THE WORK OF DRINFELD

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1. Notation

Let $k = \mathbf{F}_q$, X/k a smooth projective, geometrically connected curve over k. Let $F = k(X)$. Chooose a point $\infty \in |X|$, and assume for simplicity that $\deg \infty = 1$.

Let $F = k(X)$, F_∞ be the completion of F at ∞. Let $\mathbf{C}_\infty$ be the completion of a separable closure of F_∞, and $A = H^0(X \setminus \{\infty\}, \mathcal{O})$.

2. Elliptic modules

2.1. **Definition.** The seed of shtukas were Drinfeld's "elliptic modules". Let $\mathbf{G}_a$ be the additive group, and K a characteristic p field. We set $K\{\tau\} = K \otimes_{\mathbf{Z}} \mathbf{Z}[\tau]$, with multiplication given by

$$(a \otimes \tau^i)(b \otimes \tau^j) = ab^{p^i} \otimes \tau^{i+j}.$$

We have an isomorphism $K\{\tau\} \cong \mathrm{End}_K(\mathbf{G}_a)$ sending

$$\sum_{i=0}^{m} a_i \otimes \tau^i \mapsto \left(X \mapsto \sum_{i=0}^{m} a_i X^{p^i} \right).$$

If a_m is the largest non-zero coefficient, then the *degree* of $\sum_{i=0}^m a_i \in K\{\tau\}$ is defined to be p^m. The *derivative* is defined to be the constant term a_0.

Definition 2.1. Let $r > 0$ be an integer and K a characteristic p field. An *elliptic A-module of rank r* is a ring homomorphism

$$\phi \colon A \to K\{\tau\}$$

such that for all non-zero $a \in A$, $\deg \phi(a) = |a|_\infty^r$.

We can also make a relative version of this definition.

Definition 2.2. Let S be a scheme of characteristic p. An *elliptic A-module of rank r over S* is a $\mathbf{G}_a$-torsor $\mathcal{L}/S$, with a morphism of rings $\phi \colon A \to \mathrm{End}_S(\mathcal{L})$ such that for all points $s \colon \operatorname{Spec} K \to S$, the fiber $\mathcal{L}_s$ is an elliptic A-module of rank r.

Remark 2.3. The function $a \mapsto \phi(a)'$ (the latter meaning the derivative of $\phi(a)$) defines a morphism of rings $i \colon A \to \mathcal{O}_S$, i.e. a morphism $\theta \colon S \to \operatorname{Spec} A$.

2.2. **Level structure.** Let I be an ideal of A. Let $(\mathcal{L}, \phi)$ be an elliptic module over S. Assume that S is an $A[I^{-1}]$-scheme, i.e. the map θ factors through $\theta\colon S \to \operatorname{Spec} A \setminus V(I)$.

Let $\mathcal{L}_I$ be the group scheme defined by the equation $\phi(a)(x) = 0$ for all $a \in I$. This is an étale group scheme over S with rank $\#(A/I)^r$. An I-level structure on $(\mathcal{L}, \varphi)$ is an A-linear isomorphism

$$\alpha\colon (I^{-1}/A)^r_S \xrightarrow{\sim} \mathcal{L}_I.$$

Choose $0 \subsetneq I \subsetneq A$. We have a functor

$$F^r_I\colon A[I^{-1}] - \mathbf{Sch} \to \mathbf{Sets}$$

sending S to the set of isomorphism classes of elliptic A-modules of rank r with I-level structure, with θ being the structure morphism.

Theorem 2.4 (Drinfeld)**.** *F^r_I is representable by a smooth affine scheme M^r_I over $A[I^{-1}]$.*

3. Analytic theory of elliptic modules

3.1. **Description in terms of lattices.** Let Γ be an A-lattice in $\mathbf{C}_\infty$. (This means a discrete additive subgroup of $\mathbf{C}_\infty$ which is an A-module.) Then we define

$$e_\Gamma(x) = x \prod_{x \in \Gamma - 0} (1 - x/\gamma).$$

Drinfeld proved that this is well-defined for all $x \in \mathbf{C}_\infty$, and induces an additive surjection:

$$e_\Gamma\colon \mathbf{C}_\infty/\Gamma \xrightarrow{\sim} \mathbf{C}_\infty.$$

