ARITHMETIC FUNDAMENTAL LEMMA

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1. Global motivation

1.1. **BSD Conjecture.** An early formulation of the BSD conjecture (not the one we use nowadays) said: let $E/\mathbf{Q}$ be an elliptic curve.

$$\prod_p \frac{\#E(\mathbf{F}_p)}{p} = \infty \iff \#E(\mathbf{Q}) = \infty.$$

One could view this as a local-global statement. The LHS has to do with points over finite fields, which is a purely local quantity, while the RHS is global.

1.2. **Beilinson-Bloch Conjecture.** A higher dimensional generalization was introduced by Beilinson and Bloch. Let X be a variety over a number field F.

$$\mathrm{ord}_{s=0} L(H^{2i-1}(X)(i), s) = \mathrm{rank}\, \mathrm{CH}^i(X)_0$$

where $\mathrm{CH}^i(X)_0$ is the cohomologically trivial subgroup of the Chow group of codimension i cycles. This generalizes the "rank equality" aspect of BSD.

Remark 1.1. There is a p-adic variant, where you replace the Hasse-Weil L-function by a p-adic L-function. The algebraic object of relevance is the Bloch-Kato Selmer group.

1.3. **Arithmetic Gan-Gross-Prasad.** Now we will take X to be a Shimura variety. There is an "arithmetic Gan-Gross-Prasad Conjecture" for unitary Shimura varieties. To set it up, let $F/F_0 = \mathbf{Q}$ be a quadratic imaginary extension, and $n \geq 1$. We define

$$\mathcal{M}_n := \left\{ (A, \iota, \lambda) \mid \begin{array}{c} (A, \lambda) = \text{PPAV}, \\ \iota \colon \mathcal{O}_F \to \mathrm{End}(A) \quad \text{sign} = (n-1, 1) \end{array} \right\}.$$

Then $\mathcal{M}_n(\mathbf{C})$ is a finite disjoint union of ball quotients $\mathbb{D}_{n-1}/\Gamma$. There is a forgetful map $\mathcal{M}_n \to \mathcal{A}_{g=n}$.

There's a map $\mathcal{A}_1 \times \mathcal{A}_{g-1} \to \mathcal{A}_g$. Let $\widetilde{\mathcal{M}}_n = \mathcal{M}_1^* \times \mathcal{M}_n$ where $*$ means the analogous definition for signature $(1,0)$ instead of $(0,1)$. Then there is the variant for $\mathcal{M}$,

$$\widetilde{\mathcal{M}}_{n-1} = \mathcal{M}_1^* \times \mathcal{M}_{n-1} \to \widetilde{\mathcal{M}}_n.$$

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We consider the diagonal embedding

$$\mathcal{M}_{n-1} \xrightarrow{\Delta} \widetilde{\mathcal{M}}_{n-1} \times \widetilde{\mathcal{M}}_n =: \mathcal{M}_{n,n-1}.$$

Conjecture 1.2. *Let π be a generic automorphic representation, appearing in $H^{\mathrm{mid}}(X)$. Then:*

(1) If $\mathrm{ord}_{s=center} L(H^{\mathrm{mid}}(X)[\pi], s) = 1$, then $\mathrm{rank}\,\mathrm{CH}^{n-i}(X)_0[\pi] = 1$.
(2) We have $\langle \Delta_\pi, \Delta_\pi \rangle_{BB} = L'(\pi, s = center)$.

Remark 1.3. For $n = 2$, this conjecture is essentially the Gross-Zagier formula.

For f in a suitable Hecke algebra, one can consider

$$\langle f * \Delta, \Delta \rangle_{GS}.$$

where $\langle \cdot, \cdot \rangle_{GS}$ is the Gillet-Soulé pairing on arithmetic Chow groups (defined unconditionally). Here Δ is viewed in the arithmetic Chow group $\widehat{\mathrm{CH}}(\widetilde{M}_{n,n-1})$.

Conjecture 1.4 ("Arithmetic intersection conjecture" – Z, Rapoport-Smithling-Z)**.** *We have*

$$\langle f * \Delta, \Delta \rangle_{GS} = \text{“}\partial \mathbb{J}(f)\text{”} = \text{“}\sum_{\pi} L'(\pi, 1/2) \lambda_\pi(f).\text{”}$$

This statement for all f allows one to separate the π.

For "nice" f, one can show that this can be expanded into local quantities,

$$\sum_v \mathrm{Int}_v(f) = \sum_v \partial \mathbb{J}_v(f), \tag{1.1}$$

where if v is a p-adic place, then $\mathrm{Int}_v(f) \in \mathbf{Q} \log q_v$.

Theorem 1.5 (Z, 2019)**.** *If v is inert in F/F_0, f_v is a unit in the unramified Hecke algebra, then*

$$\mathrm{Int}_v(f) = \partial \mathbb{J}_v(f).$$

Remark 1.6. When v is split, both sides are easily shown to be 0. One still has to handle the ramified and archimedean places.

