Galois to Automorphic in Geometric Langlands

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1 The classical case, $G = \mathrm{GL}_n$

1.1 Setup

Let $X/\mathbb{F}_q$ be a proper, smooth, geometrically irreducible curve. For each $x \in |X|$ we denote

- $\mathcal{O}_x = \widehat{\mathcal{O}}_{X,x}$,
- $\mathcal{F}_x$ its field of fractions,
- k_x its residue field, and
- $F = \mathbb{F}_q(X)$ its function field.

Finally, we write

$$\mathbb{A}_F := \prod_{x\in|X|}' F_x \supset \mathcal{O}_F := \prod_{x\in|X|} \mathcal{O}_x.$$

1.2 Automorphic side

Fix a prime $\ell \neq p$. The main player is the space of *unramified automorphic functions*

$$\mathcal{A} := \mathrm{Funct}(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)/\mathrm{GL}_n(\mathcal{O}_F), \overline{\mathbb{Q}}_\ell).$$

This admits an action of a *Hecke algebra* for each $x \in |X|$:

$$\mathcal{H}_x := \mathrm{Funct}(\mathrm{GL}_n(\mathcal{O}_x)\backslash \mathrm{GL}_n(F_x)/\mathrm{GL}_n(\mathcal{O}_x), \overline{\mathbb{Q}}_\ell).$$

The action is by convolution for $T \in \mathcal{H}_x$ and $f \in \mathcal{A}$, we have

$$(T * f)(g) = \int_{h_x\in \mathrm{GL}_n(F_x)} f(gh_x^{-1})T(h_x)\,dh_x$$

with measure normalized so that $\int_{\mathrm{GL}(\mathcal{O}_x)} dh_x = 1$.

Proposition 1.1. *We have*

$$\mathcal{H}_x \cong \overline{\mathbb{Q}}_\ell[T_x^1, T_x^2, \dots, T_x^n, (T_x^n)^{-1}]$$

where T_x^i is the characteristic function of the double coset

$$\mathrm{GL}_n(\mathcal{O}_x)\begin{pmatrix} \varpi_x & \\ & \ddots \end{pmatrix}\mathrm{GL}_n(\mathcal{O}_x)$$

1.3 Galois side

Let $\sigma\colon \pi_1(X) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ be an ℓ-adic representation; we can think of this equivalently as a local system $E_\sigma \in \mathrm{Loc}_n(X)$.

We define

$$\mathcal{G} := \{\sigma\colon \pi_1(X) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell) \text{ geometrically irreducible}\}/\cong .$$

Recall that for each $x \in |X|$ we have a $\mathrm{Frob}_x \in \pi_1(X)$.

Theorem 1.2 (Drinfeld, Lafforgue, Frenkel-Gaitsgory-Vilonen)**.** *To every $\sigma \in \mathcal{G}$ there corresponds a non-zero $f_\sigma \in \mathcal{A}$ such that*

$$T_x^i * f_\sigma = \mathrm{Tr}(\wedge^i \sigma(\mathrm{Frob}_x)) f_\sigma.$$

Moreover, f_σ is unique up to scalar and cuspidal.

Here $\wedge^i\sigma$ is the ith exterior power of σ and "cuspidal" means that

$$\int_{U(F)\backslash U(A_F)} f_\sigma(ug)\, du = 0$$

for all $g \in \mathrm{GL}_n(A_F)$ and U the unipotent radical of proper standard parabolic subgroup of GL_n.

2 Geometric reformulation

2.1 Geometrization of adeles

Let Bun_n be the stack of rank n vector bundles on X.

Theorem 2.1 (Weil's uniformization theorem)**.** *We have*

$$\mathrm{Bun}_n(\mathbb{F}_q) = \mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)/\mathrm{GL}_n(\mathcal{O}_F).$$

This allows us to interpret $\mathcal{A} = \mathrm{Funct}(\mathrm{Bun}_n(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$. An obvious categorification of this is sheaves on Bun_n.

2.2 Geometrization of Hecke operators

A geometric version of Hecke algebra is the moduli stack of modifications

$$\mathrm{Hecke}^i = \left\{ (x, M, M', \beta) : \begin{array}{c} x \in X, \\ M, M' \in \mathrm{Bun}_n \\ \beta \colon M \hookrightarrow M' \\ M'/M \cong k_x^{\oplus i} \end{array} \right\}$$

This admits maps

$$\begin{array}{ccc} & \mathrm{Hecke}^i & \\ {\scriptstyle h_i^{\leftarrow}} \swarrow & & \searrow {\scriptstyle h_i^{\rightarrow} \times \pi_i} \\ \mathrm{Bun}_n & & \mathrm{Bun}_n \times X \end{array}$$

where

- $h_i^{\leftarrow}(x, M, M', \beta) = M$,
- $h_i^{\rightarrow}(x, M, M', \beta) = M'$, and
- $\pi_i(x, M, M', \beta) = x$.