Given Γ, we define a function

$$\phi_\Gamma\colon A \to \operatorname{End}_{\mathbf{C}_\infty}(\mathbf{G}_a)$$

by the following rule. For $a \in A$, there exists $\phi_\Gamma(a)$ such that

$$\phi_\Gamma(a) e_\Gamma(x) = e_\Gamma(ax) \text{ for all } x \in \mathbf{C}_\infty.$$

If Γ is replaced by $\lambda\Gamma$, for $\lambda \in \mathbf{C}^*_\infty$, then ϕ_Γ doesn't change. Therefore, ϕ_Γ is a function on homothety classes of A-lattices.

Theorem 3.1 (Drinfeld)**.** *The function $\Gamma \mapsto \phi^\Gamma$ induces a bijection between*

$$\left\{ \begin{array}{c} \textit{rank } r \textit{ projective } A\textit{-lattices} \\ \textit{in } \mathbf{C}_\infty/\textit{homothety} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{rank } r \textit{ elliptic } A\textit{-modules} \\ \textit{over } \mathbf{C}_\infty \textit{ such that } \phi(a)' = a \\ /\textit{isomorphism} \end{array} \right\}$$

Remark 3.2. Under this bijection, an I-level structure equivalent to an A-linear isomorphism $(A/I)^r \cong \Gamma/I\Gamma$ for the lattices.

3.2. **Uniformization.** We now try to parametrize the objects on the left hand side of (3.1). First we parametrize the isomorphism classes. Let Y be a projective A-module of rank r. Then we have a bijection

$$\left\{ \begin{array}{c} \text{homothety classes of } A\text{-lattices in } \mathbf{C}_\infty \\ \text{isomorphic to } Y \text{ as } A\text{-modules} \end{array} \right\} \leftrightarrow \mathbf{C}_\infty^\times \backslash \mathrm{Inj}(F_\infty \otimes_A Y, \mathbf{C}_\infty) / \mathrm{GL}_A(Y).$$

Next we observe that there is a bijection

$$\mathbf{C}_\infty^\times \backslash \mathrm{Inj}(F_\infty \otimes_A Y, \mathbf{C}_\infty) \leftrightarrow \mathbf{P}^{r-1}(\mathbf{C}_\infty) \setminus \bigcup (F_\infty\text{-rational hyperplanes}),$$

given by sending $u \in \mathrm{Inj}(F_\infty \otimes_A Y, \mathbf{C}_\infty)$ to $[u(e_1), \dots, u(e_r)]$. This is called the *Drinfeld upper half plane* Ω^{r-1}.

As Spec $A = X \setminus \infty$, a projective A-module of rank r is the same as a vector bundle of rank r on $X \setminus \infty$. We saw yesterday that there is an isomorphism (Weil's uniformization)

$$\left\{ \begin{array}{c} \text{rank } r \text{ vector bundles on } X \setminus \infty \\ \text{plus generic trivialization} \end{array} \right\} /\text{isom.} \leftrightarrow \mathrm{GL}_r(\mathbf{A}_F^\infty) / \prod_{v \neq \infty} \mathrm{GL}_r(\mathcal{O}_v).$$

Set $\mathrm{GL}_r(\widehat{A}) := \prod_{v \neq \infty} \mathrm{GL}_r(\mathcal{O}_v)$, and

$$\mathrm{GL}_r(\widehat{A}, I) := \ker \left(\mathrm{GL}_r(\widehat{A}) \to \mathrm{GL}_r(\widehat{A}/I) \right).$$

In conclusion, there is a natural bijection

$$M_I^r(\mathbf{C}_\infty) \cong \mathrm{GL}_r(F) \backslash (\mathrm{GL}_r(\mathbf{A}_F^\infty) / \mathrm{GL}_r(\widehat{A}, I) \times \Omega^r(\mathbf{C}_\infty))$$

This can be upgraded into an isomorphism of rigid analytic spaces:

