ANALYTIC RTF: SPECTRAL SIDE

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1. Decomposition of the kernel

Recall that we defined

$$\mathbb{J}(f,S)=\int_{[A]\times[A]}\mathbb{K}_f(h_1,h_2)|h_1h_2|^s\eta(h_2)\,dh_1dh_2.$$

We have an action of $G(\mathbf{A})$, and hence $C_c^\infty(G(\mathbf{A}))$, on the space of automorphic functions $L_0^2([G])$. We are going to try to decompose the kernel functions into three parts:

$$\mathbb{K}_f(x_1,x_2)=\mathbb{K}_{f,\mathrm{cusp}}+\mathbb{K}_{f,\mathrm{sp}}+\mathbb{K}_{f,\mathrm{Eis}}$$

corresponding to cuspidal, "special", and Eisenstein. This idea is essentially due to Selberg.

1.1. **The cuspidal part.** We have

$$\mathbb{K}_{f,\mathrm{cusp}}=\sum_\pi\mathbb{K}_{f,\pi}$$

where

$$\mathbb{K}_{f,\pi}(x,y)=\sum_\phi\pi(f)\phi(x)\overline{\phi(y)}.$$

where ϕ runs over an orthonormal basis.

1.2. **The special part.** Using the determinant map, we have a map

$$\chi\colon[G]\to F^\times\backslash\mathbf{A}^\times/(\mathbf{A}^\times)^2\to\{\pm1\}.$$

Then

$$\mathbb{K}_{f,\mathrm{sp},\chi}(x,y)=\pi(f)\chi(x)\overline{\chi(y)}.$$

This is the same expression as for the cuspidal part, actually - it just looks much simpler because it is 1-dimensional.

1.3. **The Eisenstein part.** The Eisenstein part will be defined later.

1.4. **Goals:**

(1) Identify $f\in\mathcal{H}$ such that $\mathbb{K}_{f,\mathrm{Eis}}=0$.
(2) For such f, show that

$$\mathbb{J}_\pi(f)=\sum_\phi\frac{\mathcal{P}(\pi(f)\varphi)\overline{\mathcal{P}_\eta(\varphi)}}{\langle\varphi,\varphi\rangle}.$$

2. Satake isomorphism

Let $\mathcal{H}_G$ be the spherical Hecke algebra of G. By definition,

$$\mathcal{H}_G = \bigotimes_{x\in|X|}{}' \mathcal{H}_x.$$

For $A \subset G$ the split torus, we have $A \cong \mathbf{G}_m$. Then $\mathcal{H}_A = \bigotimes' \mathcal{H}_{A,x}$, and the local Hecke algebras are all isomorphic to

$$\mathcal{H}_{A,x} \cong \mathbf{Q}[F_x^\times/\mathcal{O}_x^\times] \cong \mathbf{Q}[t_x^{-1}, t_x]$$

where $t_x = \mathbf{1}_{\varpi_x^{-1}\mathcal{O}_x^\times}$.

The Weyl group action is, in this normalization,

$$\iota_x(t_x) = q_x t_x^{-1}.$$

The *Satake homomorphism*

$$\mathrm{Sat}_x \colon \mathcal{H}_x \to \mathcal{H}_{A,x}$$

sends $h_x \mapsto t_x + q_x t_x^{-1}$. In fact Sat_x is an isomorphism onto the subgroup of Weyl invariants.

The local Satake homomorphisms extend to a global one:

$$\mathrm{Sat}\colon \mathcal{H} \to \mathcal{H}_A^\iota.$$

3. Eisenstein ideal

3.1. **Definition of Eisenstein ideal.** We can identify $\mathbf{A}^\times/\mathcal{O}^\times \cong \mathrm{Div}(X)$. There is a map $\mathrm{Div}(X) \to \mathrm{Pic}(X)$. Now, $\mathcal{H}_A \cong \mathbf{Q}[\mathrm{Div}(X)] \to \mathbf{Q}[\mathrm{Pic}(X)]$.

The Weyl involution descends to ι_{Pic} on $\mathbf{Q}[\mathrm{Pic}(X)]$,

$$\mathbf{1}_{\mathcal{L}} \mapsto q^{\deg \mathcal{L}} \mathbf{1}_{\mathcal{L}^{-1}}.$$

Thus we have a map

$$a_{\mathrm{Eis}}\colon \mathcal{H} \xrightarrow{\mathrm{Sat}} \mathcal{H}_A^\iota \to \mathbf{Q}[\mathrm{Pic}(X)]^\iota.$$

Definition 3.1. We define the Eisenstein ideal to be $\mathcal{I}_{\mathrm{Eis}} := \ker a_{\mathrm{Eis}}$.

