ANALYTIC RTF: GEOMETRIC SIDE

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1. The big picture

Yesterday we defined a certain "geometric" quantity $\mathbb{I}_r(f)$. Today we will define an "analytic" quantity $\mathbb{J}_r(f)$. Both of these have two expansion:

$$\sum_{u\in\mathbf{P}^1(F)-0} I_\gamma(u,f) ===\!= I_r(f) ===\!= \sum_\pi I_r(\pi,f). \tag{1.1}$$

and

$$\sum_{u\in\mathbf{P}^1(F)-0} J_\gamma(u,f) ===\!= J_r(f) ===\!= \sum_\pi J_r(\pi,f). \tag{1.2}$$

The left sides of (1.1), (1.2) are expansions in terms of orbital integrals. The right side are the quantities that we want to compare: $\mathbb{J}_r(\pi,f) \sim L^{(r)}(\pi_F,1/2)$, and $\mathbb{I}_r(\pi,f) = \langle [\mathrm{Sht}_T]_\pi, f * [\mathrm{Sht}_T]_\pi\rangle$. The functions f are "test functions" which provide the flexibility to isolate the terms of interest.

2. The relative trace formula

Let F be a global field, the function field of X for $X/\mathbf{F}_q$. Let G/F be a reductive group, and $H_1, H_2 \hookrightarrow G$ subgroups over F. We'll write $[G] := G(F)\backslash G(\mathbf{A})$, and similarly for H_i.

2.1. **The kernel.** Let $\mathbf{A} = \mathbf{A}_F$, $[G] = G(F)\backslash G(\mathbf{A})$. For $f \in C_c^\infty(G(\mathbf{A}))$, we define the kernel function

$$\mathbb{K}_f(g_1,g_2) := \sum_{\gamma\in G(F)} f(g_1^{-1}\gamma g_2).$$

The point is that $G(\mathbf{A})$ acts on $C^\infty(G(F)\backslash G(\mathbf{A}))$, and for $\phi \in C^\infty(G(F)\backslash G(\mathbf{A}))$ we have

$$\pi(f)\cdot\phi = \int_{G(F)\backslash G(\mathbf{A})} \mathbb{K}_f(g_1,g_2)\phi(g_2)\,dg_2. \tag{2.1}$$

The relative trace formula involves the quantity

$$\int_{[H_1]\times[H_2]} \mathbb{K}_f(h_1,h_2)\,dh_1dh_2 \tag{2.2}$$

2.2. **The geometric expansion.** The geometric expansion of (2.2) is

$$\int_{[H_1]\times[H_2]} \sum_{\gamma\in G(F)} f(h_1^{-1}\gamma h_2)\, dh_1 dh_2$$

$$= \sum_{\gamma\in H_1(F)\backslash G(F)/H_2(F)} \int_{[H_1]\times[H_2]} \sum_{\delta\in H_1(F)\gamma H_2(F)} f(h_1^{-1}\delta h_2)\, dh_1 dh_2$$

Rearranging, one rewrites this as

$$= \sum_{\gamma\in H_1(F)\backslash G(F)/H_2(F)} \int_{(H_1\times H_2)_\gamma(F)\backslash H_1(\mathbf{A})\times H_2(\mathbf{A})} f(h_1^{-1}\gamma h_2).$$

Here $_\gamma$ denotes the stabilizer of γ:

$$(H_1 \times H_2)_\gamma := \{(h_1, h_2) \colon h_1^{-1}\gamma h_2 = \gamma\}.$$

2.3. **The spectral expansion.** Now we rewrite (2.1) in a different way. The idea is to decompose

$$\mathbb{K}_f \approx \sum_{\pi \text{ cuspidal}} \sum_{\phi} \pi(f)\phi \otimes \overline{\phi}$$

where ϕ runs over orthogonal basis of π. This is a bit of lie, as one also needs to consider the residual and Eisenstein parts, but it roughly works. Using this, we can rewrite

$$(2.1) = \sum_{\pi} \sum_{\phi} \int_{[H_1]} \pi(f)\phi\, dh_1 \cdot \int_{[H_2]} \overline{\phi}\, dh_2$$

Here the term $\int_{[H_1]} \pi(f)\phi$ is a "period" $\mathcal{P}_{H_1}(\pi(f)\phi)$.

2.4. **The split subtorus.** For $G = \mathrm{PGL}_2$, set $H_1 = H_2 = A$ to be the diagonal torus of G. For $f \in C_c^\infty(G(\mathbf{A}))$ we get a kernel function $\mathbb{K}_f$. We consider the integral

$$\int_{[A]\times[A]} \mathbb{K}_f(h_1, h_2)|h_1 h_2|^s \eta(h_2)$$

where if $h = \begin{pmatrix} x & \\ & y \end{pmatrix}$ then $|h| = |x/y|$, and $\eta \colon \mathbf{A}_F^\times \to \{\pm 1\}$ is the character corresponding by class field theory to F'/F.

