GEOMETRIC INTERPRETATION OF ORBITAL INTEGRALS

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1. Geometric expansion

This is is a talk about "geometrization of the geometric side of the analytic RTF". Yesterday we introduced $\mathbb{J}(f,s)$. This has a geometric expansion and a spectral expansion; we will focus on the geometric expansion:

$$\mathbb{J}(f,s) = \sum_{u \in \mathbf{P}^1(F) - \{1\}} \mathbb{J}(u,f,s).$$

1.1. Orbital integrals.

1.1. Orbital integrals. The regular semisimple orbital integrals correspond to $u \neq 0, \infty$:

$$\mathbb{J}(u,f,s) = \mathbb{J}(\gamma,f,s) = \int_{A(\mathbf{A}) \times A(\mathbf{A})} f(h_1^{-1}\gamma h_2)|h_1 h_2|^s \eta(h_2)\, dh_1 dh_2$$

where $\mathrm{inv}(\gamma) = u$. For these γ, there are no convergence issues because the conjugacy class is closed in $G(\mathbf{A})$, and f has compact support in $G(\mathbf{A})$. So no regularization is needed in this case.

We can restrict our attention to Hecke functions of the form $f = h_D$, for D an effective divisor.

1.2. Observation. We can compute the orbital integral on GL_2, as follows. If $\widetilde{\gamma}$ is a lift of γ, and $D = \sum n_x x$, we can define

$$\widetilde{h} := \bigotimes_x \widetilde{h}_{n_x,x} \in \mathcal{H}_x(\mathrm{GL}_2)$$

where

$$\widetilde{h}_{n_x,x} = \mathbf{1}_{\mathrm{Mat}_2(\mathcal{O}_x)_{\mathrm{val}(\det)=n_x}} \in \mathcal{H}_x(\mathrm{GL}_2).$$

Remark 1.1. The $\widetilde{h}_D$ is not a pullback of h_D; rather, it is a lift.

Lemma 1.2. *We have*

$$\mathbb{J}(\gamma, h_D, s) = \int_{\Delta(Z(\mathbf{A}))\backslash(\widetilde{A}\times\widetilde{A})(\mathbf{A})} \widetilde{h}_D(h_1^{-1}\widetilde{\gamma}h_2)|\alpha(h_1)\alpha(h_2)|^s \eta(\alpha(h_2))\, dh_1 dh_2.$$

Here $\widetilde{A}$ is the diagonal torus in GL_2, and $\alpha\colon \begin{pmatrix} a & \\ & d \end{pmatrix} \mapsto a/d.$

Proof. Clear. □

1.3. **Geometrization.** Note that $\widetilde{h}_D(h_1^{-1}\widetilde{\gamma}h_2)$ only depends on the value of h_1 and h_2 in $\widetilde{A}(\mathbf{A})/\widetilde{A}(\mathbf{O})$. Since $\widetilde{A} \cong \mathbf{G}_m^2$, we have

$$\widetilde{A}(\mathbf{A})/\widetilde{A}(\mathbf{O}) \cong (\mathbf{G}_m(\mathbf{A})/\mathbf{G}_m(\mathbf{O}))^2 = (\operatorname{Div} X)^2.$$

The condition that $\widetilde{h}_D = 1$ defines a subset of $\Delta(\operatorname{Div} X)\backslash(\operatorname{Div} X)^4$. We'll first describe the subset in $(\operatorname{Div} X)^4$ before quotienting by center. It will be denoted

Definition 1.3. We define $\widetilde{\mathcal{N}}_{D,\widetilde{\gamma}} \subset (\operatorname{Div} X)^4$ to be the set of $(E_1, E_2, E_1', E_2') \in \operatorname{Div}(X)^4$ which are all effective, such that the rational map $\mathbf{O}^2 \xrightarrow{\widetilde{\gamma}} \mathbf{O}^2$ induces a holomorphic map

$$\begin{array}{ccc} \mathbf{O}^2 & \xrightarrow{\widetilde{\gamma}} & \mathbf{O}^2 \\ \uparrow & & \uparrow \\ \mathcal{O}(-E_1) \oplus \mathcal{O}(-E_2) & \xrightarrow{\phi_{\widetilde{\gamma}}} & \mathcal{O}(-E_1') \oplus \mathcal{O}(-E_2') \end{array}$$

such that $\operatorname{Div} \varphi_{\widetilde{\gamma}} = D$. Finally, we define

$$\mathcal{N}_{\widetilde{\gamma},D} := \widetilde{\mathcal{N}}_{\widetilde{\gamma},D}/\Delta(\operatorname{Div} X).$$

