.

Example 16.2.2. Let $E = \mathbb{R}$. The unitary groups $U(p,q)$ for $p + q = n$ constitute a clas of inner forms. If ϕ is a discrete parameter for $U(n)$ (a quasisplit group), then $\#S_\phi = 2^n$ and $\#\overline{S}_\phi = 2^{n-1}$. For $G = U(p,q)$ we have $\#\Pi_\phi(G) = \binom{p+q}{q}$. So if you add up the contributions for all inner forms then you cover the size of the $\overline{S}_\phi$. This suggests that we should treat $U(p,q)$ and $U(q,p)$ as being distinct, even though they have the same L-group.

Vogan introduced the notion of inner twists to codify this phenomenon.

Definition 16.2.3. For G^* a quasisplit form over E, an *inner twist* is an isomorphism class of maps

$$\xi\colon G^*_{\overline{E}} \xrightarrow{\sim} G$$

such that $\xi^{-1} \circ \sigma(\xi)$ is an inner automorphism of $G^*_{\overline{E}}$ for every $\sigma \in \Gamma$, and isomorphisms are diagram isomorphisms.

Example 16.2.4. Contrast this with the notion of *inner form*, which is an inner twist (G, ξ) but forgetting ξ. This is badly behaved; for instance, you can think of GL_n as an inner twist in two ways: via identity or transpose maps. If you didn't consider both you wouldn't have a chance of parametrizing the L-packet in a canonical way.

Unfortunately, the refinement of inner twists is also not quite enough.

Example 16.2.5. Let $E = \mathbb{R}$ and $G = \mathrm{SL}_2/\mathbb{R}$ (viewed as an inner twist of itself via the identity map). There is a discrete series L-packet $\{\pi^+, \pi^-\}$ such that for $g = \begin{pmatrix} i & \\ & -i \end{pmatrix}$, the automorphism $\mathrm{Ad}(g)$ preserves the inner twist but acts on $\mathrm{SO}_2(\mathbb{R})$ as $x \mapsto x^{-1}$. One can show that $\mathrm{Ad}(g)$ exchanges π^+ and π^-, which shows that inner twists are also not sufficiently rigid to provide a canonical parametrization of L-packets.

16.2.3 Extended pure inner forms

Definition 16.2.6. We need to refine further: a *pure inner twist* is the isomorphism class of a pair (ξ, z) where

- ξ is an inner twist of G^*, and
- $z \in Z^1(\Gamma, G^*)$ is such that $\xi^{-1} \circ \sigma(\xi) = \mathrm{Ad}(z(\sigma))$ for all $\sigma \in \Gamma$. (This adds an extra rigidification.)

These are parametrized by $H^1(\Gamma, G^*)$. Pure inner forms are defined analogously.

Conjecture 16.2.7. *Let G^* be a quasisplit connected reductive group over E. We have a commutative diagram*

$$\begin{array}{ccc} \coprod_{(\xi,z)} \Pi_\phi(\xi, z) & \xrightarrow{\sim} & \mathrm{Irr}(\pi_0(S_\phi)) \\ \downarrow{\scriptstyle (\xi,z)\mapsto z} & & \downarrow \\ H^1(\Gamma, G^*) & \xrightarrow[\sim]{} & \pi_0(Z(\widehat{G})^\Gamma)^{*=dual} \end{array}$$

Here $\Pi_\phi(\xi, z)$ is the L-packet corresponding to inner forms determined by (ξ, z).

Recall that inner forms for G are parametrized by $H^1(\Gamma, G^*_{\mathrm{ad}})$. Here we see a problem: the conjecture only gives access to the image of $H^1(\Gamma, G^*) \to H^1(\Gamma, G^*_{\mathrm{ad}})$. So this conjecture doesn't reach all inner forms. Therefore, we need to expand the notion of pure inner forms.

Definition 16.2.8. Recall the set

$$B(G^*)_{\mathrm{basic}} := G^*(\breve{E})/\text{conjugacy}$$

It turns out that any $b \in B(G^*)_{\mathrm{basic}}$ determines an inner twist (b, ξ) with corresponding inner form J_b. This pair (b, ξ) is called an *extended pure inner twist*.

As was discussed in Rapoport's talk, Kottwitz defined a map

$$\kappa \colon B(G^*)_{\mathrm{basic}} \to \pi_1(G)_\Gamma \cong X^*(Z(\widehat{G})^\Gamma).$$

♠♠♠ TONY: [isomorphism seems hard to believe] This is compatible with the map from Conjecture 16.2.7 for the inclusion $H^1(\Gamma, G^*) \hookrightarrow B(G^*)$:

$$\begin{array}{ccc} H^1(\Gamma, G^*) & \hookrightarrow & B(G^*)_{\text{basic}} \\ \downarrow \sim & & \sim \downarrow \kappa \\ \pi_0(Z(\widehat{G})^{\Gamma})^* & \hookrightarrow & X^*(Z(\widehat{G})^{\Gamma}) \cong \pi_1(G)_{\Gamma} \end{array}$$

Conjecture 16.2.9 (Conjecture F). *Assume ϕ is a discrete parameter, i.e. $S_\phi/Z(\widehat{G})^\Gamma$ is finite. Then there exists a unique bijection*

$$\iota_\omega \colon \coprod_{(\xi,b)} \Pi_\phi(\xi, b) \xrightarrow{\sim} \operatorname{Irr}(S_\phi)$$

such that the following diagram commutes:

$$\begin{array}{ccc} \coprod_{(\xi,b)} \Pi_\phi((\xi, b)) & \xrightarrow[\sim]{\iota_\omega} & \operatorname{Irr}(S_\phi) \\ \downarrow (\xi,b) \mapsto b & & \downarrow \textit{restriction} \\ B(G^*)_{\text{basic}} & \xrightarrow[\kappa]{\sim} & X^*(Z(\widehat{G})^{\Gamma}) \end{array}$$

Chapter 17

Discussion Session: Constructing the Eigensheaf

These are notes from a 20-minute evening discussion by Dennis Gaitsgory, giving an alternate perspective on the construction of the Hecke eigensheaf for GL_n.

17.1 The dream

This will be a slightly different perspective on Stefan's talk. Some thing which did exist at the time of the paper the discusses do now exist. We are trying to geometrize functions on certain sets, which are tabulated below.

Artin stacks	Non-existent spaces	Sets
		$N(K)\backslash N(\mathbb{A})/N(\mathcal{O})$ $P(K)\backslash P(\mathbb{A})/P(\mathcal{O})$ $G(K)\backslash G(\mathbb{A})/G(\mathcal{O})$

We want to geometrize these functions to certain spaces. An immediate issue is that the spaces of interest don't exist. (Some of them are now known as "pre-stacks".) The obvious one is Bun_n (an existing space is a special case of a non-existence space).

Artin stacks	Non-existent spaces	Sets
	 Bun_n	$N(K)\backslash N(\mathbb{A})/N(\mathcal{O})$ $P(K)\backslash P(\mathbb{A})/P(\mathcal{O})$ $G(K)\backslash G(\mathbb{A})/G(\mathcal{O})$

As was discussed by Stefan, it is easy to construct the right function on $N(K)\backslash G(\mathbb{A})/G(\mathcal{O})$. We want to produce a function on $G(K)\backslash G(\mathbb{A})/G(\mathcal{O})$. The most naïve thing would be to pushforward (i.e. integrate), but that won't work.

We can at least take the pushforward to $P(K)\backslash P(\mathbb{A})/P(\mathcal{O})$, which is basically the Fourier transform. Then we want to descend the function somehow. To geometrize it, we propose a new space $\mathrm{Bun}_n^{\prime,\mathrm{rat}}$. While Bun_n classifies bundles $\{M\}$, we want our new space $\mathrm{Bun}_n^{\prime,\mathrm{rat}}$ to classify $\{(M, \Omega^{\otimes(n-1)} \dashrightarrow M)\}$.

Artin stacks	Non-existent spaces	Sets
		$N(K)\backslash N(\mathbb{A})/N(\mathcal{O})$
	$\mathrm{Bun}_n^{\prime,\mathrm{rat}}$	$P(K)\backslash P(\mathbb{A})/P(\mathcal{O})$
	Bun_n	$G(K)\backslash G(\mathbb{A})/G(\mathcal{O})$

Next we propose a space $\overline{\mathcal{Q}}^{\mathrm{rat}}$ which should classify M along with flags whose first subquotient is Ω^{n-1}, second subquotient $\Omega^{n-2}, \ldots \mathcal{O}$. We can define this in terms of the Plucker embedding. So $\overline{\mathcal{Q}}^{\mathrm{rat}}$ parametrizes bundles M plus maps

Artin stacks	Non-existent spaces	Sets
	$\overline{\mathcal{Q}}^{\mathrm{rat}}$	$N(K)\backslash N(\mathbb{A})/N(\mathcal{O})$
	$\mathrm{Bun}_n^{\prime,\mathrm{rat}}$	$P(K)\backslash P(\mathbb{A})/P(\mathcal{O})$
	Bun_n	$G(K)\backslash G(\mathbb{A})/G(\mathcal{O})$

satisfying some conditions.

$$\begin{aligned} \Omega^{n-1} &\dashrightarrow M \\ \Omega^{n-1+n-2} &\dashrightarrow M \\ \Omega^{n-1+n-2+n-3} &\dashrightarrow M \\ &\vdots \\ \Omega^{n(n-1)/2} &\dashrightarrow M \end{aligned}$$

Now for the spaces that actually exist. They parametrize the corresponding things with rational maps replaced by regular maps.

Artin stacks	Non-existent spaces	Sets
$\overline{\mathcal{Q}}$	$\overline{\mathcal{Q}}^{\mathrm{rat}}$	$N(K)\backslash N(\mathbb{A})/N(\mathcal{O})$
Bun_n'	$\mathrm{Bun}_n^{\prime,\mathrm{rat}}$	$P(K)\backslash P(\mathbb{A})/P(\mathcal{O})$
Bun_n	Bun_n	$G(K)\backslash G(\mathbb{A})/G(\mathcal{O})$

17.2 Laumon's sheaf

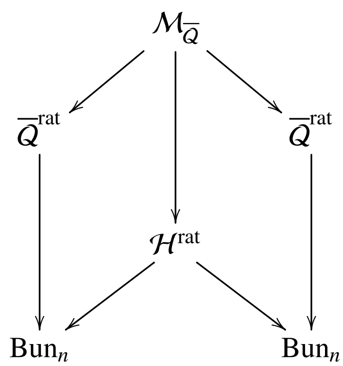
Now we construct the sheaves. First, there is a Hecke stack

$$\mathrm{Bun}_n \xleftarrow{\;h^{\leftarrow}\;} \mathcal{H}^{\mathrm{rat}} \xrightarrow{\;h^{\rightarrow}\;} \mathrm{Bun}_n$$

whose which gives a correspondence $h^{\to}_! \circ (h^{\leftarrow})^*$ geometrizing convolutions.

Here is an algorithm to produce something that has the Hecke property.

Key idea. If V is any G-representation, then multiplying by the regular representation of G produces an "eigensheaf" because of the identity $R_G \otimes V \cong \underline{V} \otimes R_G$.



We now define *Laumon's sheaf* $\mathcal{L}^{\mathrm{rat}}_E \in D(\mathcal{M}^{\mathrm{rat}}_{\overline{Q}})$. Motivated by the above heuristic, we start with the regular representation. This is a two-sided G-representation. Given a $\widehat{G}$-local system E, it induces a bundle $R^E_{\widehat{G}}$ on X by the "external interpretation" as a functor from $\widehat{G}$-repesentations to vector bundles. But since $R_{\widehat{G}}$ has an action of $\widehat{G}$ on both sides, there is still a $\widehat{G}$-action on $R^E_{\widehat{G}}$ to which we can still apply Geometric Satake, obtaining a perverse sheaf on Bun_G. So we define:

$$\mathcal{L}^{\mathrm{rat}}_E := \text{``}\bigotimes_{x \in |X|}{}' \, \mathrm{GeomSat}(R^E_{\widehat{G}})\text{''}$$

using that $R_{\widehat{G}}$ is a bi-module, so there is one thing left after twisting by the local system E to apply Geometric Satake to.

Then

$$\mathcal{F} \mapsto h^{\to}_!(h^{\leftarrow *}(\mathcal{F}) \otimes \mathcal{L}^{\mathrm{rat}})$$

will have the desired Hecke property.