Theorem 3.3 (Drinfeld). *We have an isomorphism of rigid analytic spaces over* F_∞*:*

$$(M_I^r)^{\mathrm{an}} = \mathrm{GL}_r(F) \backslash (\mathrm{GL}_r(\mathbf{A}_F^\infty) / \mathrm{GL}_r(\widehat{A}, I) \times \Omega^r(\mathbf{C}_\infty)).$$

4. Cohomology of M_I^2 and global Langlands for GL_2

4.1. **Cohomology of the Drinfeld upper half plane.** We now outline Drinfeld's proof of global Langlands for GL_2 using the moduli space of elliptic modules. Set $r = 2$, and $\Omega := \Omega^2$. Then one has

$$\Omega(\mathbf{C}_\infty) = \mathbf{P}^1(\mathbf{C}_\infty) \backslash \mathbf{P}^1(F_\infty).$$

There is a map λ from $\Omega(\mathbf{C}_\infty)$ to the Bruhat-Tits tree, sending (z_0, z_1) to the homothety class of the norm on F_∞^2 defined by

$$(a_0, a_1) \in F_\infty^2 \mapsto |a_0 z_0 + a_1 z_1|.$$

The pre-image of a vertex is $\mathbf{P}^1$ minus $q+1$ open unit disks, and the pre-image of an open edge is an annulus (which can be thought of as $\mathbf{P}^1$ minus 2 open disks). There is an admissible covering of Ω given by $\{U_e := \lambda^{-1}(e)\}$ as e runs over the closed edges. We have an exact sequence

$$H^1(\Omega, \mathbf{Z}/n) \to \prod_{e \in E} H^1(U_e, \mathbf{Z}/n) \to \prod_{v \in V} H^1(U_v, \mathbf{Z}/n)$$

and using this, we get that for $\ell \neq p$, the vector space $H^1_{\text{ét}}(\Omega, \overline{\mathbf{Q}}_\ell)$ is naturally the space of harmonic cochains on the Bruhat-Tits tree, which is the set of functions c from oriented edges to $\overline{\mathbf{Q}}_\ell$ satisfying

(1) $c(-e) = -c(e)$ and
(2) $\sum_{e\in E(v)} c(e) = 0$.

Any such harmonic cochain defines a ($\overline{\mathbf{Q}}_\ell$-valued) measure on $\partial\Omega = \mathbf{P}^1(F_\infty)$. In other words, we have

$$H^1_{\text{ét}}(\Omega, \overline{\mathbf{Q}}_\ell) = (C^\infty(\mathbf{P}^1(F_\infty), \overline{\mathbf{Q}}_\ell)/\overline{\mathbf{Q}}_\ell)^* \cong \mathrm{St}^*.$$

The isomorphism is $\mathrm{GL}_2(F_\infty)$-invariant.

4.2. **Cohomology of M_I^2.** Now we use the uniformization of M_I^2. We can rewrite it as follows:

$$M_I^{2,\text{an}} = \left(\Omega \times \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}_F)/\mathrm{GL}_2(\widehat{A}, I)\right)/\mathrm{GL}_2(F_\infty).$$

(Some elementary trickery is required to go from the previous formulation to the one above.) Now you use the Hochschild-Serre spectral sequence for $Y \to Y/\Gamma$ to get a long exact sequence

$$0 \to H^1(\Gamma, H^0(Y, \overline{\mathbf{Q}}_\ell)) \to H^1(Y/\Gamma, \overline{\mathbf{Q}}_\ell) \to H^1(X, \overline{\mathbf{Q}}_\ell)^\Gamma \to \dots.$$

From this we deduce

$$H^1_{\text{ét}}(M_I^2 \otimes_F \overline{F}, \overline{\mathbf{Q}}_\ell) \cong \mathrm{Hom}_{\mathrm{GL}_2(F_\infty)}(\mathrm{St}, C^\infty(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}_F)/\mathrm{GL}_2(\widehat{A}, I)) \otimes \mathrm{sp}$$

where sp is a 2-dimensional representation of $\mathrm{Gal}(\overline{F}_\infty/F_\infty)$, which should be the Galois representation corresponding to Steinberg. This isomorphism is compatible for the action of $\mathrm{GL}_2(\mathbf{A}_F) \times \mathrm{Gal}(\overline{F}_\infty/F_\infty)$.