There's an issue with this integral. Since $A \cong \mathbf{G}_m/F$, $[A] := F^\times\backslash\mathbf{A}_F^\times$ is not compact, since

$$[A]/\prod_x \mathcal{O}_x^\times = \mathrm{Pic}(X).$$

This has infinitely many connected components, which are finite since they are isomorphic to $\mathrm{Pic}^0(X)$. To regularize the integral, define

$$[A]_n = \left\{ \begin{pmatrix} x & \\ & y \end{pmatrix} : v(x/y) = n \right\}.$$

We have a map

$$v \colon \mathbf{A}_F^\times/\prod \mathcal{O}_x^\times \to \mathbf{Z}$$

sending $v(\pi_x) = \log_q(q_x)$ for any $x \in |X|$.

The $[A_n]$ are compact, so we can talk about

$$\int_{[A]_{n_1}\times[A]_{n_2}} \mathbb{K}(h_1,h_2)|h_1h_2|^s\eta(s)\,ds.$$

This is actually a polynomial in q^s.

Proposition 2.1. *For each f, there exists N such that $|n_1|+|n_2|\geq N$ implies*

$$\int_{[A]_{n_1}\times[A]_{n_2}} \mathbb{K}_f(h_1,h_2)\eta(h_2)|h_1h_2|^s = 0.$$

Assuming this claim, we can define the regularized integral

$$\int_{[A_1]\times[A_2]}^{\mathrm{reg}} := \sum_{n_1,n_2}\int_{[A]_{n_1}\times[A]_{n_2}}$$

3. Spectral expansion

The goal of this section is to establish the identity

$$\mathbb{J}_r(f) = \sum_{u\in\mathbf{P}^1(F)-0} J_\gamma(u,f).$$

3.1. **The invariant map.** We have seen that in the RTF, we care about the double coset space

$$H_1(F)\backslash G(F)/H_2(F).$$

For $G=\mathrm{PGL}_2$, $H_1=H_2=A$ we can define an invariant map

$$A(F)\backslash \mathrm{PGL}_2(F)/A(F)\to \mathbf{P}^1-\{1\}$$

by

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\mapsto \frac{bc}{ad}.$$

Proposition 3.1. *For $x\in\mathbf{P}^1(F)-\{1\}$, we have*

$$\mathrm{inv}^{-1}(x) = \begin{cases} \textit{single orbit} & x\neq 0,\infty\\ \begin{pmatrix}1 & \\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ & 1\end{pmatrix}, \begin{pmatrix}1 & \\ & 1\end{pmatrix} & x=0\\ \begin{pmatrix} & 1\\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 1 & \end{pmatrix}, \begin{pmatrix} & 1\\ 1 & \end{pmatrix} & x=\infty\end{cases}$$

Also, γ is regular semisimple iff $\mathrm{inv}(\gamma)\neq 0,\infty$.

3.2. **Expansion.** This lets us write

$$\mathbb{J}(f,s) = \sum_{\gamma \in A(F)\backslash G(F)/A(F)} J(\gamma, f, s)$$

where

$$J(\gamma, f, s) = \int_{[A]\times[A]} \mathbb{K}_{f,\gamma}(h_1, h_2)|h_1 h_2|^s \eta(h_2)\, dh_2$$

and

$$\mathbb{K}_{f,\gamma}(h_1, h_2) = \sum_{\delta \in A(F)\gamma A(F)} f(h_1^{-1}\delta h_2^{-1}).$$

For $u \in \mathbf{P}^1(F) - \{1\}$, we define

$$\mathbb{J}(u, f, s) = \sum_{\substack{\gamma \in A(F)\backslash G(F)/A(F) \\ \mathrm{inv}(\gamma)=u}} \mathbb{J}(\gamma, f, s).$$

3.3. **Higher derivatives.** Now we define

$$\mathbb{J}_r(f) = \left(\frac{d}{ds}\right)^r J(f,s)|_{s=0}.$$

Similarly, we have a decomposition

$$\mathbb{J}_r(u, f) = \left(\frac{d}{ds}\right)^r \mathbb{J}(u, f, s)|_{s=0}.$$

Also

$$\mathbb{J}_r(f) = \sum_{u \in \mathbf{P}^1(F)-\{1\}} \mathbb{J}_r(u, f).$$

4. The case $r = 0$

The goal is to establish the identity

$$\sum_{u \in \mathbf{P}^1(F)-0} I_\gamma(u, f) = \mathbb{I}_r(f).$$

Yesterday we defined

$$\mathbb{I}_0(f) := \langle \mathrm{Sht}_T^0, f * \mathrm{Sht}_T^0 \rangle.$$

Let F'/F be a quadratic extension. Define $T = \mathrm{Res}_{F'/F}\, \mathbf{G}_{m,F'}/\mathbf{G}_{m,F}$. It turns out that we also have an equality

$$\mathbb{I}_0(f) = \int_{[T]\times[T]} \mathbb{K}_f(h_1, h_2).$$

The Waldspurger formula can be reinterpreted in these terms:

$$\sum_{\gamma \in A(F)\backslash G(F)/A(F)} \mathbb{J}_0(\gamma, f) = \mathbb{J}_0(f) \sim L(\pi, 0).$$