The upshot is

$$\mathbb{J}(\gamma, h_D, s) = \sum_{E_1,E_2,E_1',E_2' \in \mathcal{N}_{\widetilde{\gamma},D}} q^{-\deg(E_1-E_2+E_1'-E_2')s}\eta(E_1)\eta(E_2)$$

Since η is a quadratic character, we can rewrite this as

$$\mathbb{J}(\gamma, h_D, s) = \sum_{E_1,E_2,E_1',E_2' \in \mathcal{N}_{\widetilde{\gamma},D}} q^{-\deg(E_1-E_2+E_1'-E_2')s}\eta(E_1-E_1')\eta(E_2-E_1') \quad (1.1)$$

Here $h_1 \leftrightarrow (E_1', E_2')$ and $h_2 \leftrightarrow (E_1, E_2)$.

The idea of geometrization is that the formula (1.1) should be expressible as the sum, over k-points of a scheme, of the value at that point of the function associated to a sheaf on the scheme. Through this we can relate the formula to Lefschetz cohomology.

2. The moduli spaces

Definition 2.1. Let $\widehat{X}_d \to \operatorname{Pic}_X^d$ be the moduli space of sections, i.e.

$$\widehat{X}_d(S) = \left\{(\mathcal{L}, s) \colon \begin{array}{c} \mathcal{L} = \text{degree } d \text{ line bundle on } X \times S \\ s \in H^0(X \times S, \mathcal{L}) \end{array}\right\}.$$

Let $X_d = \operatorname{Sym}^d X = X^d//S_d$. This is a scheme, with a natural embedding $X_d \hookrightarrow \widehat{X}_d$ sending

$$(t_1, \ldots, t_d) \mapsto (\mathcal{O}(t_1 + \ldots + t_d), 1).$$

This is an isomorphism onto the open subscheme of X^d where the section is not the zero section.

Note that $\widehat{X}_d \setminus X_d \xrightarrow{\sim} \mathrm{Pic}_X^d$. The composition

$$X_d \hookrightarrow \widehat{X}_d \to \mathrm{Pic}_X^d$$

is the Abel-Jacobi map.

Definition 2.2. For $d = \deg D$, let

$$\Sigma_d = \left\{ \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \mid d_{ij} \in \mathbf{Z}_{\geq 0}, d_{11} + d_{22} = d_{12} + d_{21} = d \right\}$$

Given $\underline{d} \in \Sigma_d$, we define the moduli space $\widetilde{\mathcal{N}}_{\underline{d}}$ classifying

- four line bundles K_1, K_2, K_1', K_2' such that
$$\deg K_i' - \deg K_j = d_{ij}.$$
- A map $\varphi \colon K_1 \oplus K_2 \to K_1' \oplus K_2'$, which we can write as
$$\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$$
with $\varphi_{ij} \colon K_i \to K_j'$, satisfying some technical conditions. One example is if $d_{11} < d_{22}, d_{12} < d_{21}$
$$\varphi_{11} \neq 0, \varphi_{12} \neq 0 \tag{2.1}$$
and $\varphi_{21}, \varphi_{22}$ are not both 0.

There is an obvious action of Pic_X on $\widetilde{\mathcal{N}}_{\underline{d}}$, and we define

$$\mathcal{N}_{\underline{d}} = \widetilde{\mathcal{N}}_{\underline{d}} / \mathrm{Pic}_X .$$

Definition 2.3. We define the moduli space $\mathcal{A}_d$ classifying (Δ, a, b) where

- $\Delta \in \mathrm{Pic}_X^d$ and
- $a, b \in H^0(X, \Delta)$ are global sections not vanishing simultaneously.

Remark 2.4. The scheme $\mathcal{A}_d$ is covered by two pieces

$$X_d \times_{\mathrm{Pic}_X^d} \widehat{X}_d$$

and

$$\widehat{X}_d \times_{\mathrm{Pic}^d} X_d.$$

The morphism $X_d \to \mathrm{Pic}_X^d$ is representable, the fibers being vector spaces, hence $\mathcal{A}_d$ is a scheme.

