The Relative Fargues-Fontaine Curve

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for a talk by Matthew Morrow

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The goals of this talk are to:

1. Define the relative curves Y_S and $X_S = Y_S/\varphi^{\mathbb{Z}}$ for any $S \in \mathrm{Perf}_{\mathbb{F}_p}$. (To recover Y^{ad} and X^{ad} from before, put $S = \mathrm{Spa}\,\mathbb{C}_p^\flat$.) These will be adic spaces over $\mathrm{Spa}\,\mathbb{Q}_p := \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$.

2. Describe the relation to untilting and the diamond formula

$$\text{“}Y_S = S \times \mathrm{Spa}\,\mathbb{Q}_p\text{”}.$$

1 Construction of the relative curves Y_S and X_S

1.1 The affinoid perfectoid case

Suppose now that $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbb{F}_p}$ is affinoid perfectoid of characteristisc p. Fix once and for all a pseudo-uniformizer $\varpi \in R$. We define the ring

$$\mathbb{A}(= \mathbb{A}_{R^+}) = W(R^+) \ni p, [\varpi]$$

with the $(p, [\varpi])$-adic topology.

Definition 1.1. We define

$$Y_{(R,R^+)} := \mathrm{Spa}(\mathbb{A}, \mathbb{A}) \setminus V(p[\varpi]).$$

This is, at the moment, a pre-adic space (i.e. the structure presheaf is not yet known to be a sheaf) over $\mathrm{Spa}\,\mathbb{Q}_p$.

The points of this are continuous valuations

$$\|\cdot\| \colon A \to \Gamma \cup \{0\}$$

such that

- $\|a\| \leq 1$ for all $a \in A$,

- $\|p[\varpi]\| \neq 0$.

Definition 1.2. (1) Given $(\|\cdot\|, \Gamma) \in Y_{(R,R^+)}$, its *maximal generalization* is the rank-one point $(\|\cdot\|_{\max}, \mathbb{R}_{\geq 0}) \in Y_{(R,R^+)}$ given by the rule

$$\|a\|_{\max} := p^{-\sup\{r/s \in \mathbb{Q}_{\geq 0}\colon \|[\varpi]\|^r \geq \|a\|^s\}}.$$

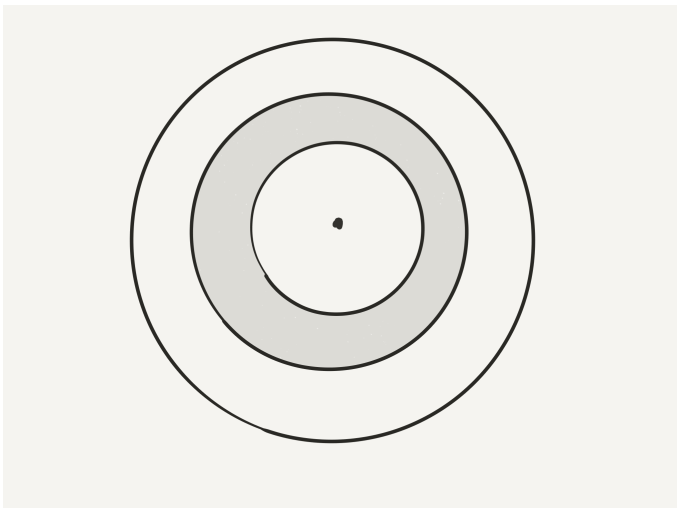
This is the "closest point to $\|a\|$ in the line of Γ generated by $\|[\varpi]\|$".

(2) The *radius* of $(\|\cdot\|, \Gamma)$ is then $\delta(\|\cdot\|) := \|p\|_{\max} \in (0,1)$ (the value in in $[0,1]$ by definition, and cannot be either endpoint because p is topologically nilpotent and not killed). This defines a continuous radius function

$$\delta\colon Y_{(R,R^+)} \to (0,1).$$

(3) Given a closed interval $I \subset (0,1)$, the associated *annulus* is

$$Y^I_{(R,R^+)} := \text{interior of } \delta^{-1}(I) \overset{\text{open}}{\subset} Y_{(R,R^+)}.$$



Lemma 1.3. *(i) We have*

$$Y_{(R,R^+)} = \bigcup_{I \subset (0,1)} Y^I_{(R,R^+)}.$$

(ii) If $I = [p^{-r/s}, p^{-r'/s'}]$ with $r, s, r', s' \in \mathbb{N}$, then $Y^I_{(R,R^+)}$ is the rational subdomain

$$\mathrm{Spa}(\mathbb{A}, \mathbb{A})\langle \frac{[\varpi]^r}{p^s}, \frac{p^{s'}}{[\varpi]^{r'}} \rangle \subset Y_{(R,R')}.$$

