Vector Bundles on the Fargues-Fontaine Curve

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1 Preliminaries on the Fargues-Fontaine curve

Let E be a local ring with residue field $\mathbb{F}_q$. (We can imagine $E = \mathbb{Q}_p$ or $\mathbb{F}_q((t))$.) Let F be an algebraically closed perfectoid extension of $\mathbb{F}_q$. (In terms of the notation of Colmez's talk, $F = C^\flat$.)

We form the adic curve $X^{\text{ad}} := Y^{\text{ad}}/\varphi^{\mathbb{Z}}$. This has a map to the scheme-theoretic Fargues-Fontaine curve $X := \operatorname{Proj} P$, where

$$P = \bigoplus_{d \geq 0} B^{\varphi = \varpi^d}$$

where $B = O(Y^{\text{ad}})$ is a Fréchet algebra.

Theorem 1.1. *X is a regular noetherian scheme of dimension 1.*

If we fixed $\infty \in |X|$ (corresponding to an untilt C) then

$$X \setminus \{\infty\} = \text{Spec } B_e$$

where $B_e = B[1/t]^{\varphi=1}$.

Theorem 1.2 (Fargues-Fontaine)**.** *The ring B_e is a PID.*

We want to discuss the classification of vector bundles on the Fargues-Fontaine curve. Thanks to the following theorem, we can think interchangeably about the analytic or algebraic curve for this purpose.

Theorem 1.3 (Fargues-Fontaine, Hartl-Pink, Kedlaya-Liu)**.** *GAGA for X: the map $X^{\text{ad}} \to X$ induces an equivalence of categories*

$$\text{Bun}_X \cong \text{Bun}_{X^{\text{ad}}}.$$

2 Constructing vector bundles

2.1 Line bundles

We already know about the line bundles $\mathcal{O}(d)$ for $d \in \mathbb{Z}$. Are these all of them?

The answer is **yes**: $\operatorname{Pic} X \xrightarrow{\sim} \mathbb{Z}$ by $d \mapsto [\mathcal{O}(d)]$. This is saying that the curve has a well-defined notion of degree. This is extremely non-trivial: the usual theory of degree does not apply, because the curve X lives over $\mathbb{Q}_p$ but its residue fields are generally huge (infinite-dimensional over $\mathbb{Q}_p$).

What we are saying here is that if one *redefines* the degree of a point in an appropriate way then it is a theorem that the divisor of any function has degree 0, and that allows us to define a coherent notion of degree.

2.2 Higher rank vector bundles

We give an analytic construction of vector bundles on X^{ad}. The key point is that Y^{ad} lives over $\operatorname{Spa} L$ where $L = \breve{E} := \widehat{E^{\mathrm{unr}}}$. So one can pull back φ-equivariant bundles on $\operatorname{Spa} L$ to Y^{ad} to get a functor

$$(\varphi\text{-bundles on } \operatorname{Spa}(L, \mathcal{O}_L)) \to (\varphi\text{-bundles on } Y^{\mathrm{ad}}),$$

which then descend to bundles on X^{ad}. By GAGA (Theorem 1.3) this is the same as bundles on X.

This construction gives a functor

$$(\varphi - \text{bundles on } \operatorname{Spa}(L, \mathcal{O}_L)) \to \operatorname{Bun}_X .$$

But of course φ–bundles on $\operatorname{Spa}(L, \mathcal{O}_L)$ are simply classical *L-isocrystals*: finite-dimensional L-vector spaces equipped with bijective semi-linear "Frobenius" endomorphism φ.

This can be made concrete. For $D \in \varphi - \mathrm{Mod}_L$ we get a graded P-module

$$\mathcal{E}(D) := \bigoplus_{d \geq 0} (D \otimes_L B)^{\varphi = \varpi^d}.$$

One then takes the associated quasicoherent sheaf on X. (But it is not clear from this description that this is a vector bundle.)

Warning 2.1. $\mathcal{O}(1)$ depends on the choice of ϖ.

3 Geometric properties of $\mathcal{E}(D)$

If D is simple of slope $-\lambda$ ($\lambda \in \mathbb{Q}$) then we get a vector bundles $\mathcal{E}(D) =: \mathcal{O}_X(\lambda)$. The *Dieudonné-Manin theorem* gives a classification of irreducible isocrystals in terms of the slope.

What is $\mathcal{O}_X(\lambda)$ concretely? If $\lambda \in \mathbb{Z}$ then it is easy to show that

$$\mathcal{O}_X(\lambda) = \widetilde{P[\lambda]}$$

and this is a line bundle since P is degenerated in degree 1. In general, if $\lambda = \frac{d}{h}$ in reduced form then let E_h/E be the unramified extension of degree h. Then we get a curve $X_{E_h,F} \cong X \otimes_E E_h$ which is a finite étale covering of X with Galois group $\mathbb{Z}/h$. At the level of adic spaces the covering can be described simply as

$$Y^{\mathrm{ad}}/\varphi^{h\mathbb{Z}} \to Y^{\mathrm{ad}}/\varphi^{\mathbb{Z}}.$$

Then $\mathcal{O}_X(\lambda) := \pi_{h*}\mathcal{O}_{X_h}(d)$. Why? This comes from an understanding of the irreducible isocrystals.