Remark 4.1. This is cheating a little; we really need to work with a compactification of M_I^2 instead.

Drinfeld shows that

$$\varinjlim_I H^1(\overline{M}_I^2 \otimes_F \overline{F}, \overline{\mathbf{Q}}_\ell) = \bigoplus_\pi \pi^\infty \otimes \sigma(\pi)$$

where π runs over cuspidal automorphic representations of $\mathrm{GL}_2(\mathbf{A}_F)$ with $\pi_\infty \cong$ St. Here $\sigma(\pi)$ is a $\mathrm{Gal}(\overline{F}/F)$ representation. Moreover, Drinfeld shows that at unramified places, π_v and $\sigma(\pi_v)$ correspond by local Langlands.

4.3. **The local Langlands correspondence.** Using this, one can construct the local Langlands correspondence for GL_2 over K, a characteristic p local field. Indeed, let π be a supercuspidal representation of $\mathrm{GL}_2(K)$. Write $K = F_v$ for a global F. Choose a global automorphic representation Π such that $\Pi_v \cong \pi$ and $\Pi_\infty \cong$ St. By the work of Drinfeld, we get $\sigma(\Pi)$ and we know that Π_w and $\sigma(\Pi)_w$ have the same ϵ-factors and L-functions at all w outside some finite set S. Then for any global Hecke character χ, we have

$$\prod_w L_w(\Pi_w \otimes \chi_w) = \prod_w \epsilon_w(\Pi_w \otimes \chi_w) \prod_w L_w(\Pi_w^\vee \otimes \chi_w^{-1} \otimes \omega_{\Pi_w})$$

and similarly for $\sigma(\Pi)$. We can divide these two equalities by the product for $w \notin S$, getting an equality of two finite products

$$\prod_{w\in S} \epsilon'_w(\Pi_w \otimes \chi_w) = \prod_{w\in S} \epsilon'_w(\sigma(\Pi)_w \otimes \chi_w)$$

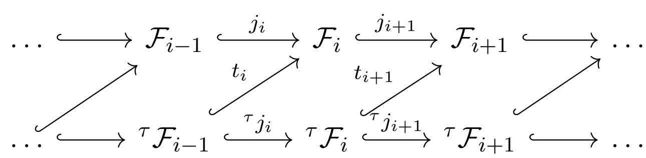
where $\epsilon'_w(\tau) = \epsilon_w(\tau)\frac{L(\tau^\vee \otimes \omega_\tau)}{L(\tau)}$.

Now for a trick: we can choose χ such that $\chi_v = 1$ and χ_w is very ramified for all other $w \in S - v$, thus forcing the L-factors at those w to be 1. Then $\epsilon'_w(\Pi_w \otimes \chi_w) = \epsilon(\Pi_w \otimes \chi_w)$ only depends on χ_w. In this way one can isolate an equality for the ϵ and L-factors of $\Pi_v = \pi$.

5. Elliptic sheaves

(This material is from a discussion session.) We will explain the connection between elliptic modules and shtukas. The relation passes through an intermediate object called an "elliptic sheaf".

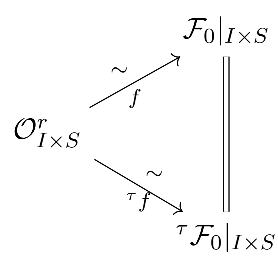
Definition 5.1. An *elliptic sheaf of rank $r > 0$ with pole at ∞* is a diagram



(here as usual $\tau^*\mathcal{F} = (\mathrm{Id}_X \times \mathrm{Frob}_S)^*\mathcal{F})$ with $\mathcal{F}_i$ bundles of rank r, such that j and t are $\mathcal{O}_{X\times S}$-linear maps satisfying

1. $\mathcal{F}_{i+r} = \mathcal{F}_i(\infty)$ and $j_{i+r} \circ \ldots \circ j_{i+1}$ is the natural map $\mathcal{F}_i \hookrightarrow \mathcal{F}_i(\infty)$.
2. $\mathcal{F}_i/j(\mathcal{F}_{i-1}$ is an invertible sheaf along Γ_∞.
3. For all i, $\mathcal{F}_i/t_i(\tau^*\mathcal{F}_{i-1}) =$ is an invertible sheaf along Γ_z for some $z\colon S \to X \setminus \infty$ (independent of i).
4. For all geometric points $\overline{s}$ of S, the Euler characteristic $\chi(\mathcal{F}_0|_{X_{\overline{s}}}) = 0$.

