Banach-Colmez Spaces

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We fix C to be the completion of an algebraic closure of $\mathbb{Q}_p$.

1 Definition of Banach-Colmez Spaces

Definition 1.1. A *Banach sheaf* is a contravariant functor $\mathcal{F}$ from Perf_C to topological $\mathbb{Q}_p$-vector spaces such that

1. if $X \in \mathrm{Perf}_C$ is affinoid perfectoid then $\mathcal{F}(X)$ is a Banach space
2. $\mathcal{F}$ is a sheaf on $\mathrm{Perf}_{C,\text{pro-étale}}$.

Morphisms are morphisms of functors (i.e. natural transformations).

A sequence of Banach sheaves is exact if it is is exact as a sequence of sheaves on $\mathrm{Perf}_{C,\text{pro-étale}}$.

Example 1.2. Let V be a finite-dimensional $\mathbb{Q}_p$ vector space. Then we have a constant Banach sheaf $\underline{V}$. This is represented by $\mathrm{Spa}(\mathrm{Funct}(V, C), \mathrm{Funct}(V, \mathcal{O}_C))$. The sections can be described explicitly as $\mathcal{F}(X) = \mathrm{Funct}(|X|, V)$.

Example 1.3. Let W be a finite-dimensional C-vector space. We can form $\mathcal{F} = W \otimes \mathcal{O}$, which is representable by $W \otimes \mathbb{G}_a$ (but this is not a perfectoid space).

Definition 1.4. An *effective Banach-Colmez space* is a Banach sheaf which is an extension

$$0 \to \underline{V} \to \mathcal{F} \to W \otimes \mathcal{O} \to 0.$$

Where W is a C-vector space and V is a $\mathbb{Q}_p$-vector space.

Definition 1.5. A *Banach-Colmez space* is a Banach sheaf $\mathcal{F}$ which is a quotient of an effective Banach-Colmez space by a $\mathbb{Q}_p$-vector space:

$$0 \to \underline{V'} \to \underbrace{\mathcal{F}'}_{\text{effective}} \to \mathcal{F} \to 0$$

The category of such is denoted $\mathcal{BC}$. If $\mathcal{F}$ is a BC and V, W, V' are as before we call $\dim_C W$ the *dimension* of the presentation of $\mathcal{F}$ and we call $\dim_{\mathbb{Q}_p} V - \dim_{\mathbb{Q}_p} V'$ the *height* of the presentation.

Example 1.6. We have $\dim \underline{\mathbb{Q}_p} = 0$ and $\operatorname{ht} \underline{\mathbb{Q}_p} = 1$; while $\dim \mathcal{O} = 1$ and $\operatorname{ht} \mathcal{O} = 0$.

Theorem 1.7 (Colmez)**.** *$\mathcal{BC}$ is an abelian category. The functor $\mathcal{F} \mapsto \mathcal{F}(C)$ is exact and conservative. Moreover, the dimension and the height do not depend on the presentation. Objects of $\mathcal{BC}$ have a Harder-Narasimhan filtration for the slope function $\mu = -\frac{\mathrm{ht}}{\dim}$.*

2 Examples as universal covers of p-divisible groups

2.1 Universal cover of a p-divisible group

Let G be a p-divisible group over $\mathcal{O}_C$. We can take $\mathcal{G}$ to be the formal completion of G along its unit section, which is a formal group scheme over $\operatorname{Spf} \mathcal{O}_C$. We can then take the generic fiber

$$\mathcal{G}_\eta^{\mathrm{ad}} := \mathcal{G}^{\mathrm{ad}} \times_{\mathrm{Spa}(C,\mathcal{O}_C)} \mathrm{Spa}(\mathcal{O}_C, \mathcal{O}_C).$$

There is an exact sequence of étale sheaves on the big étale site of $\mathrm{Spa}(C)$

$$0 \to \underline{\mathcal{G}_\eta^{\mathrm{ad}}[p^\infty]} \to \mathcal{G}_\eta^{\mathrm{ad}} \xrightarrow{\log} \operatorname{Lie} G[1/p] \otimes \mathbb{G}_a \to 0.$$

Here $\mathcal{G}_\eta^{\mathrm{ad}}[p^\infty] = \mathbb{Q}_p/\mathbb{Z}_p \otimes T_p(G)$. Taking the inverse limit of this sequence along the pth power map, we get a short exact sequence

$$0 \to \underline{T_p(G)[p^{-1}]} \to \widetilde{\mathcal{G}_\eta^{\mathrm{ad}}} := \varprojlim_{x \mapsto px} \mathcal{G}_\eta^{\mathrm{ad}} \to \mathrm{Lie}(G)[1/p] \otimes \mathbb{G}_a \to 0.$$

Definition 2.1. This $\widetilde{\mathcal{G}_\eta^{\mathrm{ad}}}$ is called the *universal cover* of the p-divisible group G.