Definition 2.5. We define a map

$$f_{\underline{d}} \colon \mathcal{N}_{\underline{d}} \to \mathcal{A}_d$$

sending

$$(K_1, K_2, K_1', K_2') \mapsto (K_1' \otimes K_2' \otimes K_1^\vee \otimes K_2^\vee, \varphi_{11} \otimes \varphi_{22}, \varphi_{12} \otimes \varphi_{21}).$$

Proposition 2.6. *$\mathcal{N}_{\underline{d}}$ enjoys the following properties.*

1. *$\mathcal{N}_{\underline{d}}$ is a geometrically connected scheme over k.*
2. *If $d \geq 4g - 3$, $\mathcal{N}_{\underline{d}}$ is smooth of dimension $2d - g + 1$.*
3. *The morphism $f_{\underline{d}}$ is proper.*

Proof. Use the non-vanishing conditions to find a covering of $\mathcal{N}_{\underline{d}}$ analogous to the covering of $\mathcal{A}_d$ discussed above. This imnplies (1) + (3). (Properness reduces to properness of $X_{d_{ij}}$.) For (2), by Riemann-Roch the map $\widehat{X}_{d_{ij}} \to \mathrm{Pic}_X^{d_{ij}}$ is smooth of relative dimension $1-g+d_{ij}$ if d_{ij} is large. If d is large then at least one of the relevant d_{ij} is large, and you use that one to run this argument. □

3. Geometrization of the analytic RTF

We now define a crucial local system $L_{\underline{d}}$ on $\mathcal{N}_{\underline{d}}$. By geometric class field theory there is a rank 1 local system on the Picard scheme of X corresponding to the quadratic character η. We first define a local system L_d on $\widehat{X}_d$ as the pullback of this local system via the map $\widehat{X}_d \to \mathrm{Pic}_X \to \mathrm{Pic}_X^{\mathrm{coarse}}$.

There is an open embedding

$$\mathcal{N}_{\underline{d}} \hookrightarrow (\widehat{X}_{d_{11}} \times \widehat{X}_{d_{22}}) \times_{\mathrm{Pic}_X^d} (\widehat{X}_{d_{12}} \times \widehat{X}_{d_{21}}) \tag{3.1}$$

given by the universal φ_{ij}'s. Finally, we define the rank 1 local system $L_{\underline{d}}$ on $\mathcal{N}_{\underline{d}}$ to be the restriction of the local system

$$L_{d_{11}} \boxtimes \mathbf{Q}_\ell \boxtimes L_{d_{12}} \boxtimes \mathbf{Q}_\ell \text{ on } (\widehat{X}_{d_{11}} \times \widehat{X}_{d_{22}}) \times_{\mathrm{Pic}_X^d} (\widehat{X}_{d_{12}} \times \widehat{X}_{d_{21}})$$

to $\mathcal{N}_{\underline{d}}$ via (3.1).

Definition 3.1. We define

$$\delta\colon \mathcal{A}_d \to \widehat{X}_d$$

to be the morphism sending

$$(\Delta, a, b) \mapsto (\Delta, a-b).$$

We also define

$$\mathcal{A}_D := \delta^{-1}(\mathcal{O}(D), 1) \cong \Gamma(X, \mathcal{O}_X(D)).$$

and the invariant map

$$\mathrm{inv}_D\colon \mathcal{A}_D(k) \to \mathbf{P}^1(F) - \{1\}$$

sending $a \mapsto 1 - a^{-1}$, viewing a as a rational function in F.

Proposition 3.2. *Assume $u \neq 0, \infty$.*

(1) If $u \notin \mathrm{Im}\,\mathrm{inv}_D$, then $\mathbb{J}(u, h_D, s) = 0$.

(2) If $u = \mathrm{inv}_D(a)$ for $a \in \mathcal{A}_D(k)$, then

$$\mathbb{J}(u, h_D, s) = \sum_{\underline{d} \in \Sigma_d} q^{(2d_{12}-d)s} \,\mathrm{Tr}(\mathrm{Frob}_a, Rf_{\underline{d},*} L_{\underline{d}})_{\overline{a}}$$

Proof. (2) We have a bijection

$$\mathcal{N}_{D,\widetilde{\gamma}} \xrightarrow{\sim} \mathcal{N}_a(k)$$

where $\mathcal{N}_a(k) = \coprod_{\underline{d} \in \Sigma_d} f_{\underline{d}}^{-1}(a)$ sending

$$(E_1, E_2, E_1', E_2') \mapsto (\mathcal{O}(-E_1), \mathcal{O}(-E_2), \mathcal{O}(-E_1'), \mathcal{O}(-E_2'), \varphi_{\widetilde{\gamma}}).$$

Use (1.1) and the definition of $L_{\underline{d}}$. □