Proof. With I as in (ii), it follows from the definitions (modulo interior issues) that

$$Y^I_{(R,R^+)} = \{\|\cdot\|\colon \|p\| \in I \subset \Gamma\}$$

which is the claimed rational subdomain. (Note that here we are normalizing $\|\cdot\|$ so that $\|[\varpi]\| = p^{-1}$.) □

Theorem 1.4. *$Y_{(R,R^+)}$ is an adic space (i.e. the presheaf is sheafy).*

Idea of proof. Pick some perfectoid field $E \supset \mathbb{Q}_p$ and check that $Y^I_{(R,R^+)} \times_{\operatorname{Spa}\mathbb{Q}_p} \operatorname{Spa} E$ is affinoid perfectoid. (Although we have used finite extensions of $\mathbb{Q}_p$ for our E, references in the literature allow the field E to be perfectoid precisely to deal with this issue.) By definition, this is saying that $Y^I_{(R,R^+)}$ is *pre-perfectoid*. That implies the sheaf property by results of Scholze, or Kedlaya-Liu. □

Remark 1.5. Descent of the sheafiness is not hard, but proving that the base change is affinoid perfectoid requires work (to show that the ring of power-bounded elements is bounded), and the original result that affinoid perfectoid spaces are sheafy is hard.

1.2 Forming the quotient

We have an action φ on $W(R)$ inducing an action of φ on $Y_{(R,R^+)}$ such that

$$\delta(\varphi(y)) = \delta(y)^{1/p}.$$

Note that in the unit disk picture, this is expanding towards the boundary. Therefore the action of φ on $Y_{(R,R^+)}$ is properly continuous, so we can define

$$\boxed{X_{(R,R^+)} := Y_{(R,R^+)}/\varphi^{\mathbb{Z}}}$$

as an adic space over $\operatorname{Spa}\mathbb{Q}_p$.

Remark 1.6. Suppose $I = [a,b] \subset (0,1)$ such that $b^p < a \le b < a^{1/p}$, so that the interval is translated to a disjoint interval under $x \mapsto x^{1/p}$. Then $Y^I_{(R,R^+)}$ maps isomorphically to an open subset of $X_{(R,R^+)}$.

1.3 The map θ

Suppose that $(R,R^+) = (B^\flat, B^{+\flat})$ for some $\operatorname{Spa}(B,B^+) \in \operatorname{Perf}$ over $\operatorname{Spa}\mathbb{Q}_p$ (e.g. $S = \operatorname{Spa}\mathbb{C}_p^\flat$). Then Fontaine's map

$$\theta\colon W(B^{+\flat}) = \mathbb{A} \to B^+$$

induces a closed immersion

$$\theta\colon \operatorname{Spa}(B,B^+) \hookrightarrow Y_{(R,R^+)}. \tag{1}$$

Lemma 1.7. *The composition*

$$\operatorname{Spa}(B,B^+) \hookrightarrow Y_{(R,R^+)} \twoheadrightarrow X_{(R,R^+)}$$

is a closed immersion.

Proof sketch. Explicitly check that θ has image in an annulus which is small enough so that it maps isomorphically down to the Fargues-Fontaine curve. □

1.4 Construction for general $S \in \mathrm{Perf}_{\mathbb{F}_p}$

One checks that the process $(R, R^+) \mapsto Y^I_{(R,R^+)}$ (for any $I \subset (0, 1)$) behaves well under taking rational subdomains. Then it's easy to glue to define Y_S and $X_S = Y_S/\varphi^{\mathbb{Z}}$ (as adic spaces over $\mathrm{Spa}\,\mathbb{Q}_p$) for any $S \in \mathrm{Perf}_{\mathbb{F}_p}$.

One has an obvious analogue of θ if S is a tilt. That is, if $S = (S^\#)^\flat$ then we get a closed embedding

$$\theta\colon S^\# \hookrightarrow Y_S$$

by gluing.

2 Diamonds and untilting

2.1 Diamonds parametrize untilts

Definition 2.1. For X an analytic adic space (i.e. covered by adic spectra of Tate rings) over $\mathrm{Spa}\,\mathbb{Z}_p$, we define

$$X^\diamond\colon \mathrm{Perf}_{\mathbb{F}_p} \to \mathrm{Sets}$$

by

$$\begin{aligned} T &\mapsto \{\text{Untilts over } X \text{ of } T\} \\ &= \left\{ (T^\#, \iota)\colon \begin{array}{c} T^\# \in \mathrm{Perf}_X \\ \iota\colon T^{\#\flat} \cong T \end{array} \right\}. \end{aligned}$$

Lemma 2.2. *If X is perfectoid then* $X^\diamond = \mathrm{Hom}(-, X^\flat)$.

Therefore the formation of diamonds can be thought of as an extension of tilting to adic spaces.

Proof. Let's check that the functors of points agree. For a test space T, $X^\diamond(T)$ is an untilt over X of T. Given such an untilt, we can tilt to obtain a map $T \to X^\flat$.