To relate these Hecke stacks to the classical Hecke algebra, we define a local Hecke stack Hecke_x^i by the cartesian diagram

$$\begin{array}{ccc} \mathrm{Hecke}_x^i & \longrightarrow & \mathrm{Hecke}^i \\ \downarrow & & \downarrow \\ x & \longrightarrow & X \end{array}$$

On rational points, we have a diagram

$$\begin{array}{ccc} & \mathrm{Hecke}_x^i(\mathbb{F}_q) & \\ \swarrow & & \searrow \\ \mathrm{Bun}_n(\mathbb{F}_q) & & \mathrm{Bun}_n(\mathbb{F}_q) \end{array}$$

Then the classical Hecke operators can be interpreted as

$$T_x^i * f := (h_i^{\rightarrow})_!(h_i^{\leftarrow})^* f$$

for $T \in \mathcal{H}_x$ and $f \in \mathcal{A}$.

2.3 Hecke eigensheaves

Based on this we make the following definition.

Definition 2.2. (Geometric Hecke operators) We define

$$\mathbb{T}^i\colon D_c^b(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell) \to D_c^b(\mathrm{Bun}_n \times X, \overline{\mathbb{Q}}_\ell).$$

by

$$\mathcal{F} \mapsto (h_i^{\rightarrow} \times \pi_i)_!(h_i^{\leftarrow})^*\mathcal{F}[i(n-i)].$$

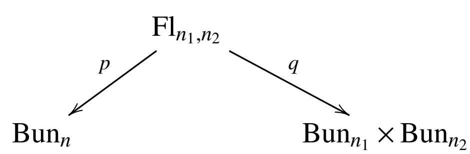
Let $\sigma \in \mathcal{G}$. An object $\mathcal{F} \in D_c^b(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ is called a *Hecke eigensheaf* with respect to σ if for all $i = 1, \ldots, n$ we have $\mathbb{T}^i(\mathcal{F}) \cong \mathcal{F} \boxtimes \bigwedge^i E_\sigma$.

Theorem 2.3 (Frenkel-Gaitsgory-Vilonen)**.** *For every $\sigma \in \mathcal{G}$, there exists a non-zero Hecke eigensheaf* Aut_σ *which is cuspidal.*

What is the meaning of cuspidality? You can consider the moduli spaces of flags Fl_{n_1,n_2} for $n_1 + n_2 = n$, which parametrize

$$\{0 \to E_1 \to E \to E_2 \to 0\}$$

with rank $E_i = n_i$. This admits maps



where $p = E$ and $q = (E_1, E_2)$. A $\mathcal{F} \in D_c^b(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ is *cuspidal* if $q_! p^* \mathcal{F} = 0$ for all n_1, n_2.

Remark 2.4. Frenkel-Gaitsgory-Vilonen show that Aut_σ is a perverse sheaf and is irreducible on each connected component Bun_n^d. If we demand that Aut_σ be irreducible, then it is unique.

3 Geometric Satake

Let $k = \overline{k}$. Let G be a connected reductive group over k and $\widehat{G}/\overline{\mathbb{Q}}_\ell$ be the dual group over $\overline{\mathbb{Q}}_\ell$.

Definition 3.1. The *affine Grassmannian* is $\mathrm{Gr}_G := LG/L^+G$ where the loop group LG is defined by $LG(R) := G(R((t)))$ and $L^+G(R) := G(R[[t]])$.

Definition 3.2. We define the category $\mathrm{Sat} := \mathcal{P}_{L^+G}(\mathrm{Gr}_G)$, perverse sheaves equivariant for the "arc group" L^+G.

Properties.

1. There is a bijection
$$L^+G\backslash \mathrm{Gr}_G(k) \leftrightarrow X_*(T)^+$$
with $L^+G \cdot t^\lambda \leftarrow \lambda$ for $t \in T(k((t)))$. Here for $\lambda\colon \mathbb{G}_m \to T$ we get $k((t))^* \to T(k((t)))$ sending $t \mapsto t^\lambda$.

2. Denoting $L^+G \cdot t^\lambda$ by O_λ, we have
$$\overline{O_\lambda} = \bigcup_{\mu\leq\lambda} O_\mu.$$

3. We have
$$\mathrm{Gr}_G(R) = \left\{ (\mathcal{E},\beta)\colon \begin{array}{c} \mathcal{E} = G-\text{bundle on } D_R \\ \beta\colon \mathcal{E}|_{D_R^0} \cong G \times D_R^0 \end{array} \right\}.$$
Here we are using the notation
   - $D = \mathrm{Spec}\ k[[t]]$, $D^0 = \mathrm{Spec}\ k((t))$,
   - $D_R = \mathrm{Spec}\ R[[t]]$, $D_R^0 = \mathrm{Spec}\ R((t))$.