Theorem 3.2. *For $f \in \mathcal{I}_{\mathrm{Eis}}$,*

$$\mathbb{K}_{f,\mathrm{Eis}}(x,y) = 0.$$

Before we can prove this, we need to say what $\mathbb{K}_{f,\mathrm{Eis}}$ is. And before that, we need to define the Eisenstein series.

3.2. **Eisenstein series.** The Eisenstein representations are induced from A, so we should first parametrize the representations of A, which are necessarily characters. Note that

$$A(\mathbf{A}) = \mathbf{A}^\times \cong \mathbf{A}^1 \times \alpha^{\mathbf{Z}}$$

for some choice of $\alpha \in \mathbf{A}$ with $|\alpha| = q$. For a character

$$\chi \colon F^\times \backslash \mathbf{A}^1 \to \mathbf{C}^\times$$

we can extend to a character

$$\chi_0 \colon F^\times \backslash \mathbf{A} \to \mathbf{C}^\times$$

by sending $\chi(\alpha) = 1$.

More generally, for any $u \in \mathbf{C}$ we get a character

$$\chi_u \colon F^\times \backslash \mathbf{A}^1 \to \mathbf{C}^\times$$

sending $\chi_u(a) = \chi_0(a)|a|^u$.

For $B = A \ltimes U$ and $U = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$, we define the modular character

$$\delta_B \colon B(\mathbf{A}) \to \mathbf{A}^\times$$

by

$$\delta_B \begin{pmatrix} a & b \\ & d \end{pmatrix} = a/d.$$

Finally, we define

$$\phi \colon B(\mathbf{A}) \to \mathbf{C}^\times$$

by $b \mapsto \chi_0(a/d)$.

Definition 3.3. We define the *induced (Eisenstein) representation* $V_{\chi,u}$ by

$$V_{\chi,u} = \{\varphi \in C^\infty(G(\mathbf{A})) \mid \varphi(bg) = \chi(b)|\delta_B(b)|^{1/2+u}\varphi(g) \forall b \in B(\mathbf{A})\}.$$

3.3. **The Eisenstein kernel.** Take φ_i orthogonal basis of V_χ. Then

$$\mathbb{K}_{f,\mathrm{Eis}}(x,y) = \sum_\chi \mathbb{K}_{f,\mathrm{Eis},\chi}(x,y)$$

and

$$\mathbb{K}_{f,\mathrm{Eis},\chi}(x,y) = \frac{\log q}{2\pi i} \int_{0+0i}^{0+2\pi i/\log q} \sum_{i,j} (\rho_\chi \varphi_j, \varphi_i) E(x, \varphi_i, u, \chi) \overline{E(y, \varphi_j, u, x)}\, du$$

3.4. **Proof of Theorem 3.2.** Any $f \in \mathcal{H}$ is unramified, so $f_{\chi,u}$ is periodic under $u \mapsto u + \frac{2\pi i}{\log q}$. If χ is unramified, then

$$\mathbb{K}_{f,\mathrm{Eis},\chi} = \frac{\log q}{2\pi i} \int_0^{\frac{2\pi i}{\log q}} (\rho_{\chi,u}(f)\mathbf{1}_K, \mathbf{1}_K) \ldots \, du.$$

It is a property of the Satake transform that

$$\mathrm{Tr}(\rho_{\chi,u}(f)) = \chi_{u+1/2}(\mathrm{Sat}(f)) \tag{3.1}$$

Inflate the character $\chi_{u+1/2} \colon F^\times \backslash \mathbf{A}^\times / \mathbf{O}^\times \to \mathbf{C}^\times$ to $\chi_{u+1/2} \colon \mathbf{A}^\times / \mathbf{O}^\times \to \mathbf{C}^\times$. Thus we get a character of $\mathcal{H}_A$. Then (3.1) reads

$$\mathrm{Tr}(\rho_{\chi,u}(f)) = \chi_{u+1/2}(a_{\mathrm{Eis}}(f)) = 0.$$

□

4. Relation to L-functions

4.1. **Normalization of L-function.** We have

$$L(\pi_{F'}, s) = L(\pi, s) L(\pi \otimes \eta, s).$$

The functional equation reads

$$L(\pi_{F'}, s) = \epsilon(\pi, s) L(\pi_{F'}, 1 - s)$$

where

$$\epsilon(\pi, s) = q^{-8(q-1)(s-1/2)}.$$

Definition 4.1. We define the *normalized L-function*

$$\mathcal{L}(\pi_{F'}, s) = \epsilon(\pi_{F'}, s)^{-1/2} \frac{L(\pi_{F'}, s)}{L(\pi, \mathrm{Ad}, 1)}.$$