17.3 The Whittaker sheaf

Now what do we apply this to? There's a substack $i \colon Q \hookrightarrow \overline{Q}$ which parametrizes the locus where everything is a bundle map (no zeros or poles) - this corresponds to bundles which admit a full flag whose subquotients are really identified with Ω^*. There's a map from

$Q \xrightarrow{ev} \mathbb{A}^1$ by sending the extensions

$$0 \to \Omega \to ? \to \mathcal{O} \to 0$$
$$0 \to \Omega^2 \to ? \to \Omega \to 0$$
$$\vdots$$

to the sum of the extension classes in $H^1(X, \Omega)$. Then the dream Whitaker sheaf $Whit_E \in D(\overline{Q}^{\text{rat}})$ is built from an Artin-Schreier sheaf on $\mathbb{A}^1$ by

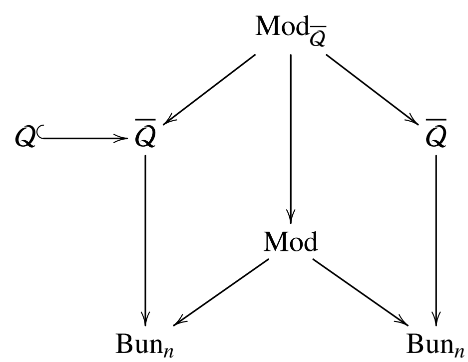
$$h^{\rightarrow}_{Q!}(\pi^*(\mathcal{L}_E^{\text{rat}} \otimes h^{\leftarrow *}_{Q}(i_! \circ ev^*(\text{Artin-Schreier})))$$

Why this formula? At any point $x \in |X|$ the Hecke property $R_G \otimes V = \underline{V} \otimes R_G$ is implies the Hecke eigensheaf property at that point. If you do this at all points, then you force the eigensheaf property at all points.

The problem is that in the process of using this sheaf we used spaces that are not defined (namely we used $i \colon Q \to \overline{Q}^{\text{rat}}$).

What we do to is produce $Whit_E = Whit_E^{\text{rat}}|_{\overline{Q}}$. We'll construct this in a moment. However, the fact that $Whit_E \cong Whit_E^{\text{rat}}|_{\overline{Q}}$ is a theorem. (There's something to check that the dream coincides with reality.)

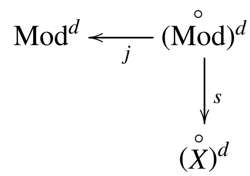
The last thing is to define $Whit_E$. Consider a similar diagram with Hecke replaced by Mod, which classifies $(M, M', M \hookrightarrow M')$:



The definition is similar to before:

$$Whit_E := h^{\rightarrow}_{Q!}(\pi^*(\mathcal{L}_E \otimes h^{\leftarrow *}_{Q}(i_! \circ ev^*(\text{Artin-Schreier})))$$

but we have to describe the Laumon sheaf $\mathcal{L}_E \in D(\text{Mod})$. We divide into components: $\text{Mod} = \bigsqcup_d \text{Mod}^d$ and there are maps



We then define the Laumon sheaf to be the middle extension sheaf $\mathcal{L}_E^d := j_{!*}(s^*(E^{(d)}))$.

Part V

Day Five

Chapter 18

Formulation of Fargues' Conjecture

Let $E = \mathbb{F}_q((t))$ or a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$, and ϖ be a uniformizer for E. We fix an algebraic closure $\overline{\mathbb{F}}_q$. Let G/E be a quasi-split reductive group.

(Several of the assertions here are not yet proved.)

18.1 The stack Bun_G

18.1.1 Structure as a diamond

For a perfectoid space $S \in \mathrm{Perf}_{\mathbb{F}_p}$, we can associate the "relative Fargues-Fontaine curve" X_S, which is an adic space over E. Then Bun_G is the stack on $\mathrm{Perf}_{\mathbb{F}_q}$ for the pro-étale topology, with functor of points

$$\mathrm{Bun}_G(S) = \{G\text{-bundles}/X_S\}.$$

Theorem 18.1.1. *We have the following.*

1. *The diagonal Δ_{Bun_G} is represented by a diamond. (In equal characteristic, it is even a perfectoid space.)*

2. *For all vector bundles $\mathcal{E}$ on X_S, we define the sheaf*

$$\begin{array}{l} \mathrm{Quot}^{\lambda}_{\mathcal{E}/X/S} \; : T/S \to \{\textit{locally free quotients of } \mathcal{E}|_{X_T}\} \\ \quad\;\;\downarrow \\ \quad\;\; S \end{array}$$

   This is represented by a diamond over S. (Again, in equal characteristic it is even represented by a perfectoid space.)

Remark 18.1.2. The theorem reflects the general phenomenon that one doesn't need stacks in equal characteristic.

This gives a "smooth" presentation of Bun_G by perfectoid spaces. We want to use it later to give a more constructive proof that the B^+_{dR}-affine Grassmannian is a diamond.

18.1.2 Points

For $b \in G(\breve{E})$ we obtain a point

$$x_b\colon \operatorname{Spa}(\overline{\mathbb{F}}_q) \to \operatorname{Bun}_{G,\overline{\mathbb{F}}_q} \tag{18.1}$$

Here $\operatorname{Spa}(\overline{\mathbb{F}}_q)$ is the sheaf on $\operatorname{Perf}_{\overline{\mathbb{F}}_p}$ which assigns to each perfectoid space a point, so it is tautologically the final object. The map is given by $\mathcal{E}_b$, i.e. assigns to S over $\operatorname{Spa}(\overline{\mathbb{F}}_q)$ the G-bundle $\mathcal{E}_b/X_S$.

This induces a bijection

$$B(G) \xrightarrow{\sim} |\operatorname{Bun}_{G,\overline{\mathbb{F}}_q}|$$

(modulo a conjecture in the equal characteristic case $E = \mathbb{F}_q((\varpi))$). Since we have defined $\operatorname{Bun}_{G,\overline{\mathbb{F}}_q}$ as a sheaf it may not be clear what is meant by its points: that meaning is

$$|\operatorname{Bun}_{G,\overline{\mathbb{F}}_q}| = \left(\coprod_{F \text{ perfectoid}/\overline{\mathbb{F}}_q} \operatorname{Bun}(F) \right) / \sim .$$

18.1.3 Connected components

We now discuss the topology on Bun_G. Again, it is not obvious what this means. The answer is that the topology is determined by declaring the open sets to be those coming from open substacks. (Conjecturally the topology on $B(G)$ is such that the closure of a point is the set of points with HN polygon over it.)

We have seen that there is a Kottwitz map

$$\kappa\colon B(G) \to \pi_1(G)_\Gamma$$

where $\Gamma = \operatorname{Gal}(\overline{E}/E)$.

Theorem 18.1.3. *(Assume that $Z_{G_{sc}}$ is étale if $E = F_q((\varpi))$.) The map κ is locally constant on* Bun_G.

This gives a decomposition

$$\operatorname{Bun}_G = \coprod_{\alpha \in \pi_1(G)_\Gamma} \operatorname{Bun}^\alpha$$

where $\operatorname{Bun}^\alpha = \kappa^{-1}(\alpha)$, which is open and closed.

18.1.4 Harder-Narasimhan filtration

Fix as usual a triplet $(A \subset T \subset B)$ where B is a Borel, T is a maximal torus inside B, and A is a maximal split torus inside T. There is a "Harder-Narasimhan polygon map"

$$HN\colon |\operatorname{Bun}_G| = B(G) \to X_*(A)^+_{\mathbb{Q}}$$

and a theorem of Kedlaya-Liu implies that this is semi-continuous. What is important is that this implies Bun^{ss} is open. Moreover,

1. each $|\mathrm{Bun}_G^{\alpha,ss}|$ is a single point, represented by a basic element of $B(G)_{\mathrm{basic}}$ via

$$\kappa\colon B(G)_{\mathrm{basic}} \xrightarrow{\sim} \pi_1(G)_\Gamma.$$

   In other words, there is a unique semi-stable point in each component, which is the image of the basic locus.

2. For all $\nu \in X_*(A)^+_{\mathbb{Q}}$, $|\mathrm{Bun}_G^{\alpha,HN=\nu}|$ is either empty or a singleton.

18.1.5 Uniformization

When b is basic, the map (18.1) giving the point $\mathcal{E}_b$ descends through the quotient by $J_b(E) = \mathrm{Aut}(\mathcal{E}_b)$:

$$x_b\colon [\mathrm{Spec}\,(\overline{\mathbb{F}}_q)/\underline{J_b(E)}] \xrightarrow{\sim} \mathrm{Bun}_{G,\overline{\mathbb{F}}_q}^{\kappa(b),ss}$$

where the left side is the classifying stack of pro-étale $\underline{J_b(E)}$-torsors. These J_b are extended pure inner forms, as in Ana's talk.

Remark 18.1.4. The dimension of the non-semistable Harder-Narasimhan strata goes to $-\infty$ when you go deeper in the Weyl chamber.

18.2 Through the looking glass

18.2.1 The mirror curve

The moduli space of effective degree 1 Cartier divisors on the curve is not itself a curve (unlike in the classical setting). We call it the "mirror curve". To describe it, recall the diamond formula for the Fargues-Fontaine curve:

$$\begin{array}{ccc} X_S^\diamond & = & (S \times \mathrm{Spa}(E)^\diamond)/\varphi_S^{\mathbb{Z}} \\ \downarrow & & \\ \mathrm{Spa}(E)^\diamond & & \end{array}$$

The mirror curve is a characteristic p version which sits over S:

$$\begin{array}{c} (S \times \mathrm{Spa}(E)^\diamond)/\varphi_{E^\diamond}^{\mathbb{Z}} \\ \downarrow \\ S \end{array}$$

These two diamonds have the same topological space, the same étale site, and are locally isomorphic, but they are not isomorphic (for instance, $X_S^\diamond$ has no natural map to S). The point is that $\varphi_S \circ \varphi_{E^\diamond}$ is the identity on $S \times \mathrm{Spa}(E)^\diamond$; therefore, quotienting by one or the other gives us something interesting, while quotienting by both at the same time does nothing.

Let us carefully highlight the difference between φ_S and $\varphi_{E^\diamond}$. Consider a T-valued point of $S \times \mathrm{Spa}(E)^\diamond$ (so T is a perfectoid space over S).

$$\begin{array}{ccc} T & \longrightarrow & S \times \mathrm{Spa}(E)^\diamond \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

The Frobenius φ_S acts on the S coordinate, translating the structure morphism $T \to S$ by φ_S.

On the other hand, by definition $\mathrm{Spa}(E)^\diamond(T)$ is the set of untilts $(T^\#, \iota)$ of T over E. The Frobenius φ_E acts by translating the untilt isomorphism $\iota \colon T \cong T^{\#,\flat}$ by φ_T.

Example 18.2.1. If $E = \mathbb{F}_q((\varpi))$, then $Y_S = \mathbb{D}_S^*$. This has two maps

$$\begin{array}{ccc} & Y_S = D_S^* & \\ \swarrow & & \searrow \\ S & & \mathbb{D}^*_{\mathbb{F}_q} = \mathrm{Spa}(E) \end{array}$$

We have $X_S = \mathbb{D}_S^*/\varphi_S$. In this case the mirror curve is

$$\mathrm{Div}^1_{X/S} = \mathbb{D}_S^{*,1/p^\infty}/\varphi_E^{\mathbb{Z}}.$$

18.2.2 The mirror curve as the moduli space of divisors

The remarkable fact is that the mirror curve can be identified with the moduli space of degree 1 divisors on X:

$$\boxed{S \times \mathrm{Spa}(E)^\diamond/\varphi_{E^\diamond}^{\mathbb{Z}} \xrightarrow{\sim} \mathrm{Div}^1_{X/S} = \left\{ \begin{matrix} \mathcal{L} = \deg 1 \text{ line bundle on } X_S \\ f \in H^0(X_S, \mathcal{L}) \text{ fiberwise non-zero}/S \end{matrix} \right\}}$$

Another way to say this is that

$$\mathrm{Div}^1_{X/S} \cong (B_S^{\varphi=\varpi} \setminus \{0\})/\underline{E}^*,$$

where the right hand side is viewed as a projective space over a Banach-Colmez space.

More generally, for all $d \geq 1$ we define the moduli space

$$\begin{array}{l} \mathrm{Div}^d_{X/S} = \{\deg d \text{ effective Cartier divisors on } X_S\} \\ \big\downarrow \\ S \end{array}$$

and

$$\mathrm{Div}^d_{X/S} = \mathrm{Div}^d_X \times_{\mathbb{F}_q} S.$$

The Div_X^d is not a diamond, but it is an "absolute diamond". This just means that it is not representable by a diamond, but its pullback to any perfectoid space is a repesentable by a diamond.

Example 18.2.2. The situation is similar for $\mathrm{Spa}(\overline{\mathbb{F}}_q)$: it is not a diamond, but the "diagonal is a diamond", i.e. the pullback to any perfectoid space is a diamond. This is just the statement that the final object, which takes every perfectoid space to a point, is not a diamond; but after base-changing to a perfectoid space S, obviously S is the final object in the category of perfectoid spaces over S.