Consequences:

1. $\mathcal{O}_X(\lambda) \in \mathrm{Bun}_X$ (since it's the pushforward of a line bundle via a finite étale map).
2. rank $\mathcal{O}_X(\lambda) = h$ and $\deg \mathcal{O}_X(\lambda) = d$, so the slope of $\mathcal{O}_X(\lambda)$ is λ.
3. $\mathcal{O}_X(\lambda)$ is semistable. (This can be checked after pulling back to the finite étale cover Y_{E_h}, where it becomes a direct sum of copies of line bundles of degree d.) In fact it is even stable, by the classification theorem.

4 Classification of vector bundles

4.1 The classification theorem

Theorem 4.1 (Fargues-Fontaine)**.** *The functor*

$$\varphi - \mathrm{Mod}_L \to \mathrm{Bun}_X$$

sending $D \mapsto \mathcal{E}(D)$ is essentially surjective. In other words, any vector bundle is isomorphic to a direct sum of $\mathcal{O}_X(\lambda)$:

$$\mathcal{E} \cong \mathcal{O}_X(\lambda_1) \oplus \ldots \oplus \mathcal{O}_X(\lambda_n) \text{ for some } \lambda_1 \geq \ldots \geq \lambda_n \in \mathbb{Q}.$$

This expression is unique.

Warning 4.2. This is *not* true over non-algebraically-closed fields.

Remark 4.3. An important consequence of the classification is that if $\mathcal{E} \in \mathrm{Bun}_X$ is non-zero then

$$\deg \mathcal{E} \geq 0 \implies H^0(\mathcal{E}) \neq 0.$$

This is really hard. To give an example, there is a huge space of extensions

$$0 \to \mathcal{O}(-1) \to \mathcal{E} \to \mathcal{O}(1) \to 0.$$

The classification implies that for any such extension $H^0(\mathcal{E}) \neq 0$. This is difficult; proving it is basically tantamount to proving the theorem.

We also have

$$\mathcal{O}(\lambda)^\vee = \mathcal{O}(-\lambda)$$

and

$$\mathcal{O}(\lambda) \otimes \mathcal{O}(\mu) = \mathcal{O}(\lambda + \mu)^{\oplus ?}$$

which is the best that one can hope for in consideration of the ranks.

4.2 Cohomology and consequences

We have

$$\mathrm{Hom}(\mathcal{O}(\lambda), \mathcal{O}(\mu)) = 0 \quad \text{if } \mu < \lambda.$$

On the other hand,

$$\mathrm{Ext}^1(\mathcal{O}(\lambda), \mathcal{O}(\mu)) = 0 \quad \text{if } \mu > 0.$$

These cohomology groups are really big; for instance

$$H^0(\mathcal{O}(1)) = \mathrm{Hom}(\mathcal{O}, \mathcal{O}(1)) = B^{\varphi=\varpi}$$

and

$$H^1(\mathcal{O}(-1)) \cong C/E,$$

which in particular is infinite-dimensional over E.

The "fundamental exact sequence"

$$0 \to E \to B_e \to B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \to 0$$

is equivalent to the statements $H^0(\mathcal{O}) = E$ and $H^1(\mathcal{O}) = 0$.

Corollary 4.4. *We have* $\pi_1(X) \cong \mathrm{Gal}(\overline{E}/E)$.

Proof. We have an obvious functor from finite extensions of E to finite étale covers of X, which on fields is $E' \mapsto X \otimes_E E'$. We want to show that this induces an equivalence of categories.

For $f : Y \to X$ with Y connected, we want to show that $f_* \mathcal{O}_Y$ is trivial vector bundle, because then we can try to recover E' as its global sections (it will be a finite-dimensional E-vector space with an E-algebra structure).

Using that that $\mathcal{E}$ has an algebra structure, the classification theorem implies that all slopes of $\mathcal{E}$ are ≤ 0 because if it has some component with positive slope λ, then

$$\mathcal{O}(\lambda) \otimes \mathcal{O}(\lambda) \to \mathcal{E}$$

must be zero since the description of cohomology tells us that a vector bundle on the Fargues-Fontaine curve cannot admit a non-zero map to a vector bundle of smaller slope. Since $\mathcal{E}$ is self-dual we also get that all slopes are non-negative, so all slopes are 0. The classification theorem then implies that the bundle is trivial. □

5 Link with p-divisible groups

Let $E = \mathbb{Q}_p$ for simplicity. Fix $\infty \in |X|$ with residue field C (an algebraically closed and complete extension of E). We can then form B_{dR}, etc. (see Colmez's lecture).

