Limit linear series and distribution of Weierstrass points

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1 Statement of Main Theorem

First let me recall the meaning of equidistribution. Let S be a compact topological space and μ a measure on S. Suppose that for each n, we are given a multiset $W_n \subset S$ of finite size.

Definition 1.1. We say that W_n is *equidistributed* in (S, μ) if for any continuous function $f : S \to \mathbb{R}$,

$$\frac{1}{|W_n|} \sum_{x \in W_n} f(x) \to \int f_S \, d\mu.$$

Example 1.2. For $S = S^1$ with the Lebesgue measure, the "multiples" of an irrational α are equidistributed.

Let L be an ample line bundle of degree d and rank $r \geq 1$ on X. For $x \in X(\overline{K})$, you can consider the set

$$\{\text{ord}_x(f) \mid f \in H^0(L)\} = \{a_0^x < a_1^x < \ldots < a_r^x\}.$$

We define the *Weierstrass multiplicity of x* to be

$$\omega(x) = \sum_{i=0}^{r} (a_i^x - i).$$

There is a formula for the number of Weierstrass points: $(r+1)d + r(r+1)(g-1)$. This is a good way to write the result because the points are the zeros of the Wronskian, which is a section of $L^{\otimes(r+1)} \otimes \omega_X^{r(r+1)/2}$.

Theorem 1.3 (Amini)**.** *Let X/K be smooth proper urve and L an ample bundle on X. Let W_n be the set of Weierstrass points of $L^{\otimes n}$. Then W_n is equidistributed in the Berkovich analytic spectrum $X^{\text{an}}_{\overline{K}}$ with respect to the Zhang measure.*

We will try to give another formulation of this theorem (and measure) which is more explicit. This is a non-archimedean version of a theorem proved by Mumford and Neeman for a Riemann surface.

2 Reformulation

2.1 Preliminaries

Let X/K be a smooth proper curve over a complete discrete valuation field. Possibly by passing to a finite extension K'/K, we can consider a semistable, regular integral model

$$\begin{array}{ccccc} \mathcal{X}_s & \longrightarrow & \mathcal{X} & \longleftarrow & X_\eta = X_{K'} \\ \downarrow & & \downarrow & & \downarrow \\ s & \longrightarrow & \operatorname{Spec} R' & \longleftarrow & \eta \end{array}$$

The special fiber is nodal. We form the dual graph of the special fiber, which has a vertex for each component and an edge for each node. We define

$$\ell(\{u,v\}) = \frac{1}{[K':K]}.$$

This now a weighted graph. Let Γ be the associated metric graph, which is precisely the geometric realization of this simplicial complex. If you make a further field extension, then the metric graph may grow some edges but the *core* is well-defined and stable.

Let $x \in X(K')$. Then we get a section $s \in \mathcal{X}(R')$. There exists some vertex v such that $\operatorname{Im}(s) \in X_v^{\mathrm{sm}}$, and we take the $\tau(x) := v$. This gives a map $\tau\colon X(\overline{K}) \to \Gamma$.

2.2 Zhang measure

We have a genus function $g\colon \Gamma \to \mathbb{Z}_{\geq 0}$, such that $g(v)$ is the genus of X_v. The Zhang measure is the sum of a "discrete part" and a "continuous part."

$$\mu_{Zh} = \frac{1}{g(x)}\left(\sum_{x\in\Gamma} g(x)\delta_x + \sum_{e \text{ edge of } \Gamma} F(e)d\theta\right)$$

where $d\theta$ is the Lebesgue measure on the interval of length $\frac{1}{[K':K]}$.

Let's explain what is the meaning of $F(e)$. Suppose $e = \{u,v\}$. Then $F(e)$ is the probability of a random walk arriving at v starting at u, without going through the edge $\{u,v\}$. (This is something like the "resistance between u and v" after deleting the edge e.)

Corollary 2.1. *We have*

1. $$\frac{\#\{\textit{Weierstrass points } x \textit{ with } \tau(x) \in X_v^{sm}\}}{\#W_n} \to \frac{g(v)}{g(X)}.$$

2. $$\frac{\#\{\textit{Weierstrass points } x \textit{ with } \tau(x) \in X_v\}}{\#W_n} \to \frac{g(v)}{g(X)} + \frac{1}{g(X)}\sum_{e\sim v} F(e).$$