Theorem 18.2.3. *We have an isomorphism*

$$\frac{\mathrm{Spa}(E)^\diamond \times \ldots \times \mathrm{Spa}(E)^\diamond}{\varphi_{E^\diamond}^{\mathbb{Z}} \times \ldots \times \varphi_{E^\diamond} \rtimes S_d} \xrightarrow{\sim} \mathrm{Div}_X^d \cong (B_S^{\varphi=\varpi^d} \setminus \{0\})/\underline{E}^*$$

as absolute diamonds.

We have a map $\mathrm{Div}_X^d \to \mathrm{Pic}_X^d := [\mathrm{Spa}\,\mathbb{F}_q/\underline{E}^*]$

$$\begin{array}{ccccc} \frac{\mathrm{Spa}(E)^\diamond \times \ldots \times \mathrm{Spa}(E)^\diamond}{\varphi_{E^\diamond}^{\mathbb{Z}} \times \ldots \times \varphi_{E^\diamond} \rtimes S_d} & \xrightarrow{\sim} & \mathrm{Div}_X^d & \xrightarrow{\sim} & (B_S^{\varphi=\varpi^d} \setminus \{0\})/\underline{E}^* \\ & \searrow^{AJ^d} & & & \downarrow \\ & & & & \mathrm{Pic}_X^d := [\mathrm{Spa}\,\mathbb{F}_q/\underline{E}^*] \end{array}$$

which plays the role of the *Abel-Jacobi map* AJ^d. This looks like it's over a point; but when you pull back to any S you see the Abel-Jacobi map for the relative curve.

Remark 18.2.4. The identification $\mathrm{Pic}_X^d := [\mathrm{Spa}\,\mathbb{F}_q/\underline{E}^*]$ depends on a choice of $\mathcal{O}(d)$, which depends on a choice of uniformizing element.

18.3 Hecke correspondences

For $\mu \in X_*(T)/\Gamma$ (just assume G is split), we have a Hecke correspondence $\mathrm{Hecke}^{\le\mu}$

$$\begin{array}{ccc} & \mathrm{Hecke}^{\le\mu} & \\ {}^{h^{\leftarrow}}\swarrow & & \searrow^{h^{\rightarrow}} \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times_{\mathbb{F}_q} \mathrm{Div}_X^1 \end{array}$$

where $\mathrm{Hecke}^{\le\mu}$ is the moduli stack

$$\mathrm{Hecke}^{\le\mu}(S) = \left\{ \begin{array}{c} \mathcal{E}_1, \mathcal{E}_2 = G\text{-bundles} \\ D \in \mathrm{Div}_{X/S}^1 \\ \mu\colon \mathcal{E}_1 \overset{\le\mu}{\dashrightarrow} \mathcal{E}_2 \text{ such that} \\ \mathrm{coker}\,\mu \text{ supported on } D \end{array} \right\}$$

The map $h^{\leftarrow}$ takes $(\mathcal{E}_1, \mathcal{E}_2, D, \mu)$ to $\mathcal{E}_2$ and $h^{\rightarrow}$ takes it to $(\mathcal{E}_1, D)$. The map $h^{\rightarrow}$ is locally (in the pro-étale topology) a fibration in $\mathrm{Gr}_G^{B_{\mathrm{dR}},\le\mu}/\varphi_{E^\diamond}^{\mathbb{Z}}$.

Example 18.3.1. If $E = \mathbb{F}_q((\varpi))$ and G is a reductive group over E, then the affine Grassmannian Gr is an ind-scheme over E, whose functor of points is the sheafification of the presheaf $R \mapsto G(R((T)))/G(R[[T]])$. For R an E-algebra,

$$\mathrm{Gr}^{B_{\mathrm{dR}}} = \varprojlim_{\mathrm{Frob}} \mathrm{Gr}^{\mathrm{ad}}$$

and ind-perfectoid space. We know geometric Satake for IC_μ.

18.4 The conjecture

18.4.1 Setup

Assume $l \neq p$.

- Set ${}^LG = \widehat{G} \rtimes W_E$, where $\widehat{G}$ is the $\overline{\mathbb{Q}}_\ell$-Langlands dual of G.
- Let $\phi\colon W_E \to {}^LG$ be a Langlands parameter and

  $$S_\phi = \mathrm{Aut}(\phi) = \{g \in \widehat{G} \mid g\phi g^{-1} = \phi\}.$$

  We have $Z(\widehat{G})^\Gamma \subset S_\phi$. Suppose ϕ is discrete, so $S_\phi/Z(\widehat{G})^\Gamma$ is finite.
- Fix a Whittaker datum (B, ψ).

18.4.2 The conjecture

There exists a "perverse" Weil sheaf $\mathcal{F}_\phi$ on $\mathrm{Bun}_{G,\overline{\mathbb{F}}_q}$ (with "Weil" structure coming from $\overline{\mathbb{F}}_q$) equipped with an action of S_ϕ, satisfying the following properties.

1. For all $\alpha \in \pi_1(G)_\Gamma$, the action of $Z(\widehat{G})^\Gamma$ on $\mathcal{F}_\phi|_{\mathrm{Bun}^\alpha}$ is given by α via the identification $\pi_1(G)_\Gamma = X^*(Z(\widehat{G})^\Gamma)$.
2. Suppose that ϕ is moreover cuspidal, meaning that the composite map

   $$\begin{array}{ccc} I_E & \longrightarrow & \widehat{G} \\ & {\scriptstyle \phi|_{I_E}}\searrow \quad \nearrow & \\ & {}^LG & \end{array}$$

   has finite image. Then $\mathcal{F}_\phi$ is cuspidal, meaning that

   $$\mathcal{F}_\phi = j_! j^* \mathcal{F}_\phi \quad \text{for} \quad j\colon \mathrm{Bun}^{ss} \hookrightarrow \mathrm{Bun}\,.$$

3. For all $b \in G(\breve{E})_{\text{basic}}$, consider the map:

$$x_b \colon [\operatorname{Spa} \overline{\mathbb{F}}_q / \underline{J_b}(E)] \hookrightarrow \operatorname{Bun}_{G,\overline{\mathbb{F}}_q}$$

The pullback $x_b^* \mathcal{F}_\phi$ has an action of $J_b(E) \times S_\phi$ (and the action of $J_b(E)$ is smooth because $\ell \neq p$). We conjecture that

$$x_b^* \mathcal{F}_\phi \cong \bigoplus_{\substack{\rho \in \widehat{S}_\phi \\ \rho|_{Z(\widehat{G})^\Gamma} = \kappa(b)}} \rho \otimes \pi_{\phi,b,\rho}$$

where $\pi_{\phi,b,\rho}$ is a representation of $J_b(E)$. (The direct sum is finite since ϕ is discrete.) Whatever "perverse" means, it should imply $\pi_{\phi,b,\rho}$ is admissible.

We also predict that $\{\pi_{\phi,b,\rho}\}_\rho$ is an L-packet of a local Langlands correspondence for the extended pure inner form J_b of G. Moreover (which is why we need to fix the Whittaker datum) $\pi_{\phi,1,1}$ is the unique generic representation of its L-packet.

4. (Hecke eigensheaf property) For $\mu \in X_*(T)/\Gamma$, there exists $r_\mu \in \operatorname{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L G)$ with the following "eigenvalue" property. For the Weil sheaf $r_\mu \circ \phi \colon W_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ on $\operatorname{Spa}(E)^\diamond / \varphi_E^{\mathbb{Z}} \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q = \operatorname{Div}^1_{\overline{\mathbb{F}}_q}$, which is equipped with an action of S_ϕ, we have an isomorphism

$$h_!^{\rightarrow}(h^{\leftarrow *} \mathcal{F}_\phi \otimes IC_\mu) \xrightarrow{\sim} S_\phi \boxtimes r_u \circ \phi$$

as Weil sheaves enriched with S_ϕ-action.

5. ("naïve" character sheaf property) For elliptic $\delta \in G(E)$, which implies that $\delta \in G(\breve{E})$ is basic, we get a map

$$\{G(E)\}_{\text{ellip}} \to B(G)_{\text{basic}}$$

which induces

$$x_\delta \colon \operatorname{Spa}(\overline{\mathbb{F}}_q) \to \operatorname{Bun}_{G,\overline{\mathbb{F}}_q}.$$

Then $x_\delta^* \mathcal{F}_\phi$ has Frobenius and Weil structure on S_ϕ. We ask that Frob act like $\delta \in J_\delta(E)$, meaning that if

$$T_\phi \colon \{G(E)\}_{\text{ellip}} \to \overline{\mathbb{Q}}_\ell$$

is the stable distribution over $G(E)$ attached to ϕ, then

$$\delta \mapsto \operatorname{Tr}(\mathrm{Frob}, x_\delta^* \mathcal{F}_\phi).$$

6. (local/global compatibility) "The Caraiani-Scholze sheaf is purely local linked to $\mathcal{F}_\phi$".

Chapter 19

Proof of Langlands for GL(2), II

19.1 Overview

Let $X/\mathbb{F}_q$ be a smooth, projective, geometrically connected curve. The aim is to show that if E is a geometrically irreducible local system of rank 2 on X, then there is a Hecke eigensheaf $\mathrm{Aut}_E =: A_E$ on Bun_2 with eigenvalue E. Under the function-sheaf correspondence this Aut_E gives the automorphic function f_E.

Let us recall the strategy from the very beginning: the rank 1 case. If L is a local system of rank 1 then we knew how to construct a local system on the symmetric power

$$\begin{array}{c} X^{(d)} \\ \downarrow \\ \mathrm{Pic}^d(X) \end{array}$$

We can think of $X^{(d)}$ as the classifying space of line bundles of degree *d plus a section*. There we constructed the sheaf $L^{(d)}$, the symmetric products of L. The idea is that $X^{(d)} \to \mathrm{Pic}^d(X)$ is a fiber bundle with fibers being projective spaces (for large enough d), so any local system descends.

In Stefan's talk, we saw how to construct a candidate function/sheaf A'_E on a space Bun'_2 lying over Bun_2:

$$\begin{array}{c} \mathrm{Bun}'_2 = \{\Omega \hookrightarrow \mathcal{E}\}. \\ \downarrow \\ \mathrm{Bun}_2 \end{array}$$

Just as in the rank 1 case, over a large open subset the fibers are projective spaces, so if the A'_E were a local system we were be done by descent. Unfortunately it is not a local system, so we need to find some other way to descend A'_E to A_E on Bun_2.

Aims. The rest of the argument breaks up into three steps.

1. Show that A'_E is a perverse sheaf.
2. Descend A'_E to A_E on Bun_2.
3. Show that A_E is a Hecke eigensheaf. (This is sort of independent of the other two steps.)

Some of the constructions only work over large open subsets because the map is only a fibration on such. We will happily ignore these issues.

19.2 Construction of the Laumon sheaf

We briefly remind you about the construction of the Laumon sheaf. There is a moduli space Mod^d parametrizing degree d modifications of vector bundles:

$$\mathrm{Mod}^d = \left\{ \begin{matrix} \mathcal{E}, \mathcal{E}' \in \mathrm{Bun}_2 \\ \iota \colon \mathcal{E}' \hookrightarrow \mathcal{E} \\ \deg(\operatorname{coker} \iota) = d \end{matrix} \right\}.$$

We have a map $\mathrm{Mod}^d \to \mathrm{Coh}^d$ sending $(\mathcal{E}' \subset \mathcal{E}) \mapsto (\mathcal{E}/\mathcal{E}')$. This is a smooth map, since the fibers are parametrized by a choice of $\mathcal{E} \in \mathrm{Bun}_2$ plus choices of points in projective spaces specifying the modifications.

But while Mod^d is infinite-dimensional, the space Coh^d is a finite-dimensional space related to symmetric powers of the curve. So we can think of it as a finite-dimensional model for Mod^d. A resolution of singularities for Coh^d is given by specifying a "flag" of torsion sheaves with subquotients of length one (which will be uniquely determined for most torsion sheaves).

$$\begin{array}{ccc} \widetilde{\mathrm{Coh}}^d & = \left\{ \begin{matrix} \mathcal{T}_1 \subset \ldots \subset \mathcal{T}_d \\ \deg \mathcal{T}_{i+1}/\mathcal{T}_i = 1 \end{matrix} \right\} \\ \downarrow \pi & \downarrow \\ \mathrm{Coh}^d & = \{T_d\} \end{array}$$

Oh Coh^d, we defined the Laumon sheaf L_E^d as follows. We have a commutative diagram

$$\begin{array}{ccccc} \widetilde{\mathrm{Coh}}_0^d & \xrightarrow{\mathrm{gr}} & (\mathrm{Coh}_0^1)^d & \longrightarrow & X^d \\ \downarrow \pi & & & & \downarrow \\ \mathrm{Coh}_0^d & & \longrightarrow & & X^{(d)} \end{array}$$

Then we defined

$$\mathcal{L}_E := R\pi_*(\mathrm{gr}^* E^{\boxtimes d})^{S_d}.$$

This is formally similar to what we did in the case of GL_1. Yesterday Stefan defined it slightly differently, as $\mathcal{L}_E = j_{!*} E^{(d)}|_{(X^{(d)} \setminus \Delta)}$. The two definitions turn out to coincnide, giving a way of computing this middle extension sheaf.