Definition 5.2. Let $J \subset A$ be an ideal cutting out the closed subset $I \subset \mathrm{Spec}\ X$. An *$I$-level structure on an elliptic sheaf* is a diagram



Theorem 5.3. *Let $z\colon S \to \mathrm{Spec}\ A \setminus I$. Then there exists a bijection, functorial in S,*

$$\left\{\begin{array}{c} \textit{rank } r \textit{ elliptic } A\textit{-modules} \\ \textit{with } J\textit{-level structure} \\ \textit{such that } \phi(a)' = z(a) \end{array}\right\}/\textit{isom.} \leftrightarrow \left\{\begin{array}{c} \textit{rank } r \textit{ elliptic sheaves over } S \\ \textit{with zero } z \\ \textit{and } I\textit{-level structure} \end{array}\right\}/\textit{isom.}$$

We'll define the map for $S = \text{Spec } K$. Let $(\mathcal{F}_i, j_\cdot, t_\cdot)$ be an elliptic sheaf. Define $M_i = H^0(X \otimes_k K, \mathcal{F}_i)$, and

$$M = \varinjlim M_i = H^0((X - \infty) \otimes K, \mathcal{F}_i).$$

This is a module over $A \otimes_k K$.

The $t_\cdot$ induce a map $t\colon M \to M$ which satisfies

- $t(am) = at(m)$, for $a \in A$
- $t(\lambda m) = \lambda^q t(m)$, for $\lambda \in K$.

This t makes M a module over $K\{\tau\}$. Furthermore:

- Because the zero and pole are distinct, t induces an injection
$$^\tau(\mathcal{F}_i/j(\mathcal{F}_{i-1})) \to \mathcal{F}_{i+1}/j(\mathcal{F}_i).$$
This implies that $\tau\colon M_i/M_{i-1} \to M_{i+1}/M_i$ is injective.
- We claim that $M_0 = 0$. Otherwise, we would have for some $i < 0$ a non-zero x in $M_i \setminus M_{i-1}$. The previous bullet point implies that for all $m \geq 0$, we have $t^m x \in M_{i+m} \setminus M_{i+m-1}$, so $\dim M_m \geq m + 1$ because there are independent vectors $(x, tx, \ldots, t^m x)$. For really large m, we would then have $\chi(\mathcal{F}_m) = m = \dim M_m \geq m + 1$, a contradiction to $\chi(\mathcal{F}_0) = 0$.
- For all i, M_i/M_{i-1} is 1-dimensional by similar estimates as in the previous bullet point. Finally, if u is a non-zero element of M_1, we have $M \cong K\{\tau\}u$.

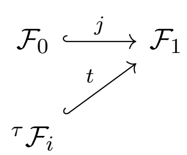
The action of A gives a ring homomorphism $A \xrightarrow{\phi} \text{End}_{K\{\tau\}}(M) = K\{\tau\}$.

- The action of A on $M/K\tau(M) \cong M_1$ being in the fiber of $\mathcal{F}_1$ at z implies $\phi(a)' = z(a)$.

One can show that if $(\mathcal{F}_i, t, j)$ is an elliptic sheaf, then for all i

$$t(\tau^*\mathcal{F}_{i-1}) = \mathcal{F}_i \cap t(\tau^*\mathcal{F}_i) \text{as subsheaves of } \mathcal{F}_{i+1}.$$

You can actually reconstruct the entire elliptic sheaf from the triangle



which is just a shtuka!

You can't go in the other direction - shtukas are more general. (You need to impose special conditions, namely "supersingular", on shtukas to construct elliptic sheaves.)