Example 2.2. For $G = \widehat{\mathbb{G}_m}^d$, if R is an affinoid perfectoid C-algebra then

$$\widetilde{\mathcal{G}}_\eta^{\mathrm{ad}}(R) \cong \varprojlim_p (R^{00})^d \cong (R^{\flat,00})^d.$$

For any G, the universal cover $\widetilde{\mathcal{G}}_\eta^{\mathrm{ad}}$ is an effective BC space.

Example 2.3. For $G = \mu_{p^\infty}$ we have $T_p(G) = \mathbb{Z}_p$ (no Galois action since we're over C) and

$$\widetilde{\mu_{p^\infty}}(R) = \varprojlim_p 1 + R^{00} \xrightarrow{\log} R$$

sending $(x_n) \mapsto \log x_0$.

Remark 2.4. The sheaf $\widetilde{\mathcal{G}}_\eta^{\mathrm{ad}}$ is always representable by a perfectoid open ball. It is a Banach-Colmez space with height and dimension equal to those of G.

2.2 p-divisible groups parametrize all Banach-Colmez spaces

Definition 2.5. Define $\mathcal{BC}^+_{p-div}$ to be the full subcategory of $\mathcal{BC}$ consisting of universal covers of p-divisible group. Define $\mathcal{BC}_{p-div}$ to be the full subcategory of $\mathcal{BC}$ obtained as a quotient of an object of $\mathcal{BC}^+_{p-div}$ by a $\mathbb{Q}_p$-vector space.

Proposition 2.6. *We have* $\mathcal{BC}_{p-div} \cong \mathcal{BC}$.

Proof. We need to show that every BC space is in $\mathcal{BC}_{p-div}$, i.e. any extension of $W \otimes \mathcal{O}$ by V can be recovered as a quotient of a universal cover of a p-divisible group by a $\mathbb{Q}_p$-vector space.

One important input we need is that $\mathrm{Ext}^1_{\mathcal{BC}}(W \otimes \mathcal{O}, V) \cong \mathrm{Hom}_C(W, V \otimes C)$. What does this even mean? It is saying that any extension

$$0 \to \underline{V} \to \mathcal{F} \to W \otimes \mathcal{O} \to 0$$

fits into a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{V} & \longrightarrow & \mathcal{F} & \longrightarrow & W \otimes \mathcal{O} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow {\scriptstyle f \in \mathrm{Hom}(W, V \otimes C)} & & \\ 0 & \longrightarrow & \underline{V} & \longrightarrow & V \otimes \widetilde{\mu_{p^\infty,\eta}}^{\mathrm{ad}} & \longrightarrow & V \otimes \mathcal{O} & \longrightarrow & 0 \end{array}$$

A result of Fargues, Scholze-Weinstein then says that for all $(W, V, f : W \to V \otimes C)$ there exists a p-divisible group G over $\mathcal{O}_C$ such that $\widetilde{\mathcal{G}}^{\mathrm{ad}}_\eta = \mathcal{F}$. (They take $V = T_p(G)[1/p]$, $W = \mathrm{Lie}(G)[1/p]$, and f is the transpose of the Hodge-Tate map of G^D). Since the fact about $\mathrm{Ext}^1_{\mathcal{BC}}$ tells us that $\mathcal{F}$ is of this form, we are done. □

Example 2.7. Let $G = \mu_{p^\infty}$. Then

$$\widetilde{\mathcal{G}}^{\mathrm{ad}}_\eta(R) = B^+_{\mathrm{cris}}(R^0/p)^{\varphi = p} = B(R)^{\varphi = p}.$$

Corollary 2.8. *Banach-Colmez spaces are diamonds over* $C^\flat$.

Now we want to describe more explicitly the objects of $\mathcal{BC}^+ = \mathcal{BC}^+_{p-div}$. Fix a section $\overline{\mathbb{F}}_p \hookrightarrow \mathcal{O}_C/p$. Given a p-divisible group G over $\mathcal{O}_C$, there exists a p-divisible group H over $\overline{\mathbb{F}}_p$ and an isogeny

$$H \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_C/p \cong G \otimes_{\mathcal{O}_C} \mathcal{O}_C/p$$

Theorem 2.9 (Scholze-Weinstein)**.** *Let R be a C-perfectoid algebra. Then*

$$\widetilde{\mathcal{G}}^{\mathrm{ad}}_\eta(R) = \widetilde{H}^{\mathrm{ad}}_\eta(R^0/p) = \mathrm{Hom}_{R^0/p}(\mathbb{Q}_p/\mathbb{Z}_p, H)[1/p] = D(H)(R^0/p)[1/p]^{\varphi = p}$$

Evaluating the associated Banach-Colmez space on C gives

$$0 \to \underline{\mathbb{Q}_p} \to (B^+_{\mathrm{cris}})^{\varphi = p} \xrightarrow{\theta} C \to 0.$$

3 Banach-Colmez Spaces and the Fargues-Fontaine curve

3.1 A t-structure

Let $X = X_{\mathbb{Q}_p, C^\flat}$. We have the abelian category Coh_X on X.