19.3 Construction of the sheaf A'_E

As Dennis discussed yesterday, one wants to consider another moduli space

$$Q = \left\{ \begin{matrix} \mathcal{E} \in \mathrm{Bun}_2 \\ \mathcal{J} \in \mathrm{Ext}^1(\mathcal{O}, \Omega) \\ \mathcal{J} \hookrightarrow \mathcal{E} \end{matrix} \right\}$$

Recall that Bun'_2 is the moduli space parametrizing $\{(\mathcal{E} \in \mathrm{Bun}_2, \Omega \hookrightarrow \mathcal{E})\}$. There is a map

$$\nu \colon Q \to \mathrm{Bun}'_2$$

by sending $(\mathcal{J} \hookrightarrow \mathcal{E})$ to $(\mathcal{E}, \Omega \hookrightarrow \mathcal{J} \hookrightarrow \mathcal{E})$.

In addition, we have maps

1. $\mathrm{ext} \colon Q \to \mathbb{A}^1$, sending the datum $(\mathcal{J} \hookrightarrow \mathcal{E})$ to the class of $\mathcal{J}$ in $\mathrm{Ext}^1(\mathcal{O}, \Omega) = H^1(\Omega) \cong H^0(\mathcal{O})$.

2. $q \colon Q \to \mathrm{Coh}$ sending the datum $(\mathcal{J} \hookrightarrow \mathcal{E})$ to the torsion sheaf $\mathcal{E}/\mathcal{J}$.

This all fits together in the following diagram.

$$\begin{array}{ccc} Q & \xrightarrow{\ \mathrm{ext} \times q\ } & \mathbb{A}^1 \times \mathrm{Coh} \\ \downarrow{\scriptstyle \nu} & & \\ \mathrm{Bun}'_2 & & \\ \downarrow & & \\ \mathrm{Bun}_2 & & \end{array}$$

Definition 19.3.1. Let $\mathcal{L}_\chi$ be an Artin-Schreier sheaf on $\mathbb{A}^1$. We define

$$A'_E = \nu_!(\mathrm{ext}^*(AS_{\mathbb{A}^1}) \otimes q^*(\mathcal{L}_E))$$

where $\mathcal{L}_E$ is the Laumon sheaf on Coh.

19.4 Perversity

We want to convine you that A'_E is a perverse sheaf. To do this, it would suffice to rewrite it as an iterated sequence of Laumon's Fourier transforms, since we know that Fourier transforms latter preserves perversity.

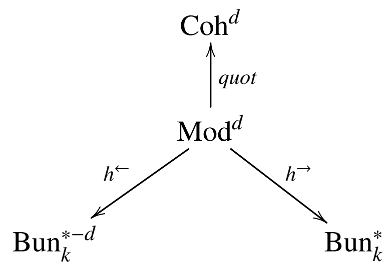
We need the following basic vanishing result.

Lemma 19.4.1. *For all $k < n$ and all bundles $\mathcal{E}, \mathcal{E}' \in \mathrm{Bun}_k$ with $\deg \mathcal{E}' \leq \deg \mathcal{E} - d$ with $d > kn(2g-2)$, we have*

$$H^*(\mathrm{Hom}^{inj}(\mathcal{E}', \mathcal{E}) \subset \mathrm{Mod}, q^*\mathcal{L}_E) = 0.$$

Example 19.4.2. Let's unravel the statement of the lemma for $n = 2$. By twisting, we can assume that $\mathcal{E}'$ is trivial. Then the claim is that on the space $\mathcal{H}^0(\mathcal{E}) - 0$, the cohomology of the Laumon sheaf vanishes.

Here is an equivalent formulation. Consider the diagram



Define the averaging functor

$$Av_E : D^b(\mathrm{Bun}_k) \to D^b(\mathrm{Bun}_k)$$

by

$$K \mapsto h_!^{\rightarrow}(h^{\leftarrow *}(K) \otimes quot^* \mathcal{L}_E).$$

Then the claim is that this is identically 0 for $d > kn(2g-2)$ and E irreducible of rank $n > k$. This goes back to Dennis's "formula" for producing Hecke eigensheaves from yesterday: the formula is to pull back, convolve with the Laumon sheaf, and push forward. This is telling us that if you perform this process when the rank is too small, so that you don't expect to get any eigensheaves, then you'll get 0.

For $n = 2, k = 1$ the statement is easy: for $E^{(d)}$ on $X^{(d)} \xrightarrow{AJ} \mathrm{Pic}^d$, we have

$$R(AJ_*)E^{(d)} = 0 \text{ if } E \text{ is irreducible.}$$

This is a result of Deligne.

Proof sketch. First check that this is a local system by checking that the map is locally acyclic. Then since Pic has abelian fundamental group, if there is a non-trivial cohomology sheaf we can tensor with a local system of rank 1 to make one summand trivial, so in particular it has sections. But then one would find a non-trivial cohomology group upstairs on $X^{(d)}$ by the Leray spectral sequence.

On the other hand, one can compute the cohomology using the Künneth formula on X. By duality the only non-zero group must be in H^1, and the cohomology of the symmetric power is the exterior power of the cohomology, which vanishes for dimension reasons. □

♠♠♠ TONY: [there was then a discussion of the Fourier transform, but I could not follow it]

19.5 The descent step

An irreducible perverse sheaf is a local system over an open subset. Then we can apply this theorem if we know that the open subset contains one pulled back from downstairs. How do we prove something like this?

The general trick is that if you have a perverse sheaf A' on a smooth space, it is locally constant if and only if the Euler characteristics of stalks are constants. Why? Assume that A' is irreducible (hence an IC sheaf); it's easy to reduce to this case. Then over an open set it's a local system. We know that A' is a middle extension (since it's an IC sheaf), so think about what happens when we form the middle extension. Since we're on a smooth space, the only action happens at the codimension one stalks, and *there you take invariants under inertia*, so the Euler characteristic can only stay constant if it extends to a local system in codimension one.

So we want the Euler characteristics to be constant along fibers of the map $\mathrm{Bun}' \to \mathrm{Bun}$. In the Frenkel-Gaitsgory-Vilonen article this is argued as follows. If we hadn't constructed our sheaf by Fourier transform, but instead by a procedure that *only uses pushforward from proper maps*, then we would know that the Euler characteristic only depends on the local isomorphism classes of the input sheaves. (In characteristic 0, this is clear from the topology perspective because, since the sheaves are locally isomorphic, we can take a small triangulation in which they are isomorphic, and then the Euler characteristics would coincide because cohomology can be computed locally. In characteristic p, it follows by reduction to the case of curves and using the Grothendieck-Ogg-Shafarevich formula.)

To summarize, the idea is to show that this is independent of E! This holds because we can rewrite the construction using proper maps only. That's where the Drinfeld compactification is useful. Recall that this was a compactification of Q by cutting up $\mathcal{E}$ into a flag with subquotients being powers of Ω; in terms of Plücker coordinates it could be described (for GL_2) as

$$\overline{Q}^d = \left\{ \begin{smallmatrix} \Omega \hookrightarrow \mathcal{E} \\ \Omega \otimes \mathcal{O} \hookrightarrow \wedge^2 \mathcal{E} \end{smallmatrix} \right\}$$

$$\begin{array}{ccc} & & \{\mathcal{J} \subset \mathcal{E}\} \\ & \swarrow & \downarrow \\ \overline{Q}^d = \left\{ \begin{smallmatrix} \Omega \hookrightarrow \mathcal{E} \\ \Omega \otimes \mathcal{O} \hookrightarrow \wedge^2 \mathcal{E} \end{smallmatrix} \right\} & & \mathrm{Mod}^d \\ \downarrow & \swarrow & \\ \mathrm{Bun} & & \end{array}$$

where the left down map is proper after dividing by $\mathbb{G}_m$s. You then rewrite the construction in terms of $\overline{Q}^d$.

The upshot of this discussion is that *we only need the result for one E* (irreducible or not). Now there are several options. For example, we could try the trivial bundle. Then we would get a purely geometric statement, which is unclear how to prove. In the paper, Frenkel-Gaitsgory-Vilonen give the following argument instead. We took the Fourier transform, so we know that if we start with a pure local system then we end up with a pure perverse sheaf. So to get constancy along fibers *it suffices to show that the trace function is constant along fibers*. So we need a local system E such that f'_E comes from f_E downstairs. That is, all we need an automorphic function for *one* local system. We could try to construct this by cyclic base change, which is what they do, but it is hard!

There's also a different argument by Gaitsgory, which goes by comparing the construction with Eisenstein series for a generic $E = \bigoplus L_i$. If you look at how people construct geometric Eisenstein series, then you see that they also use bundles with flags, so one could expect a comparison. The first problem with this approach is that the identity $j_! = j_*$ no longer holds for reducible local systems. But recall how we proved this in rank 2: it was again a computation that something was locally acyclic. So again the we can try to prove that the Euler characteristic doesn't depend on the local system. In the end, it comes down to comparing the intermediate extension and extension-by-zero using the Euler characteristic.

Anyway, this proves that A'_E descends.

19.6 The Hecke eigensheaf property

We first reduce to checking the eigensheaf property for the first Hecke operator T_1, which comes from the correspondence

$$\begin{array}{ccc} & \{(x, \mathcal{E}' \subset \mathcal{E}) \colon \mathcal{E}/\mathcal{E}' \cong k(x)\} & \\ \swarrow & & \searrow \\ \mathrm{Bun}_n^d & & \mathrm{Bun}_n^{d-1} \times X \end{array}$$

The point is that the S_2 symmetry implies that this is an eigensheaf for all Hecke operators.

Example 19.6.1. Consider modifications of length 2 for rank 2 bundles.

$$\begin{array}{ccc} & \mathrm{Hecke} = \left\{ {\mathcal{E}' \hookrightarrow \mathcal{E} \atop \text{length } \mathcal{E}/\mathcal{E}' = 2} \right\} & \\ \swarrow & & \searrow \\ \mathrm{Bun}_2^d & & \mathrm{Bun}_2^{d-2} \times X^{(2)} \end{array}$$

The Hecke eigensheaf property describes what happens if we start with A_E viewed as a perverse sheaf on Bun_2, pull it back to Hecke and convolve with the IC sheaf, and then apply proper pushforward. Consider the diagram

$$\begin{array}{ccccc}
 & \widetilde{\mathrm{Hecke}} = \{\mathcal{E}' \subset \mathcal{E}_1 \subset \mathcal{E}\} & \longrightarrow & \mathrm{Bun}_2^{d-2} \times \widetilde{\mathrm{Coh}} & \longrightarrow & \mathrm{Bun}_2 \times X \times X \\
 & \downarrow{\scriptstyle \pi} & & \downarrow & \swarrow & \\
 & \mathrm{Hecke} & \longrightarrow & \mathrm{Bun}_2^{d-2} \times \mathrm{Coh} & & \\
\swarrow & & \searrow & \downarrow & & \\
\mathrm{Bun}_2^{d} & & & \mathrm{Bun}_2^{d-2} \times X^{(2)} & &
\end{array}$$

Instead of pulling and pushing, consider going around the top of the diagram. The only difference is that we get $A_E \boxtimes E \boxtimes E$ pushing up through the top of the diagram. The map $\widetilde{\mathrm{Hecke}} \to \mathrm{Hecke}$ is a small resolution, which is locally modelled by $\widetilde{\mathrm{Coh}} \to \mathrm{Coh}$. Therefore, for $n = 2$ the fibers of π are finite except over the diagonal points $k(x)^{\oplus 2}$, where they are $\mathbb{P}^1$. So we know that the pushforward is a sum of perverse sheaves (which are their own middle extensions). Then $R\pi_*\mathbb{Q}$ has an action of S_2, and $(R\pi_*\mathbb{Q})^{S_2}$ gives the Hecke operator T^2 supported on the diagonal.

♠♠♠ TONY: [I didn't understand this example.]

So how do you check the Hecke eigensheaf property for T_1? The easiest thing to say in 2 minutes is that we just compute. Perhaps we should also say the Laumon sheaf has a Hecke property. You also check this by computation. A useful approach is to use the diagram

$$\begin{array}{ccc}
 & [\mathcal{J}_{d-1} \subset \mathcal{J}_d] & \\
\swarrow & & \searrow \\
\mathrm{Coh}_0^d & & \mathrm{Coh}_0^{d-1}
\end{array}$$

and check that it transform $\mathcal{L}_E^d$ to $\mathcal{L}_E^{d-1} \boxtimes L_E^1 = E$.

Chapter 20

Discussion Session: Diamonds for the Perplexed

These are notes for a 20-minute discussion session by Dennis Gaitsgory on the basics of diamonds and their application to the moduli stack of bundles on the Fargues-Fontaine curve.