In the other direction, given a map $T \to X^\flat$, the equivalence between perfectoid spaces over $X^\flat$ and X produces an untilt over X of T. □

In particular, if $X \in \mathrm{Perf}_{\mathbb{F}_p}$ (viewed as an analytic space over $\mathrm{Spa}\,\mathbb{Z}_p$), then $X^\diamond = \mathrm{Hom}(-, X)$, which is just X viewed as a (representable) sheaf.

Lemma 2.3. *$X^\diamond$ is a sheaf for the pro-étale topology on* $\mathrm{Perf}_{\mathbb{F}_p}$*, and even a diamond.*

Proof idea. Pick a perfectoid cover of X. To each element in the cover you apply the construction $(-)^\diamond$; this produces diamonds since they are representable. Then you check that anything pro-étale covered by diamonds is itself a diamond. □

Example 2.4. $(\mathrm{Spa}\,\mathbb{Q}_p^{\mathrm{cyc}})^\diamond \to \mathrm{Spa}\,\mathbb{Q}_p^\diamond$ is a pro-étale $\mathbb{Z}_p^*$-torsor.

Remark 2.5. For many purposes it suffices only to remember that diamonds are a full subcategory of pro-étale sheaves on $\mathrm{Perf}_{\mathbb{F}_p}$.

2.2 The diamond equation for the curve

Proposition 2.6. *Let $S \in \mathrm{Perf}_{\mathbb{F}_p}$. Then*

$$Y_S^\diamond \cong S^\diamond \times \mathrm{Spa}\,\mathbb{Q}_p^\diamond$$

(in the category of diamonds or $\widetilde{\mathrm{Perf}}_{\mathbb{F}_p,\text{pro-étale}}$).

Remark 2.7. Y_S is an analytic adic space because the annuli are Tate algebras.

Proof. Let's compare the functors of points. We have to show that for $T \in \mathrm{Perf}_{\mathbb{F}_p}$ there is a bijection

$$\{\text{untilts} / Y_S \text{ of } T\} \leftrightarrow \mathrm{Hom}(T, S) \times \{\text{untilts} / \mathrm{Spa}\,\mathbb{Q}_p \text{ of } T\}.$$

Suppose we have a pair $(f, (T^\#, \iota))$ on the right side. We can send this to $(T^\#, \iota)$:

$$(T^\#, \iota) \leftarrow (f, (T^\#, \iota)).$$

At first this seems like it's forgotten f, but that is built into the meaning of $T^\#$, because we need to specify the structure of $T^\#$ as a space over Y_S. This structure is via

$$T^\# \overset{\theta}{\hookrightarrow} Y_{T^{\#\flat}} \overset{\iota}{\cong} Y_T \overset{f}{\to} Y_S.$$

(The embedding θ is (1)). In the other direction we send

$$(T^\#, \iota) \mapsto (T \overset{\iota}{\cong} T^{\#\flat} \to S, (T^\#, \iota))$$

where the map $T^{\#\flat} \to S$ is defined is follows. Reduce to the affinoid case $S = \mathrm{Spa}(R, R^+)$. Then we can compose the map $T^\# \to Y_S$ to get $T^\# \to Y_S \to \mathrm{Spa}(W(R^+), W(R^+))$, at which point the universal property of the Witt vectors gives

$$T^{\#\flat} \to \mathrm{Spa}(R, R^+).$$

(At the level of rings formation of Witt vectors is left adjoint to tilting, so at the level of spaces it is right adjoint.) □

Proposition 2.8. *The following are in canonical bijection with each other.*

1. *Sections of $Y_S^\diamond \to S^\diamond$.*

2. *Maps $S^\diamond \to \mathrm{Spa}(\mathbb{Q}_p)^\diamond$.*

3. *Untilts over $\mathrm{Spa}\,\mathbb{Q}_p$ of S.*

4. *Closed subsets of Y_S defined locally by a "degree 1 primitive element" (i.e. a $\xi \in W(R^+)$ of the form $[\varpi] + pu$ where $\varpi \in R$ is a pseudo-uniformizer and $u \in W(R^+)^*$).*

Proof. By proposition 2.6, (1) is the same as sections of

$$\operatorname{Spa} \mathbb{Q}_p^\diamond \to S^\diamond$$

which are the same as maps $S^\diamond \to \operatorname{Spa} \mathbb{Q}_p^\diamond$, which is (2).

The set (2) is

$$\operatorname{Hom}_{\widetilde{\mathrm{Perf}_{\mathbb{F}_p}}}(S^\diamond = \operatorname{Hom}(-, S), \operatorname{Spa} \mathbb{Q}_p^\diamond)$$

which by Yoneda is $\operatorname{Spa} \mathbb{Q}_p^\diamond(S)$, which is (3).

Finally, the identification (3) = (4) is a generalization of the final Lemma from the Peter's discussion yesterday, the idea being that $\ker \theta$ is always generated by a degree 1 primitive element. □