2. Arithmetic fundamental lemma

We will describe the local objects appearing in (1.1). Now let F/F_0 be an unramified quadratic extension of p-adic fields.

We introduce the unitary Rapoport-Zink spaces $\mathcal{N}_n$. These are used to uniformize a formal neighborhood of the supersingular locus in $\mathcal{M}_n|_{F_v}$. The formal neighborhood of the supersingular locus can be expressed as $\coprod_{\Gamma_i} \mathcal{N}_n/\Gamma_i$, in analogy to the complex uniformization of $\mathcal{M}_n(\mathbf{C})$.

$$\mathcal{M}_n^{\mathrm{ss}} = \bigcup \mathcal{N}_n/\Gamma_i$$
$$\mathcal{M}_n(\mathbf{C}) = \bigcup \mathbb{D}_{n-1}/\Gamma_i'.$$

An embedding $V^\flat \hookrightarrow V$ induces $\delta_n \colon \mathcal{N}_{n-1} \to \mathcal{N}_n$, and $\Delta \colon \mathcal{N}_{n-1} \to \mathcal{N}_n \times \mathcal{N}_{n-1}$, as before. There's an action of $U(V)$ on $\mathcal{N}_n$ and $G = U(V) \times U(V^\flat)$ on $\mathcal{N}_n \times \mathcal{N}_{n-1}$.

For $g \in G(F_0)$, we can define

$$\mathrm{Int}(g) = \text{“}(g\Delta) \cap \Delta\text{”} = \chi(\mathcal{N}_{n,n-1}, \mathcal{O}_\Delta \otimes^{\mathbb{L}} \mathcal{O}_{g\Delta}).$$

This explains one side.

The other side is expressed in terms of orbital integrals. Consider $\mathrm{GL}_{n-1} \hookrightarrow \mathrm{GL}_n$ over F_0. We can consider the conjugation action of the subgroup.

$$\mathrm{Orb}(\gamma, \mathbb{1}_{\mathrm{GL}_n(\mathcal{O}_{F_0})}) = \int_{\mathrm{GL}_{n-1}(F_0)} \mathbb{1}_{\mathrm{GL}_n(\mathcal{O}_{F_0})}(h^{-1}\gamma h)(-1)^{\mathrm{val}\det h}|h|^s dh.$$

Theorem 2.1 (Z '2019, The Arithmetic Fundamental Lemma)**.** *Let $F_0 = \mathbf{Q}_p$. For $p > n$, we have*

$$\mathrm{Int}(g) = \frac{d}{ds}|_{s=0} \mathrm{Orb}(\gamma, \mathbb{1}_{\mathrm{GL}_n(\mathcal{O}_{F_0})}, s).$$

3. The proof

The proof is based on an induction. A key step for this induction comes from global ingredients. To explain this, we consider an alternative intersection question. There is an intersection product

$$\widehat{\mathrm{CH}}^1(\mathcal{M}_n)_{\mathbf{Q}} \times Z_1(\mathcal{M}_n)_{\mathbf{Q}} \to \mathbb{R}.$$

(Right now we pretend the $\mathcal{M}_n$ is proper, for simplicity.) We define two families of cycles:

- Kudla-Rapoport divisors, denoted $Z(m)$.
- (derived) CM cycles, denoted ${}^{\mathbb{L}}\mathrm{CM}(a)$.

We will explain the definition of CM cycles for Siegel modular varieties $\mathcal{A}_g$; the versions for $\mathcal{M}_n$ are defined by pullback.

Naively, pick $a \in \mathbf{Q}[T]$ of degree $2g$, and consider

$$\mathrm{CM}(a) := \{(A, \lambda, \varphi \in \mathrm{End}^0(A)) \colon \mathrm{char}(\varphi) = a\}.$$

Suppose a is irreducible. Then the generic fiber of $\mathrm{CM}(a)$ will be 0-dimensional, but the special fibers may be positive-dimensional, depending on how far $\mathbf{Z}[a] \subset \mathbf{Q}[a]$ is from being a maximal order (which measures the non-flatness).

Consider the diagonal embedding

$$\begin{array}{ccc} \bigcup_a \mathrm{CM}(a) = \Delta_{\mathrm{Hecke}} & \longrightarrow & \mathrm{Hecke} \\ \downarrow & & \downarrow \\ \mathcal{A}_g & \xrightarrow{\quad \Delta \quad} & \mathcal{A}_g \times \mathcal{A}_g \end{array}$$

This diagram induces a virtual fundamental cycle ${}^{\mathbb{L}}\mathrm{CM}(a) \in \mathrm{CH}_1(\mathrm{CM}(a))$. The pairing doesn't factor through rational equivalence. However it factors through a partial quotient, where you kill rationally trivial 1-cycles supported in a special fiber.

The global incarnation has to do with intersection numbers $Z(m) \cdot {}^{\mathbb{L}}\mathrm{CM}(a)$. What gives some traction is the fact that $\sum_{m \geq 0} Z(m) q^m \in \widehat{\mathrm{CH}}^1(\mathrm{CM}_n)[[q]]$ is actually a modular form [BHKRY].