20.1 Diamonds for the perplexed

The "diamondification" $Y \rightsquigarrow Y^\diamond$ is a functor

$$\text{Presheaves on } \mathrm{Perf}_{\mathrm{Spa}\, E} \to \text{Presheaves on } \mathrm{Perf}_{\overline{\mathbb{F}}_q}.$$

What is $Y^\diamond$? Its functor of points is

$$Y^\diamond(R) = \begin{Bmatrix} R^\# = \text{ untilt of } R/\mathbb{Q}_p \\ \mathrm{Spa}(R^\#) \to Y/\,\mathrm{Spa}\, E \end{Bmatrix}.$$

Suppose Y is representable, i.e. is given by a perfectoid space over $\mathrm{Spa}\, E$. In this case, we claim that $Y^\diamond$ is also representable, namely by $Y^\flat$.

Proof. The R-points are untilts $R^\#$ of R over $\mathrm{Spa}\, E$. There is a theorem due to Scholze that perfectoid spaces over a perfectoid Y are equivalent to those over $Y^\flat$.

$$\begin{array}{ccc} R^\# & \longleftrightarrow & R \\ \downarrow & & \downarrow \\ Y & & Y^\flat \end{array}$$

□

Definition 20.1.1. A *diamond* $\mathcal{Z}$ is a presheaf on $\mathrm{Perf}_{\mathbb{F}_q}$ such that there exists a map $Y \to \mathcal{Z}$ such that Y is representable and the morphism is representable and quasi-profinite covering.

The point is not a diamond.

Lemma 20.1.2. *Let* $S \in \mathrm{Perf}_{\mathbb{F}_q}$. *Then*

$$(Y_{S,E})^{\diamond} = S \times (\mathrm{Spa}\, E)^{\diamond}$$

Proof. Assume $S = \mathrm{Spa}(R, R^+)$. Let's check the functor of points on $\mathrm{Spa}\, B \in \mathrm{Perf}_{\mathbb{F}_p}$. For the left side, we have by definition

$$Y_{S,E}^{\diamond}(B) = \left\{ \begin{matrix} B^{\#} / \mathrm{Spa}(E) \\ \iota \colon (B^{\#})^{\flat} \cong B \\ \mathrm{Spa}\, B^{\#} \to Y_{S,E} \end{matrix} \right\}.$$

At the level of rings, a map $\mathrm{Spa}\, B^{\#} \to Y_{S,E}$ is the same as $W_E(R^+) \to B^{\#+}$ satisfying an invertibility condition, which by the adjunction is the same as $R^+ \to B^{\#+\flat} = B^+$.

On the right side, we have by definition

$$S \times (\mathrm{Spa}\, E)^{\diamond}(B) = \left\{ \begin{matrix} B^{\#} / \mathrm{Spa}(E) \\ \iota \colon (B^{\#})^{\flat} \cong B \\ R^+ \to B^+ \end{matrix} \right\}.$$

These look more or less the same. The only thing left is to carefully track the invertibility condition that comes packaged in with the map $\mathrm{Spa}\, B^{\#} \to Y_{S,E}$. The condition is that the map $W_E(R^+) \to B^{\#+}$ must not kill ϖ or $[\varpi^{\flat}]$. But ϖ is a unit in E, hence also in $B^{\#}$, so that is built into the the fact that $B^{\#}$ is over $\mathrm{Spa}(E)$, and the analogous condition for $\varpi^{\flat}$ comes from similar reasoning with respect to the isomorphism $\iota \colon (B^{\#})^{\flat} \cong B$.

□

20.2 The stack Bun_G

20.2.1 The classical version

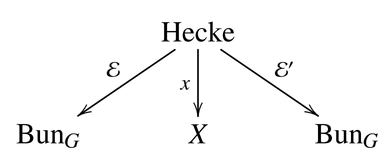
Bun_G is a presheaf on $\mathrm{Perf}_{\mathbb{F}_q}$, with $\mathrm{Bun}_G(S)$ is the groupoid of G-bundles on Y_S equivariant with respect to Frob_S.

There is a map $\mathrm{Gr}_G^{B_{\mathrm{dR}}} \to \mathrm{Bun}_G \times (\mathrm{Spa}\, E)^{\diamond}$. This is supposed to be smooth after taking some kind of Bott-Samuelson resolution of the affine Grassmannian.

In the usual world of algebraic geometry, there is a Hecke stack Hecke which parametrizes

$$S \mapsto \left\{ (\mathcal{E}, \mathcal{E}', x, \iota) \colon \begin{matrix} x \colon S \to X \\ \mathcal{E}, \mathcal{E}' = G\text{-bundles}/S \times X \\ \iota \colon \mathcal{E} \cong \mathcal{E}'|_{S \times X - \Gamma_x} \end{matrix} \right\}.$$

We have maps



20.2.2 The diamond version

We denote a mirror curve $X' = \operatorname{Spa} E^\diamond / \operatorname{Frob}$. What does this mean?

$$X'(S) = \left\{ \begin{matrix} S^\#/E \\ \iota\colon (S^\#)^\flat \cong S \end{matrix} \right\} / \sim$$

where the equivalence relation is for the action of Frobenius on ι.

Again we have maps

$$\begin{array}{ccccc} & & \mathrm{Hecke} & & \\ & \swarrow & \downarrow & \searrow & \\ \mathrm{Bun}_G & & X' & & \mathrm{Bun}_G \end{array}$$

where Hecke has the functor of points

$$\mathrm{Hecke}(S) = \left\{ \begin{matrix} \mathcal{E}, \mathcal{E}' = G\text{-bundles} / Y_S, \operatorname{Frob}_S\text{-equivariant} \\ \iota\colon (S^\#)^\flat \cong S \leftrightarrow S^\# \xrightarrow{\theta} Y_S \\ \mathcal{E} \cong \mathcal{E}' \text{ away from } \theta \text{ and Frobenius translates} \end{matrix} \right\} / \sim$$

The datum is up to Frobenius, hence the map to X'.

Chapter 21

Discussion Session: Function Field Analogues

These are notes for a discussion session given by Urs Hartl on function field analogues. For references, see articles 7, 12 (dictionary), 13, 15, 17, 23, 24 (survey) on www.math.uni-muenster.de/u/urs.hartl/Public.html.en.

21.1 The Fargues-Fontaine Curve

In constructing the curve, we start with a base local field which is either of mixed or equal characteristics.

Number Field	Function Field
$\mathbb{Q}_p$	$\mathbb{F}_p((z))$

There is also the input of a perfectoid field. Whereas in the mixed case we have a field $\mathbb{C}_p/\mathbb{Q}_p$ and a $\mathbb{C}_p^\flat$ over a characteristic p field, in the equal case we start with two symmetric local rings in characteristic p.

Number Field	Function Field
$\mathbb{Q}_p$	$\mathbb{F}_p((z))$
$\mathbb{C}_p/\mathbb{Q}_p \leftrightarrow \mathbb{C}_p^\flat/(\text{char p})$	$\mathbb{F}_p((z)) \leftrightarrow \mathbb{F}_p((\zeta))$

While in characteristic 0 we build the Witt vectors of some algebra R, in equal characteristic we build the power series ring.

Number Field	Function Field	Remarks
$\mathbb{Q}_p$	$\mathbb{F}_p((z))$	
$\mathbb{C}_p/\mathbb{Q}_p \leftrightarrow \mathbb{C}_p^\flat/(\text{char p})$	$\mathbb{F}_p((z)) \leftrightarrow \mathbb{F}_p((\zeta))$	
$W(R^+)$	$R^+[[z]]$	R^+ = algebra $/\mathbb{F}_p[[\zeta]]$

We can imagine for simplicity that $R^+ = \mathbb{F}_p[[\zeta]]$, so $R = \mathbb{F}_p((\zeta))$. To build the curve we take the adic spectrum, and then poke out two points.

Number Field	Function Field
$\mathbb{Q}_p$	$\mathbb{F}_p((z))$
$\mathbb{C}_p/\mathbb{Q}_p \leftrightarrow \mathbb{C}_p^\flat/(\text{char p})$	$\mathbb{F}_p((z)) \leftrightarrow \mathbb{F}_p((\zeta))$
$W(R^+)$	$R^+[[z]]$ (imagine $R^+ = \mathbb{F}_p[[\zeta]]$)
$Y_{(R,R^+)} = \operatorname{Spa}(W(R^+), W(R^+)) \setminus V(p[\varpi^\flat])$	$Y_{(R,R^+)} = \operatorname{Spa}(R^+[[z]], R^+[[z]]) \setminus V(z\zeta)$

There are two Frobenii acting in the equal characteristic case. We denote by φ_R the one which takes $\zeta \mapsto \zeta^p$ and $z \mapsto z$. In equal characteristic we have $Y^{\text{ad}}_{(R,R^+)} = \operatorname{Spa}(R, R^+) \times_{\mathbb{F}_p} \operatorname{Spa} E$. Now we quotient by Frobenius.

Number Field	Function Field
$\mathbb{Q}_p$	$\mathbb{F}_p((z))$
$\mathbb{C}_p/\mathbb{Q}_p \leftrightarrow \mathbb{C}_p^\flat/(\text{char p})$	$\mathbb{F}_p((z)) \leftrightarrow \mathbb{F}_p((\zeta))$
$W(R^+)$	$R^+[[z]]$ (imagine $R^+ = \mathbb{F}_p[[\zeta]]$)
$Y_{(R,R^+)} = \operatorname{Spa}(W(R^+), W(R^+)) \setminus V(p[\varpi^\flat])$	$Y_{(R,R^+)} = \operatorname{Spa}(R^+[[z]], R^+[[z]]) \setminus V(z\zeta)$
$X^{\text{ad}}_{(R,R^+)} = Y_{(R,R^+)}/\varphi_R$	$X^{\text{ad}}_{(R,R^+)} = Y_{(R,R^+)}/\varphi_R$

Vector bundles on X^{ad} are vector bundles $\mathcal{E}$ on Y^{ad} plus an equivariant structure $\varphi_{\mathcal{E}} \colon \varphi^*\mathcal{E} \cong \mathcal{E}$. In equal characteristic,

$$H^0(Y^{\text{ad}}_{(R,R^+)}, \mathcal{O}_{Y^{\text{ad}}}) = \left\{ \sum_{i=-\infty}^{\infty} b_i z^i, \quad b_i \in R \mid \text{ convergent on } 0 < |z| < 1 \right\}.$$

The radius function is the distance to $z = 0$. We define Y^I to be the (adic) spectrum of the ring of power series convergent on $\{|z| \in I\}$.

We have a bundle $\mathcal{O}(d)$ corresponding to $\mathcal{E} = (\mathcal{O}_{Y^{\text{ad}}}, \varphi_{\mathcal{E}} = \cdot z^{-d})$. It is relatively easy to write down global sections in equal characteristic. In mixed characteristic it is much harder, because the elements are not truly power series.

Number Field	Function Field
$\mathbb{Q}_p$	$\mathbb{F}_p((z))$
$\mathbb{C}_p/\mathbb{Q}_p \leftrightarrow \mathbb{C}_p^\flat/(\text{char p})$	$\mathbb{F}_p((z)) \leftrightarrow \mathbb{F}_p((\zeta))$
$W(R^+)$	$R^+[[z]]$ (imagine $R^+ = \mathbb{F}_p[[\zeta]]$)
$Y_{(R,R^+)} = \operatorname{Spa}(W(R^+), W(R^+)) \setminus V(p[\varpi^\flat])$	$Y_{(R,R^+)} = \operatorname{Spa}(R^+[[z]], R^+[[z]]) \setminus V(z\zeta)$
$X^{\text{ad}}_{(R,R^+)} = Y_{(R,R^+)}/\varphi_R$	$X^{\text{ad}}_{(R,R^+)} = Y_{(R,R^+)}/\varphi_R$
$\mathcal{O}(d) = (\mathcal{O}_{Y^{\text{ad}}}, \varphi_{\mathcal{E}} = \cdot p^{-d})$	$\mathcal{O}(d) = (\mathcal{O}_{Y^{\text{ad}}}, \varphi_{\mathcal{E}} = \cdot z^{-d})$
Hard to write sections	Easy to write sections

21.2 p-divisible groups

The basic analogues are tabulated below.

Number Field	Function Field
p-divisible groups / R^+	divisible local Anderson modules / R^+
Dieudonné modules	effective local shtukas / R^+

However there are some differences. For instance, the functor from p-divisible groups to Dieudonné modules is not known to be fully faithful in general. However, the functor from divisible local Anderson modules to effective local shtukas is fully faithful.

Now what are the things on the right side anyway? Divisible local Anderson modules are too messy to define. However, we can say define an effective local shtukas.