Definition 3.1. We define an abelian category.

$$\mathrm{Coh}_X^{0,-} = \left\{ \mathcal{F} \in D^b(X) \colon \begin{array}{c} H^i(\mathcal{F}) = 0 \forall i \neq -1, 0 \\ \mu(H^0(\mathcal{F})) \geq 0 \\ \mu(H^{-1}(\mathcal{F})) < 0 \end{array} \right\}$$

The point is that we can think of coherent sheaves with positive/negative slopes as a torsion pair, since the classification theorem tells us that $\mathrm{Hom}(\mathcal{E}, \mathcal{E}') = 0$ and $\mathrm{Ext}^1(\mathcal{E}', \mathcal{E}) = 0$ if $\mu(\mathcal{E}) > \mu(\mathcal{E}')$. Therefore, one (admittedly convoluted) way of describing $\mathcal{F} \in \mathrm{Coh}_X$ is as a pair $(\mathcal{F}', \mathcal{F}'')$ with $\mu(\mathcal{F}') < 0$ and $\mu(\mathcal{F}'') \geq 0$ plus an element of $\mathrm{Ext}^1_{\mathrm{Coh}(X)}(\mathcal{F}', \mathcal{F}'') = 0$.

Analogously, we can think to an object of $\mathrm{Coh}_X^{0,-}$ as a pair $(\mathcal{F}', \mathcal{F}'')$ where $\mu(\mathcal{F}') < 0$ and $\mu(\mathcal{F}'') \geq 0$, plus an element of

$$\mathrm{Ext}^1_{\mathrm{Coh}_X^{0,-}}(\mathcal{F}'', \mathcal{F}[1]) = \mathrm{Ext}^2_{\mathrm{Coh}_X}(\mathcal{F}'', \mathcal{F}') = 0.$$

Remark 3.2. We can extend additively rank, deg to $D^b(X)$. We can define $\deg^{0,-} = -\operatorname{rank}$ and $\operatorname{rank}^{0,-} = \deg$. If $\mu^{0,-} = \frac{\deg^{0,-}}{\operatorname{rank}^{0,-}}$ then objects of $\mathrm{Coh}_X^{0,-}$ have an HN filtration for this slope function.

Theorem 3.3. $\mathrm{Coh}_X^{0,-} \cong \mathcal{BC}$.

3.2 Connection to the Fargues-Fontaine curve

Let S be a perfectoid space over $C^\flat$. Then one can define a relative Fargues-Fontaine curve X_S, which you can think of as a family of the usual curves $(X_{k(s)})_{s \in S}$.

Warning 3.4. There is no map $X_S \to S$. This is already the case over a field.

- The association $S \rightsquigarrow X_S$ is functorial,
- If $S' \to S$ is pro-étale (surjective) then $X_{S'} \to X_S$ is also.

This allows us to define a morphism of sites

$$\tau \colon (\text{big pro-étale site of } X) \to (\text{sheaves on } \mathrm{Perf}_{C^\flat, \text{pro-étale}})$$

by

$$\mathcal{F} \mapsto \tau_* \mathcal{F}(S) = H^0(X, \mathcal{F}_S := \mathcal{F}|_{X_S}).$$

Proposition 3.5. *Let* $\mathcal{F} \in \mathrm{Coh}_X$. *If* $\mu(\mathcal{F}) \geq 0$ *then* $R^i \tau_* \mathcal{F} = 0$ *for all* $i \neq 0$. *If* $\mu(\mathcal{F}) < 0$ *then* $R^i \tau_* \mathcal{F} = 0$ *for all* $i \neq 1$.

Corollary 3.6. *We have*

$$\mathrm{Coh}_X^{0,-} \cong \left\{ \mathcal{F} \in D^b(X) \colon \begin{array}{c} H^i(\mathcal{F}) = 0 \textit{ if } i \neq -1, 0 \\ R^0\tau_* H^{-1}(\mathcal{F}) = 0, \\ R^1\tau_* H^0(\mathcal{F}) = 0 \end{array} \right\}.$$

In other words, the functor $R^0\tau_*\colon \mathrm{Coh}_X^{0,-} \to \widetilde{\mathrm{Perf}}_{C^\flat,\text{pro-étale}}$ (where tilde means the category of sheaves) is exact.

This induces an equivalence $\mathrm{Coh}_X^{0,-} \cong \mathcal{BC}$, implicitly using Scholze to identify sheaves on the proétale sites of C and $C^\flat$.

Example 3.7. We have

$$R^0\tau_* \mathcal{O}_X(S) = H^0(X_S, \mathcal{O}_{X_S}) = B^+(R)^{\varphi=1} = \underline{\mathbb{Q}_p}.$$

where $S = \mathrm{Spa}(R)$. Also

$$R^0\tau_* \iota_{\infty *} C = \mathcal{O}.$$

By playing with the sequence

$$\mathcal{O}_X(-1) \to \mathcal{O}_X \to i_{\infty *} C$$

which can be tilted (in the sense of torsion pairs)

$$\mathcal{O}_X \to i_{\infty *} C \to \mathcal{O}_X(-1)[1].$$

we can show that the category depends only on $C^\flat$, and that the curve can be reconstructed from the BC category.