4. Consider the diagram
$$\begin{array}{ccccc} \mathrm{Gr}_G \times \mathrm{Gr}_G & \xleftarrow{\ p\ } & LG \times \mathrm{Gr}_G & \xrightarrow{\ q\ } & LG \times^{L^+G} \mathrm{Gr}_G \\ & & & & \downarrow{\scriptstyle m} \\ & & & & \mathrm{Gr}_G \end{array}$$
For $A, A' \in \mathrm{Sat}$, we define $A\widetilde{\boxtimes}A' \in \mathcal{P}(LG \times^{L^+} \mathrm{Gr}_G)$ by the condition
$$p^*(A \boxtimes A') = q^*(A\widetilde{\boxtimes}A')$$
Define the *fusion product*
$$A * A' = m_!(A\widetilde{\boxtimes}A') \in \mathcal{P}_{L^+G}(\mathrm{Gr}_G).$$

Theorem 3.3 (Geometric Satake)**.** *We have an equivalence of categories*
$$(\mathrm{Sat}, *) \cong (\mathrm{Rep}\,\widehat{G}, \otimes)$$
such that for $\lambda \in X^*(\widehat{T})^+ \cong X_*(T)^+$ the highest weight representation V_λ corresponds to $IC_\lambda = IC(\overline{O}_\lambda, \overline{\mathbb{Q}}_\ell)$.

Under the fiber functor to vector spaces
$$\begin{array}{ccc} \mathrm{Sat} & \longrightarrow & \mathrm{Rep}(\widehat{G}) \\ & \searrow \quad \swarrow & \\ & Vec_k & \end{array}$$
this equivalence corresponds to taking cohomology.

4 Statement of global geometric Langlands for general G

4.1 The Hecke stacks

Consider the stack

$$\text{Hecke} = \left\{ (x, M, M', \beta) \colon \begin{array}{c} x \in X, \\ M, M' \in \text{Bun}_n \\ \beta \colon M|_{X-x} \cong M'|_{X-x} \end{array} \right\}$$

This has maps

$$\begin{array}{ccccc} & & \text{Hecke} & & \\ & \swarrow^{h^{\leftarrow}} & & \searrow^{h^{\rightarrow}\times\pi} & \\ \text{Bun}_G & & & & \text{Bun}_G \times X \end{array}$$

where $h^{\leftarrow}(x, M, M', \beta) = M$ and $h^{\rightarrow}(x, M, M', \beta) = (M', x)$.

There is an evaluation map

$$\text{ev} \colon \text{Hecke} \to \left[\frac{L^+G \backslash LG / L^+G}{\text{Aut}(D)} \right]$$

which is described as follows. After choosing an isomorphism $D_x \cong D$ and a trivialization of $M|_{D_x}$ and $M'|_{D_x}$ the map β describes some transition function $g_\beta \in LG$, which is $\text{ev}(x, M, M', \beta)$.

Using this we can define an operator

$$\text{Hk} : \text{Rep}(\widehat{G}) \times D^b_c(\text{Bun}_{G, \overline{\mathbb{Q}}_\ell}) \xrightarrow{\text{ev}} D^b_c(\text{Bun}_G \times X, \overline{\mathbb{Q}}_\ell)$$

sending

$$(V, \mathcal{F}) \mapsto (h^{\rightarrow} \times \pi)_!(h^{\leftarrow})^*(\mathcal{F} \otimes IC^{\text{Hk}}_V)[\text{shift}]$$

where $IC^{\text{Hk}}_V := \text{ev}^*(IC_V)$ is the pullback of the IC sheaf corresponding to the local system V under the Geometric Satake equivalence. Similarly, for $V_1, \ldots, V_d \in \text{Rep}(\widehat{G})$ we define

$$\text{Hk}_{V_1 \boxtimes \ldots \boxtimes V_d}(\mathcal{F}) \in D^b_c(\text{Bun}_G \times X^d, \overline{\mathbb{Q}}_\ell)$$

in an analogous manner. These satisfy

1. $\text{Hk}_{V_1 \boxtimes V_2}(\mathcal{F})|_{\text{Bun}_G \times \Delta(X)} \cong \text{Hk}_{V_1 \otimes V_2}(\mathcal{F})$,
2. For $s \colon X \times X \to X \times X$ the swap function, we have $s^*(\text{Hk}_{V_1 \boxtimes V_2}(\mathcal{F})) \cong \text{Hk}_{V_2 \boxtimes V_1}(\mathcal{F})$.

Let E be a $\widehat{G}$-local system on X, viewed as a tensor functor

$$E \colon \text{Rep}(\widehat{G}) \to \text{Loc}(X)$$

denoted $V \mapsto E^V$.

Definition 4.1. A *Hecke eigensheaf with eigenvalue E* is a perverse sheaf $\mathcal{F} \in \mathcal{P}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ together with isomorphisms

$$\alpha_V : \ \mathrm{Hk}_V(\mathcal{F}) \cong \mathcal{F} \boxtimes E^V \text{ for all } V \in \mathrm{Rep}(\widehat{G})$$

that are compatible with the symmetric tensor structure on $\mathrm{Rep}(\widehat{G})$ and composition of Hecke operators.

Conjecture 4.2. *To every irreducible $\widehat{G}$-local system E, there exists a non-zero Hecke eigensheaf* Aut_E *with eigenvalue E.*