Definition 21.2.1. An *effective local shtuka* is a pair $\underline{M} = (M, \varphi_M)$ with M a finite projective over $R^+[[z]]$ and

$$\varphi_M \colon \varphi^* M[\frac{1}{z-\zeta}] \cong M[\frac{1}{z-\zeta}].$$

Here

$$\varphi^* M[\frac{1}{z-\zeta}] = M \otimes_{R^+[[z]],\varphi_R} R^+[[z]].$$

Thus φ_M is the linearization of a $\varphi = \varphi_R$-linear map. We think of this as analogous to a modification between a vector bundle and its Frobenius twist on A_{inf}, which is an isomorphism away from the points $z - \zeta$. Finally, the "effective" means that we demand

$$\varphi_M(\varphi^* M) \subset M.$$

We can't really describe the functor from local divisible Anderson modules since we didn't even say what those were, but we remark that if G is such, then it is related to the corresponding M by

$$(\mathrm{Lie}\, G)^\vee = M/\varphi_M(\varphi^* M). \tag{21.1}$$

Example 21.2.2. Let $M = (R^+[[z]], \varphi_M = (z - \zeta))$. This is the analogue of the p-divisible group μ_{p^∞}. The divisible local Anderson module would be $\widehat{\mathbb{G}}_{a,R}$ with an $\mathbb{F}_p[[z]]$-action, where z acts by $[z](X) = \zeta X + X^p$ because $\varphi_M = z - \zeta$. This is a Lubin-Tate formal group.

In view of 21.1 we have $\mathrm{Lie}\, G = R^+[[z]]/(z-\zeta)$, soo $[z]|_{\mathrm{Lie}\, G} = \zeta$. This is plausible if you remember the analogy $z \leftrightarrow p$, and that for a p-divisible group the action of multiplication by p induces multiplication by p on the Lie algebra.

21.3 B_{dR}

Recall that

$$Y^{\mathrm{ad}}_{(R,R^+)} = \mathrm{Spa}(R^+[[z]], R^+[[z]]) \setminus V(z\zeta).$$

There is a closed subset $V(z-\zeta)$ which induces

$$\theta\colon R^+[[z]] \to R$$

sending $z \mapsto \zeta$.

In mixed characteristic we defined the ring

$$B_{\mathrm{dR}}^+(R) = \varprojlim W(R^+)[1/[\varpi^\flat]]/(\xi^n).$$

We think about this as the completion of $Y^{\mathrm{ad}}_{(R,R^+)}$ along a closed subvariety of codimension one:

$$B_{\mathrm{dR}}^+(R) = (\mathcal{O}_{Y^{\mathrm{ad}}_{(R,R^+)},V(\xi)})^\wedge.$$

In the equal characteristic side, we define

$$B_{\mathrm{dR}}^+ = \varprojlim R^+[[z]][1/\zeta]/(z-\zeta)^n = (\mathcal{O}_{Y^{\mathrm{ad}}_{(R,R^+)},V(z-\zeta)})^\wedge.$$

Note that this is simply isomorphic to $R[[z-\zeta]]$; this is analogous to how for $R = \mathbb{C}_p^\flat$ then on the left we get in the classical case an isomorphism of rings $B_{\mathrm{dR}}^+(\mathbb{C}_p^\flat) \cong \mathbb{C}_p[[\xi]]$.

21.4 The Period Map

Let $\underline{M}$ be a local shtuka over (R,R^+). We define its *de Rham cohomology* to be

$$H^1_{\mathrm{dR}}(\underline{M}, B_{\mathrm{dR}}^+(R)) := \varphi^* M \otimes_{R^+[[z]]} R[[z-\zeta]].$$

To put this in its proper context, let's remember the analogies going on here. Imagining $R = \mathbb{F}_q[[\zeta]]$, the ring $R^+[[z]]$ is analogous to $A_{\mathrm{inf}} = W(R^+)$ in mixed characteristic, and φ^*M is a vector bundle over it. Then $R[[z-\zeta]]$ is analogous to B_{dR}^+ in mixed characteristic, which we can thought of as the completion of A_{inf} along the point at ∞ describing an untilt.

The upshot is that if we think of M as a vector bundle on A_{inf}, then we are defining its de Rham cohomology is the restriction to a formal disk about infinity.

Think about this from the perspective of the curve. For a vector bundle on Y, restricting to a neighborhood of infinity gives a B_{dR}^+-module, while restricting to its complement gives a B_{cris}^+-module. Vector bundles on the curve correspond precisely, by a Beauville-Laszlo uniformization interpretation, to $(B_{\mathrm{dR}}^+, B_{\mathrm{cris}}^+)$ modules.

$$H^1_{\mathrm{cris}}(\underline{M}, R^+/(\zeta)[[z]]) := \varphi^* M \otimes_{R^+[[z]]} R^+/(\zeta)[[z]].$$

There is a comparison between crystalline and de Rham cohomology by the Genestevier-Lafforgue Lemma. It says the following. Let $R^+ = k[[\zeta]]$. Then there is a map

$$R^+/(\zeta)[[z]] = k[[z]] \to B_{\mathrm{dR}}^+ = k((\zeta))[[z-\zeta]]$$

given by

$$z \mapsto z = \zeta + (z - \zeta)$$

and the de Rham and crystalline cohomologies, as defined above, become isomorphic for this comparison.

There is also an étale cohomology group $H^1_{\text{ét}}(\underline{M}, \mathbb{F}_p[[z]])$. To describe it, first tensor $M \otimes_{k[[z]]} k((\zeta))^{\text{sep}}[[z]]$. Frobenius is an isomorphism after inverting $z - \zeta$, which is indeed invertible here, so we can take

$$H^1_{\text{ét}}(\underline{M}, \mathbb{F}_p[[z]]) := (M \otimes_{k[[z]]} k((\zeta))^{sep}[[z]])^{\varphi=1}.$$

Finally, let's discuss the period morphism. Consider

$$H^1_{\text{dR}}(\underline{M}, B_{\text{dR}}) := \varphi^* M \otimes_{R^+[[z]]} R[[z-\zeta]][\frac{1}{z-\zeta}].$$

There is a submodule

$$\varphi_M^{-1}(M \otimes_{R^+[[z]]} R[[z-\zeta]]) \subset H^1_{\text{dR}}(\underline{M}, B_{\text{dR}}).$$

This is called the *Hodge-Pink lattice*; it corresponds to $\text{Fil}^0 H^1_{\text{dR}}(\underline{M}, B_{\text{dR}})$ for the Hodge filtration.

The period map takes $\underline{M}$ to its Hodge-Pink lattice. But to make sense of this we have to say in which ambient space this lattice varies - that is, we have to "fix" $H^1_{\text{dR}}(\underline{M}, B_{\text{dR}})$. There is a Rapoport-Zink space of deformations of a fixed $\underline{\mathbb{M}}$, parametrizing pairs

$$(\underline{M}, \underline{M}_{R^+/\zeta} \cong \underline{\mathbb{M}}_{R^+/\zeta}).$$

By the crystalline nature of the cohomology functors, the isomorphism modulo p lifs canonically to an isomorphism $H^1_{\text{dR}}(\underline{M}, B_{\text{dR}}) \cong H^1(\underline{\mathbb{M}}, B_{\text{dR}})$.

Chapter 22

The Case of $\mathbb{G}_m$

22.1 The setup

22.1.1 Structure of Bun_G

Let $E/\mathbb{Q}_p$ be a finite field of residue field $\mathbb{F}_q$ and ϖ a uniformizer. We take $G = \mathrm{GL}_1$. The Kottwitz invariant decomposes

$$\mathrm{Bun}_G = \mathrm{Pic} = \coprod_{d \in \pi_1(\mathbb{G}_m) = \mathbb{Z}} \mathrm{Pic}^d .$$

Obviously every line bundle is semistable, so by Laurent's talk

$$\mathrm{Pic}^d = \mathrm{Pic}^{d,ss} = [\mathrm{Spa}(\mathbb{F}_q)/\underline{E}^*]. \tag{22.1}$$

The choice of ϖ allows us to define a line bundle $\mathcal{O}(1)$, hence $\mathcal{O}(d)$ for all $d \in \mathbb{Z}$, on X. On the other hand, we also have the universal degree d line bundle $\mathcal{E}_d$ on the relative curve "X_{Pic^d}".

In the identification of (22.1), $\mathcal{T}_d = \mathrm{Isom}(\mathcal{O}(d), \mathcal{E}_d)$ is the universal $\underline{E}^*$-torsor on the classifying stack $\mathrm{Pic}^d = [\mathrm{Spa}(\mathbb{F}_q)/\underline{E}^*]$.

22.1.2 The Hecke correspondences

Recall that in general we defined $\mathrm{Hecke}^{\leq \mu}$ for $\mu \in X_*(T)_+/\Gamma$; for GL_1 there is no need for $\leq$ since there are no closure relations, and $\mu = k \in \mathbb{N}$ tells us the length of the quotient.

We have a Hecke correspondence of invariant k:

$$\begin{array}{ccc} & \mathrm{Hecke}^{k,d} & \\ {}^{h^{\leftarrow}}\swarrow & & \searrow^{h^{\rightarrow}} \\ \mathrm{Pic}^{d+k} & & \mathrm{Pic}^d \times \mathrm{Div}^1 \end{array}$$

where $\mathrm{Div}^1 = \mathrm{Spa}\, E^{\diamond}/\varphi_{E^{\diamond}}$. The Hecke stack parametrizes

$$\{(\mathcal{E}_1, \mathcal{E}_2, D, f \colon \mathcal{E}_1 \to \mathcal{E}_2) \mid \text{length } \mathcal{E}_2/\mathcal{E}_1 = k\}.$$

Lemma 22.1.1. *The morphism $h^{\rightarrow}$ is an isomorphism.*

Proof. Given a line bundle and a divisor, there is a unique choice of $\mathcal{E}_2$ gven $\mathcal{E}_1$ and f; in fact we can say that $\mathcal{E}_2 = \mathcal{E}_1(kD)$. □

22.2 Lubin-Tate theory

Let's see what happens if we apply these Hecke operators to the torsors $\mathcal{T}_d$. Before doing this, we make a definition.

Definition 22.2.1. Fix a Lubin-Tate formal group $\mathcal{G}/\mathcal{O}_E$ attached to ϖ. Denote by $E(1)$ its rational Tate module, and $E(d) := E(1)^{\otimes d}$ for $d \in \mathbb{Z}$. This gives us a rank 1 local system for the pro-étale topology on the diamond $\mathrm{Spa}(E)^{\diamond}/\varphi_{E^{\diamond}}^{\mathbb{Z}}$.

Proposition 22.2.2. *There is a natural isomorphism of $\underline{E}^*$-torsors on* $\mathrm{Hecke}^{k,d}$*:*

$$h^{\leftarrow *}\mathcal{T}_{d+k} \cong h^{\rightarrow *}(\mathcal{T}_d \times^{E^*} \mathcal{LT}_{-k})$$

where $\mathcal{LT}_k$ is the $\underline{E}^*$-torsor associated with $E(k)^{\diamond}$ (i.e. $\mathrm{Isom}(\mathcal{O}, E(k)^{\diamond})$*).*

Proof. On $\mathrm{Hecke}^{k,d}$ we have the universal modification sequence

$$0 \to (p_1 \circ h^{\rightarrow})^*\mathcal{E}_d \to h^{\leftarrow *}\mathcal{E}_{d+k} \to B_{\mathrm{dR}}^+/\,\mathrm{Fil}^k \to 0.$$

On the other hand, we have the fundamental exact sequence from p-adic Hodge theory:

$$\text{“}0 \to \mathcal{O}(d) \to \mathcal{O}(d+k) \to B_{\mathrm{dR}}^+/\,\mathrm{Fil}^k \to 0.\text{”}$$

However, we need to be careful because of the Galois structure: the map $\mathcal{O}(d) \to \mathcal{O}(d+k)$ is multiplication by a section of $\mathcal{O}(1)$, so there is some Galois twisting. The correct exact sequence is actually

$$0 \to \mathcal{O}(d) \otimes_E E(k)^{\diamond} \to \mathcal{O}(d+k) \to B_{\mathrm{dR}}^+/\,\mathrm{Fil}^k \to 0.$$

Since the appearance of $E(k)^{\diamond}$ is the essential point, let us say more about it. A section of $\mathcal{O}(1)$ is a period of $E(1)$, such as the familiar element from p-adic Hodge theory:

$$t = \log[(1, \zeta_p, \zeta_{p^2}, \ldots)]$$

What we're using here is that the Dieudonné module of the Lubin-Tate formal group is $\mathcal{O}(1)$.