We write

$$\mathbb{J}(f, s) = \sum_{\pi} \mathbb{J}_\pi(f, s)$$

where

$$\mathbb{J}_\pi(f, s) = \sum_{\varphi} \frac{\mathcal{P}(\pi(f)\varphi, s) \mathcal{P}_\eta(\overline{\varphi}, s)}{\langle \varphi, \varphi \rangle}$$

for $\varphi \in \pi^K$. Here for any character χ,

$$\mathcal{P}_\chi(\varphi, s) = \int_{[A]} \varphi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \chi(h) |h|^s \, dh$$

If we write

$$I(s, \varphi, \chi) = \int_{F^\times \backslash \mathbf{A}^\times} \varphi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \chi(h) |h|^{s-1/2} \, dh \tag{4.1}$$

and $\widetilde{\varphi}(g) = \varphi({}^t g^{-1})$, then we have a functional equation

$$I(s, \varphi, \chi) = I(1 - s, \widetilde{\varphi}, \chi).$$

4.2. **Whittaker model.** To relate $\mathbb{J}_r(f)$ with derivatives of L-functions, we use "Whittaker models", which are automorphic variants of Fourier coefficients.

Let $\varphi \in V_\pi$. Then we get

$$\varphi \colon U(F)\backslash U(\mathbf{A}) \to \mathbf{C}$$

as follows. Note that $U \cong \mathbf{G}_a$, since

$$U = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}.$$

Since $U(F)\backslash U(\mathbf{A}) = \mathbf{F}\backslash\mathbf{A}$ for a character $\psi \colon F\backslash\mathbf{A} \to \mathbf{C}^\times$ we can identify

$$\widehat{(F\backslash\mathbf{A})} \cong \{\psi(\gamma x) \mid x \in \mathbf{A}, \gamma \in F\}.$$

Then we can define a *Whittaker function*

$$W_{\varphi,\psi_\gamma}(g) = \int_{F\backslash\mathbf{A}} \varphi\left(\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} g\right) \psi^{-1}(\gamma h)\, dn \tag{4.2}$$

Now we use a trick: by a change of variables, (4.2) is equal to

$$= \int_{F\backslash\mathbf{A}} \varphi\left(\begin{pmatrix} \gamma^{-1} & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right) \varphi\psi^{-1}(n)\, dn$$

Call this $W_{\varphi,\psi}(\gamma g)$. Then we have a "Fourier expansion"

$$\varphi = \sum_{\gamma \in F} W_{\varphi,\psi}(\gamma g).$$

Since f is cuspidal, the 0th Fourier coefficient vanishes. Also we have the identity

$$W_{\varphi,\psi}(ng) = \psi(n) W_\varphi(g).$$

In fact, this whole discussion applies locally, and we can define the local Whittaker function $W_{\varphi,\psi,x}$. The Whittaker function decomposes locally:

$$W_{\varphi,\psi} = \prod_{x \in |X|} W_{\varphi,\psi,x}.$$

If we write (4.1) as

$$\begin{aligned} \mathbb{I}(s,\varphi,\chi) &= \int_{F^\times\backslash\mathbf{A}^\times} \sum_{\gamma \in F^\times} W_\varphi(\gamma g) \begin{pmatrix} h & \\ & 1 \end{pmatrix} |h|^{s-1/2}\chi(h) dh \\ &= \int_{\mathbf{A}^\times} W_\varphi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \dots\, dh. \end{aligned}$$

Here we have used that since we are integrating over $F^\times\backslash\mathbf{A}^\times$ a sum over $F^\times$, we just get an integral over $\mathbf{A}^\times$ of something that decomposes locally as a product of local integrals. That's basically what an L-function is, so it is not surprising that the result is related to an L-function. However, there's an issue of test vectors. For almost all places, you get the right local factor. But at the finitely many bad places, you need to calculate a constant factor.