5.1 Miniscule modifications

For H a p-divisible group over $\overline{\mathbb{F}_p}$, there is an associated covariant Dieudonné isocrystal D, giving a bundle $\mathcal{E} = \mathcal{E}(D) \otimes \mathcal{O}(1)$. (The twisting by 1 is an artifact of the definition of duality for isocrystals, which is normalized to send an isocrystal with slope in $[0, 1]$ to another isocrystal with slope in $[0, 1]$.)

Definition 5.1. A *degree $n \in [0, ht\, H]$ miniscule modification* of $\mathcal{E}$ is a vector bundle $\mathcal{F}$ fitting into a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{E} \to i_* W \to 0$$

where $i \colon \{\infty\} \hookrightarrow X$ and $\dim_C W = n$.

The key idea is that one can make many modifications trivial using periods of p-divisible groups. The mechanism for this is the *period morphism*, which we now explain.

5.2 The period morphism

To H we can attach a *Rapoport-Zink space*, which is rigid space $\mathcal{M}/L$ classifying deformations for p-divisible groups, but where we take deformations not by isomorphisms but by *quasi-isogenies*. There exists an étale period map

$$\mathcal{M} \to Fl$$

where Fl is the flag variety of n-dimensional quotients of D, with $n = \dim H$. The period map is

$$\mathcal{G} \mapsto \mathrm{Lie}\, \mathcal{G}[1/p].$$

The key fact is that $i^*\mathcal{E} \cong D \otimes C$. Granting this fact, the map from $Fl(C)$ to the set of degree n miniscule modifications can be described as

$$x = [D \otimes C \twoheadrightarrow W] \mapsto \mathcal{E}(x) := \ker[\mathcal{E} \to i_* i^* \mathcal{E} \cong i_*(D \otimes C) \twoheadrightarrow i_* W].$$

Theorem 5.2. *If x is in the image of the period map, then $\mathcal{E}(x)$ is trivial. (So $\mathcal{E}(x) = V_p(\mathcal{G}) \otimes \mathcal{O}_X$).*

Remark 5.3. This is a "sheafy" version of the p-adic comparison theorem for p-divisible groups, whose proof is an easy consequence of the usual version.

There are essentially two cases in which the period map is surjective.

1. If we choose H to be a 1-dimensional height h formal group over $\overline{\mathbb{F}}_p$ then $Fl \cong \mathbb{P}^{h-1}$ (Gross-Hopkins, Laffaille).

2. If we choose H to be a special formal module in the sense of Drinfeld.

Therefore, in these cases p-divisible groups give us many modifications with $\mathcal{F}$ be trivial.

6 Sample of ideas in the proof of the classification

The main technical results are that for all $n \geq 1$:

1. all degree 1 modifications of $\mathcal{O}(1/n)$ are trivial (this is a trivial consequence of the theorem statement, but a significant step in the proof).
2. $\mathcal{O}(-1/n)$ is the only degree 1 modification of $\mathcal{O}^{\oplus n}$ without global sections.

Let's sketch how these facts are used in an example. We'll prove that any $\mathcal{E}$ fitting into an extension

$$0 \to \mathcal{O}(-1) \to \mathcal{E} \to \mathcal{O}(1) \to 0$$

has global sections. (This is a special case of Remark 4.3.) Suppose otherwise. There is a map $\mathcal{O}(1) \to i_*C$. Consider the composite $\mathcal{E} \to \iota_*C$ and define $\mathcal{F} := \ker(\mathcal{E} \to i_*C)$. So $\mathcal{F}$ is an extension

$$0 \to \mathcal{O}(-1) \to \mathcal{F} \to \mathcal{O} \to 0.$$

By the construction of $\mathcal{F}$ we also have an extension

$$0 \to \mathcal{F} \to \mathcal{E} \to i_*C \to 0.$$

Now for a trick: pick an embedding $\mathcal{O}(-1) \hookrightarrow \mathcal{O}$ and consider the pushout

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(-1) & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \end{array}$$

Then $\mathcal{F}' \cong \mathcal{O}^2$ because $H^1(\mathcal{O}) = 0$. So $\mathcal{F}$ is a degree 1 modification of $\mathcal{O}^2$ but it has no global sections because $\mathcal{E}$ doesn't and $\mathcal{F} \hookrightarrow \mathcal{E}$, so by fact (2) above $\mathcal{F} \cong \mathcal{O}(-1/2)$.

Now dualize the sequence

$$0 \to \mathcal{O}(-1/2) \to \mathcal{E} \to i_*C \to 0$$

to get

$$0 \to \mathcal{E}^\vee \to \mathcal{O}(1/2) \to i_*C \to 0.$$

But this is trivial by (1), so $\mathcal{E}^\vee$ is trivial, hence $\mathcal{E}$ is trivial, contradicting the assumption that $\mathcal{E}$ has no global sections.