Now compare the two sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (p_1 \circ h^{\rightarrow})^*\mathcal{E}_d & \longrightarrow & h^{\leftarrow *}\mathcal{E}_{d+k} & \longrightarrow & B_{\mathrm{dR}}^+/\,\mathrm{Fil}^k & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{O}(d) \otimes E(k)^{\diamond} & \longrightarrow & \mathcal{O}(d+k) & \longrightarrow & B_{\mathrm{dR}}^+/\,\mathrm{Fil}^k & \longrightarrow & 0 \end{array}$$

Now the point is that there is an equivalence between the isomorphisms of the middle terms and isomorphisms of the left terms. But by definition $\mathcal{T}_{d+k} = \mathrm{Isom}(\mathcal{E}_{d+k}, \mathcal{O}(d+k))$ and $\mathcal{T}_d = \mathrm{Isom}(\mathcal{E}_d, \mathcal{O}_d)$. The result follows from observing that $\mathrm{Isom}(\mathcal{O}(d) \otimes E(k)^{\diamond}, \mathcal{E}_d) = \mathrm{Isom}(\mathcal{O}(d), \mathcal{E}_d \otimes E(-k)^{\diamond})$ corresponds to the torsor $\mathcal{T}_d \times^{\underline{E}^*} \mathcal{LT}_{-k}$. □

22.3 Fargues' Conjecture for $\mathbb{G}_m$

22.3.1 The eigensheaf

Definition 22.3.1. Define the Weil $\underline{E}^*$-torsor $\mathcal{F}$ on $\mathrm{Pic}_{\overline{\mathbb{F}}_q}$ by

$$\mathcal{F}|_{\mathrm{Pic}^d_{\overline{\mathbb{F}}_q}} := \mathcal{T}_d.$$

Define a Weil descent datum

$$\mathrm{Frob}^* \mathcal{T}_d \cong \mathcal{T}_d$$

to be ϖ^{-d} times the canonical descent datum coming from the fact that $\mathcal{T}_d$ is defined over $\mathbb{F}_q$.

Now let $\phi \colon W_E \to \overline{\mathbb{Q}}_\ell^*$ be a continuous character. Let

$$\chi \colon E^* \xrightarrow{\sim} W_E^{\mathrm{ab}} \xrightarrow{\phi} \overline{\mathbb{Q}}_\ell^*$$

with the first map being the Artin map. In the notation of Fargues' conjecture, we take

$$\mathcal{F}_\phi := \mathcal{F} \times^{\underline{E}^*, \chi} \underline{\overline{\mathbb{Q}}_\ell}.$$

What is there to check?

- In this situation the group S_ϕ is the full group $\overline{\mathbb{Q}}_\ell^*$, as is $Z(\widehat{G})$. In the conjecture the S_ϕ-action is prescribed on $Z(\widehat{G})$, so there is nothing to check here.
- The cuspidality condition is irrelevant because that had something to do with restriction and pushforward to the semistable locus, but in this case the semistable locus is everything.
- The local Langlands correspondence is okay, thanks to class field theory.
- However, we do have to check a couple things: the Hecke eigensheaf property and the Satake isomorphism.

22.3.2 The Hecke eigensheaf property

The main ingredient is Proposition (22.2.2), which implies that

$$h_!^{\rightarrow}(h^{\leftarrow *}\mathcal{T}_{d+k})|_{\mathrm{Pic}^d \times \mathrm{Div}^1_X} \cong \mathcal{T}_d \boxtimes \chi_{\mathcal{LT}}^{-k} \tag{22.2}$$

where $\chi_{\mathcal{LT}} \colon W_E^{\mathrm{ab}} \to \mathbb{Q}_E^* \subset E^*$ is the Lubin-Tate character. We are using the fact that $\pi_1(\mathrm{Div}^1_X) \cong \mathrm{Gal}(\overline{E}/E)$ to identify a character of the Galois group with a local system on Div^1_X.

Remark 22.3.2. There was some confusion about whether or not this fact is obvious. Certainly $\mathrm{Spa}(E)^\diamond$ has $\pi_1 = \mathrm{Gal}(\overline{E}/E)$, since taking the diamond preserves the étale topos. But $\mathrm{Div}^1_X = \mathrm{Spa}(E)^\diamond/\varphi^{\mathbb{Z}}_{E^\diamond}$.

Now we just have to recall what Lubin-Tate theory tells us, which is:

(1) $(\mathrm{Art} \circ \chi_{\mathcal{LT}})(\sigma) = 1$ if $\sigma = \mathrm{Art}(\varpi)$,

(2) $(\mathrm{Art} \circ \chi_{\mathcal{LT}})(\sigma) = \sigma^{-1}$ if σ is in the image of I_E.

This second property and equation (22.2) imply that

$$h_!^{\rightarrow}(h^{\leftarrow *}\mathcal{F}_\phi)|_{\mathrm{Pic}^d_{\overline{\mathbb{F}}_q} \times \mathrm{Div}^1_X} \cong \mathcal{F}_\phi|_{\mathrm{Pic}^d_{\overline{\mathbb{F}}_q}} \boxtimes r_\mu \circ \phi|_{I_E}$$

where $r_\mu = (z \mapsto z^k)$, because

$$\begin{aligned} (\chi \circ \chi_{\mathcal{LT}})^{-k}|_{I_E} &= (\phi \circ \mathrm{Art} \circ \chi_{LT})^{-k}|_{I_E} \\ \text{applying property (2)} &= \phi^k|_{I_E}. \end{aligned}$$

This checks the equality on the inertia part; what's left is the Weil descent structure. By (1) and our definition of the Weil descent datum on $\mathcal{F}_\phi$, this extends to an isomorphism of Weil sheaves if $r_\mu \circ \phi$ is equipped with the natural Weil descent datum.

22.3.3 The character sheaf property

The character sheaf property concerns elliptic elements $\delta \in G(E) = E^* \subset G(\breve{E}) = \breve{E}^*$. For any such δ we get a morphism (since it's over E)

$$x_\delta \colon \mathrm{Spa}\,\overline{\mathbb{F}}_q \to \mathrm{Pic}\,.$$

We have the sheaf $\mathcal{F}_\phi$ on Pic and we can pull it back to $\mathrm{Spa}\,\overline{\mathbb{F}}_q$. It has a Frobenius action, and we have to compute what this is. Attached to δ we have the automorphism group $J_\delta(E)$, and we want that the Frobenius to act by δ.

1. If $\delta = \varpi^{-d}$ then after unwinding all our definitions it is basically tautological that

   $$x_\delta^*\mathcal{F}_\phi = \mathcal{F}_\phi|_{\mathrm{Pic}^d}.$$

   Now the result is a tautology: we *defined* the Weil sheaf structure by specifying that on Pic^d it is multiplication by ϖ^{-d}.

2. In general, one needs to make a small computation, which we'll omit.

Chapter 23

Relation with Cohomology of Lubin-Tate Spaces

The goal of this talk is to confirm Fargues' conjecture in the following (non-abelian!) case:

- $G = \mathrm{GL}_n / \mathbb{Q}_p$,
- $\mu(z) = \mathrm{diag}(z, 1, \ldots, 1)$,
- $b = \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ p^{-1} & & & \end{pmatrix} \in G(\breve{\mathbb{Q}}_p)$,
- $J_b(\mathbb{Q}_p) = D^*$ where $D/\mathbb{Q}_p$ is a division algebra of invariant $1/n$.

23.1 The Hecke stack

We have a Hecke stack

$$\begin{array}{ccc} & \mathrm{Hecke}^{\mu} & \\ {}^{h^{\leftarrow}}\swarrow & & \searrow^{h^{\rightarrow}} \\ \mathrm{Bun}_{G,\overline{\mathbb{F}}_p} & & \mathrm{Bun}_{G,\overline{\mathbb{F}}_p} \times \mathrm{Spa}\,\mathbb{Q}_p^{\diamond} \end{array}$$

where Hecke has functor of points

$$\mathrm{Hecke}^{\leq\mu}(S) = \left\{ (\mathcal{E}, \mathcal{E}', S^{\#}, u) : \begin{array}{c} \mathcal{E}, \mathcal{E}' = G\text{-bundles} \\ S^{\#} = \text{ untilt } \leftrightarrow i \colon D \hookrightarrow \mathrm{Div}^1_{X/S} \\ u \colon \mathcal{E} \overset{\leq\mu}{\dashrightarrow} \mathcal{E}' \text{ such that} \\ \operatorname{coker} \mu \text{ supported on } D \end{array} \right\}$$

We could (and usually would) write $\mathrm{Hecke}^{\leq\mu}$ but in this case there's no difference because μ is miniscule. The modification will be

$$0 \to \mathcal{E} \to \mathcal{E}' \to i_* W \to 0$$

where W is a rank 1 $S^{\#}$-module.

This should not be a perfectoid space but a "stack" because there are many automorphisms. We can address this by rigidifying, and that is how the Lubin-Tate tower shows up.

23.2 Rigidification: the Lubin-Tate tower at infinite level

Let $y_1 \colon \operatorname{Spa} \overline{\mathbb{F}}_p \to \mathrm{Bun}_{G, \overline{\mathbb{F}}_p}$ and $y_b \colon \operatorname{Spa} \overline{\mathbb{F}}_p \to \mathrm{Bun}_{G, \overline{\mathbb{F}}_p}$ be two points. (We pass to the algebraic closure because we do not want to keep track of the Weil descent datum right now; one can always go back to this later.) We define a sheaf $\mathcal{M}_\infty$ on Perf by the cartesian diagram

$$\begin{array}{ccc} \mathcal{M}_\infty & \longrightarrow & \mathrm{Hecke}^\mu \\ \downarrow & & \downarrow {\scriptstyle h^{\leftarrow} \times h_0^{\rightarrow}} \\ \operatorname{Spa} \overline{\mathbb{F}}_p & \longrightarrow & \mathrm{Bun}_{G, \overline{\mathbb{F}}_p} \times \mathrm{Bun}_{G, \overline{\mathbb{F}}_p} \end{array}$$

where $h_0^{\rightarrow} = p_1 \circ h^{\rightarrow}$. Here since y_1 corresponds to the trivial bundle, $\mathcal{M}_\infty$ parametrizes modifications of the form

$$0 \to \mathcal{O}_X^{\oplus n} \xrightarrow{u} \mathcal{O}_X(1/n) \to i_* W \to 0.$$

Note that the only thing that varies in moduli is u.

Theorem 23.2.1 (Scholze-Weinstein)**.** *Let $H_0/\overline{\mathbb{F}}_p$ be the p-divisible group which is connected of dimension 1 and height n (exactly the one corresponding to the isocrystal b).*

1. *We have*

$$\mathcal{M}_\infty(R, R^+/\mathbb{Q}_p) = \left\{ (H, \iota, \alpha) \colon \begin{array}{c} H = p\text{-div group} / R^+ \\ \alpha = \text{ quasi-isog. } : H \otimes_{R^+} R^+/p \sim H_0 \otimes_{\overline{\mathbb{F}}_p} R^+/p \\ \iota \colon T_p H \otimes \mathbb{Q}_p \cong \mathbb{Q}_p^{\oplus n} \end{array} \right\}$$

This has an action of $\mathrm{GL}_n(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$, with $\mathrm{GL}_n(\mathbb{Q}_p)$ acting on ι and $J_b(\mathbb{Q}_p)$ acting on α.

2. *$\mathcal{M}_\infty$ is a perfectoid space.*

Remark 23.2.2. The $\mathrm{GL}_n(\mathbb{Q}_p) \times J_b$-action is also clear from the description of $\mathcal{M}_\infty$ as parametrizing extensions

$$0 \to \mathcal{O}_X^{\oplus n} \xrightarrow{u} \mathcal{O}_X(1/n) \to i_* W \to 0.$$

because $\mathrm{GL}_n(\mathbb{Q}_p)$ is automorphism group of $\mathcal{E}_1 = \mathcal{O}_X^n$ and J_b is automorphism group of $\mathcal{E}_b = \mathcal{O}_X(1/n)$.

Remark 23.2.3. $\mathcal{M}_\infty$ comes equipped with a map to $\mathbb{Q}_p$ because it's fibered over Hecke^μ, which has such a map, because anything over $\mathrm{Spa}\,\mathbb{Q}_p^\diamond$ has a map to $\mathrm{Spa}\,\mathbb{Q}_p$.

Proof Sketch. 2. How do we parametrize these morphisms u? Well, u is a map of vector bundles $\mathcal{O}_X^n \to \mathcal{O}_X(1/n)$, which is the same as giving n global sections of $\mathcal{O}_X(1/n)$. So that gives a map

$$\mathcal{M}_\infty \mapsto H^0(X, \mathcal{O}(1/n))^{\oplus n}.$$

(For clarity, we spell out that $H^0(X, \mathcal{O}(1/n))^{\oplus n}$ is the sheaf that assigns to $S \in \mathrm{Perf}_{\mathbb{F}_p}$ n sections in $H^0(X_S, \mathcal{O}(1/n))$.) As covered in the discussion session on p-divisible groups, the sheaf $H^0(X, \mathcal{O}(1/n))$ is the same as $\widetilde{H}$, the universal cover of any lift $H/W(\overline{\mathbb{F}}_p)$ of H_0. (We have $H_0 \leftrightarrow b \leftrightarrow \mathcal{E}_b$, and the general theorem is that $H^0(X, \mathcal{E}_b) = \widetilde{H}$). Scholze-Weinstein proves that this map is a locally closed embedding, from which it follows that $\mathcal{M}_\infty$ is a perfectoid space.