While

$$\sum_{\gamma \in T(F)\backslash G(F)/T(F)} \mathbb{I}_0(\gamma, f) = \mathbb{I}_0(f) \sim \int_{[T]} \phi_\pi.$$

5. The equality $\mathbb{I}_0(f) = \mathbb{J}_0(f)$

The strategy to relate the things is to relate the orbital integrals. So we first need to relate the orbits.

5.1. **Matching double cosets.** Let $G = \mathrm{PGL}_2$ or $D^\times/F^\times$, where D is a quaternion algebra over F (with an embedding $F' \hookrightarrow D$).

Theorem 5.1. *We have*

$$A(F)\backslash \mathrm{PGL}_2(F)^{\mathrm{rss}}/A(F) = \coprod_{G=\mathrm{PGL}_2 \text{ or } D^\times/F^\times} T(F)\backslash G(F)^{\mathrm{rss}}/T(F).$$

Proof. We consider $G = \mathrm{PGL}_2$ or $D^\times$. Let $H = M_2(F)$ or D, so $G = H^\times$. We have an embedding $F' \hookrightarrow H$.

There exists $\epsilon \in H(F)$ such that $\epsilon x \epsilon^{-1} = \overline{x}$ for $x \in F'$. The choice of ϵ is unique up to multiplication by $(F')^\times$. By computation, $\epsilon^2 \in Z(H) = F$, so $[\epsilon^2] \in F^\times/\,\mathrm{Nm}(F')^\times$ is well-defined.

Proposition 5.2. *The element* $[\epsilon^2] \in F^\times/\,\mathrm{Nm}(F')^\times$ *determines* H*.*

We have an invariant map

$$T(F)\backslash H^\times/T(F) \xrightarrow{\text{inv}} \mathbf{P}^1(F) - \{1\}$$

sending

$$h_1 + \epsilon h_2 \mapsto \frac{h_2\overline{h_2}}{h_1\overline{h_1}}\epsilon^2.$$

The image lands in $\epsilon^2 \cdot \mathrm{Nm}((F')^\times)$, and is equal in the regular semisimple case since this is equivalent to $h_1 h_2 \neq 0$. ☐

5.2. **Matching orbital integrals.** Now that we've matched up the double cosets, we turn to showing that $\mathbb{I}_0(f) = \mathbb{J}_0(f)$ for $f = \prod_{v \in |X|} f_v \in \mathcal{H}_G$ a bi-K-invariant function. Writing out the expansion, this comes down to

$$\sum_u \mathbb{I}_0(u, f) = \sum_u \mathbb{J}_0(u, f)$$

This becomes a fundamental lemma type statement.

Consider F'_v/F_v a quadratic extension. Let $f \in C^\infty(\mathrm{PGL}_2(F_v))$ be bi-K-invariant. We have

$$A(F_v)\backslash \mathrm{PGL}_2(F)/A(F_v) = T(F_v)\backslash \mathrm{PGL}_2(F_v)/T(F_v) \coprod T(F_v)\backslash D^\times/T(F_v)$$

By Theorem 5.1, each γ on the left side matches up with a $\gamma_1 \in T(F_v)\backslash \mathrm{PGL}_2(F_v)/T(F_v)$ or $\gamma_2 \in T(F_v)\backslash D^\times/T(F_v)$. One can then compute by hand that the corresponding orbital integrals are equal.

- If $\gamma \leftrightarrow \gamma_1 \in T(F_v)\backslash \mathrm{PGL}_2(F_v)/T(F_v)$, then
$$\pm \int_{A(F)\times A(F)} f(h_1^{-1}\gamma h_2)\eta(h_2) = \int_{T(F)\times T(F)} f(h_1^{-1}\gamma h_2)$$
so $\mathbb{I}_0(u,f) = \mathbb{J}_0(u,f)$.
- If $\gamma \leftrightarrow \gamma_2 \in T(F_v)\backslash D^\times/T(F_v)$, then
$$\int_{A(F)\times A(F)} f(h_1^{-1}\gamma h_2)\eta(h_2)\, dh_2 = 0$$
so the extra double cosets do not contribute.

This shows that
$$\mathbb{I}_0(f) = \mathbb{J}_0(f).$$