□

23.3 Another Rigidification

We just related the Hecke stack to a perfectoid space at infinite level. This is a little overkill. What if we rigidify at just one vector and not the other? Suppose we just fix $\mathcal{E}' = \mathcal{O}_X(1/n)$. Then we are considering

$$0 \to \mathcal{E} \to \mathcal{O}_X(1/n) \to i_* W \to 0.$$

This is easy to parametrize because we just have to say what W is. It is a rank 1 quotient of the fiber of $\mathcal{O}_X(1/n)$ at the point D, so it's parametrized by $\mathbb{P}^{n-1}$.

To understand what $\mathcal{E}$ is, we note that $\mathcal{O}_X(1/n)$ has rank n and degree 1 while $i_* W$ has rank 0 and degree 1. By the additivity of rank and degree, we deduce that $\mathcal{E}$ has rank n and degree 0. We also know that $\mathcal{O}_X(1/n)$ is semistable. So what could a slope of $\mathcal{E}$ be? There cannot be a slope $> 1/n$ by the semistability of $\mathcal{O}_X(1/n)$. However, any other positive slope would have a larger denominator, hence larger rank. So we conclude that $\mathcal{E}$ must be semistable of slope 0. It's then proven in Kedlaya-Liu that there's some pro-étale cover trivializing it.

Remark 23.3.1. This is a really special feature of the Lubin-Tate situation.

As we said, specifying W is picking a line, i.e. 1-dimensional quotient of an n-dimensional space. So we have

$$\mathbb{P}^{n-1,\diamond}_{\breve{\mathbb{Q}}_p} \to \mathrm{Hecke}^{\mu} \xrightarrow{h^{\leftarrow}} \mathrm{Bun}_{G,\overline{\mathbb{F}}_p}$$

The preceding discussion showed that $\mathcal{E}$ is always pro-étale locally the trivial bundle, so the composite map factors through

$$y_1\colon [\operatorname{Spa}\overline{\mathbb{F}}_p/\operatorname{GL}_n(\mathbb{Q}_p)] \to \mathrm{Bun}_{G,\overline{\mathbb{F}}_p}\,.$$

Thus we get a diagram

$$\begin{array}{ccccc} & & & & [\operatorname{Spa}\overline{\mathbb{F}}_p/\operatorname{GL}_n(\mathbb{Q}_p)] \\ & & & \overset{r}{\nearrow} & \downarrow \\ \mathbb{P}^{n-1,\diamond}_{\breve{\mathbb{Q}}_p} & \longrightarrow & \mathrm{Hecke}^{\mu} & \xrightarrow{h^{\leftarrow}} & \mathrm{Bun}_{G,\overline{\mathbb{F}}_p} \end{array} \tag{23.1}$$

The map $r\colon \mathbb{P}^{n-1,\diamond}_{\breve{\mathbb{Q}}_p} \to [\operatorname{Spa}\overline{\mathbb{F}}_p/\operatorname{GL}_n(\mathbb{Q}_p)]$ corresponds by a definition to a $\operatorname{GL}_n(\mathbb{Q}_p)$-torsor on $\mathbb{P}^{n-1,\diamond}_{\breve{\mathbb{Q}}_p}$, and *it turns out to be* $\mathcal{M}_\infty$. The map to $\mathbb{P}^{n-1,\diamond}$ factors through some finite layer, i.e. we have a diagram

$$\begin{array}{ccc} \mathcal{M}_{\infty,\diamond} & \longrightarrow & \mathbb{P}^{n-1,\diamond} \\ & \searrow \quad \nearrow & \\ & \mathcal{M}_K^{\diamond} & \end{array}$$

where $K \subset \operatorname{GL}_n(\mathbb{Q}_p)$ is a compact open subgroup.

In order to match things up with the Hecke correspondence, we now base change to $\mathbb{Q}_p$ (because one of the maps of Hecke^{μ} is to $\mathrm{Bun}_{G,\overline{\mathbb{F}}_p} \times (\operatorname{Spa}\mathbb{Q}_p)^{\diamond}$).

$$[\operatorname{Spa}\breve{\mathbb{Q}}_p^{\diamond}/J_b(\mathbb{Q}_p)] \xrightarrow{(x_b,1)} \mathrm{Bun}_{G,\overline{\mathbb{F}}_p} \times \operatorname{Spa}\mathbb{Q}_p^{\diamond}.$$

We have a commutative diagram

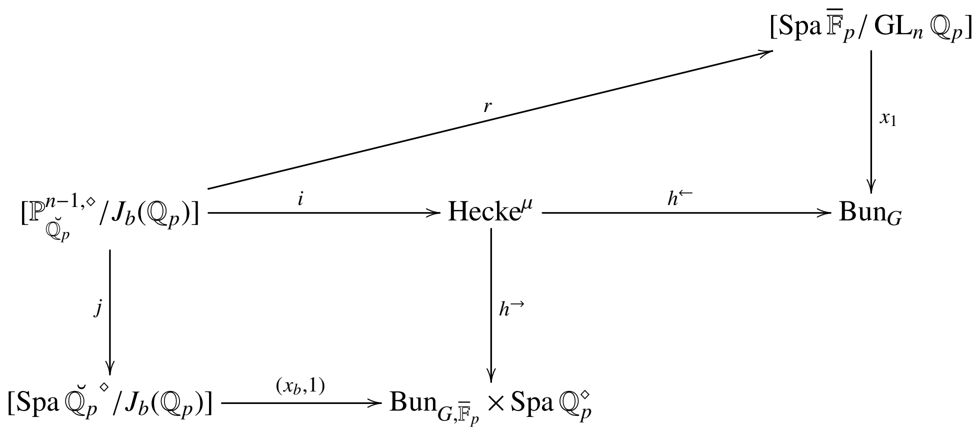
$$\begin{array}{ccc} [\mathbb{P}^{n-1,\diamond}_{\breve{\mathbb{Q}}_p}/J_b(\mathbb{Q}_p)] & \xrightarrow{\quad i \quad} & \mathrm{Hecke}^{\mu} \\ \Big\downarrow j & & \Big\downarrow h^{\rightarrow} \\ [\operatorname{Spa}\breve{\mathbb{Q}}_p{}^{\diamond}/J_b(\mathbb{Q}_p)] & \xrightarrow{(x_b,1)} & \mathrm{Bun}_{G,\overline{\mathbb{F}}_p} \times \operatorname{Spa}\mathbb{Q}_p^{\diamond} \end{array}$$

(We have written down this diagram before without modding out be J_b on the left side.) The map $i\colon [\mathbb{P}^{n-1,\diamond}_{\breve{\mathbb{Q}}_p}/J_b(\mathbb{Q}_p)] \to \mathrm{Hecke}^{\mu}$ is an open embedding. Indeed, as Peter mentioned in his talk, there is a theorem that

$$\coprod_{b \text{ basic}} \left[\frac{\operatorname{Spa} \overline{\mathbb{F}}_p}{\underline{J_b(\mathbb{Q}_p)}} \right] = \operatorname{Bun}_G^{ss}$$

and i is a base change of this map.

To summarize, we have the commutative diagram



23.4 Fargues' conjecture

Let $\phi\colon W_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ be a discrete Weil parameter. What does Fargues's conjecture say in this case? (The situation here is a little simplified by the fact that S_ϕ is trivial.) It predicts that there exists $\mathcal{F}_\phi$ on $\operatorname{Bun}_{G,\overline{\mathbb{F}}_p}$ such that (up to shifts and twists)

1. We have
$$h^{\rightarrow}_! h^{\leftarrow *} \mathcal{F}_\phi = \mathcal{F}_\phi \boxtimes \phi. \tag{23.2}$$
(This is simpler than in general because IC sheaf is constant up to shifts and twists, and also it is unnecessary to write r_μ because it is the standard representation of GL_n.)

2. We have $x_1^* \mathcal{F}_\phi = \pi$ and $x_b^* \mathcal{F}_\phi = \rho$ where π and ρ correspond to ϕ under the local Langlands correspondence.

Consequences of the conjecture. Pulling back (23.2) through $(x_b, 1)^*$ gives

$$(x_b, 1)^* h^{\rightarrow}_! h^{\leftarrow *} \mathcal{F}_\phi = (x_b, 1)^* \mathcal{F}_\phi \boxtimes \phi. \tag{23.3}$$

On the left side we get $\rho \otimes \phi$ by the second part of the conjecture. On the right side, first apply proper base change to j from the earlier diagram

$$\begin{array}{ccc} [\mathbb{P}^{n-1,\diamond}_{\breve{\mathbb{Q}}_p}/J_b(\mathbb{Q}_p)] & \xrightarrow{\;i\;} & \mathrm{Hecke}^{\mu} \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle h^{\rightarrow}} \\ [\mathrm{Spa}\,\breve{\mathbb{Q}}_p^{\diamond}/J_b(\mathbb{Q}_p)] & \xrightarrow{(x_b,1)} & \mathrm{Bun}_{G,\overline{\mathbb{F}}_p} \times \mathrm{Spa}\,\mathbb{Q}_p^{\diamond} \end{array}$$

to deduce that

$$\rho \otimes \phi = (x_b, 1)^* h^{\rightarrow}_! h^{\leftarrow *} \mathcal{F}_\phi = (x_b, 1)^* \mathcal{F}_\phi \boxtimes \phi = j_! i^* h^{\leftarrow *} \mathcal{F}_\phi. \tag{23.4}$$

Now we use the top part of the diagram

$$\begin{array}{ccccc} & & & & [\mathrm{Spa}\,\overline{\mathbb{F}}_p/\,\mathrm{GL}_n\,\mathbb{Q}_p] \\ & & \nearrow{\scriptstyle r} & & \downarrow{\scriptstyle x_1} \\ [\mathbb{P}^{n-1,\diamond}_{\breve{\mathbb{Q}}_p}/J_b(\mathbb{Q}_p)] & \xrightarrow{\;i\;} & \mathrm{Hecke}^{\mu} & \xrightarrow{\;h^{\leftarrow}\;} & \mathrm{Bun}_G \end{array}$$

to deduce that

$$j_! i^* h^{\leftarrow *} \mathcal{F}_\phi = j_! r^* x_1^* \mathcal{F}_\phi.$$

Then part 2 of the conjecture implies that this is $j_! r^* \pi$, so combining this with (23.4) gives

$$\rho \otimes \phi = j_! r^* x_1^* \mathcal{F}_\phi.$$

Recall that r corresponds to a $\mathrm{GL}_n(\mathbb{Q}_p)$-torsor on $\mathbb{P}^{n-1,\diamond}$. We can compose this with the representation associated to π to obtain a sheaf $r^*\pi$ on $[\mathbb{P}^{n-1,\diamond}_{\breve{\mathbb{Q}}_p}/J_b(\mathbb{Q}_p)]$ (recall that $\mathcal{M}_\infty \to \mathbb{P}^{n-1,\diamond}_{\breve{\mathbb{Q}}_p}$ is a $\mathrm{GL}_n(\mathbb{Q}_p)$-torsor).

Now we apply $j_!$ to get

$$\rho \otimes \phi = H^*_c(\mathbb{P}^{n-1}_{\mathbb{C}_p}, r^*\pi). \tag{23.5}$$

Here we have base-changed to $\mathbb{C}_p$ and gotten rid of J_b quotient at the cost of remembering the action of Galois and J_b. (You can get rid of quotients in your sheaves at the cost of remembering the action). So the above isomorphism is equivariant for the action of $J_b(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$.)

We ignored shifts and twists; if you keep track of them then (assuming that π is cuspidal) you get

$$\boxed{\rho \otimes \phi = H^{n-1}_c(\mathcal{M}_\infty, \mathbb{Q}_\ell)[\pi^\vee](\frac{1-n}{2}).} \tag{23.6}$$

This is a very deep theorem of Harris-Taylor. How did we get from (23.5) to (23.6)? The Hoschild-Serre spectral sequence for the fibration

$$\begin{array}{c} \mathcal{M}_{\infty,\mathbb{C}_p} \\ \downarrow \\ \mathbb{P}^{n-1}_{\mathbb{C}_p} \\ \downarrow \\ \mathbb{C}_p \end{array}$$

converges as

$$H_i(\mathrm{GL}_n(\mathbb{Q}_p), H^j_c(\mathcal{M}_{\infty,\mathbb{C}_p}, \overline{\mathbb{Q}}_\ell) \otimes \pi) \implies H^{-i+j}(\mathbb{P}^{n-1}, r^*\pi).$$

In the supercuspidal case there is no higher group cohomology, so you take the invariants in this tensor product, which gives what we claim.