The Pro-étale and v-Topologies

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1 The pro-étale topology

The pro-étale topology is a topology on the category Perf of perfectoid spaces. An important property of the category Perf which makes this theory possible is that it has all inverse limits with affinoid transition functions. (This is *not* true for the category of adic spaces.)

1.1 Pro-étale morphisms

Definition 1.1. A morphism $\text{Spa}(A_\infty, A_\infty^+) \to \text{Spa}(A, A^+)$ of perfectoid spaces is called *affinoid pro-étale* if

$$(A_\infty, A_\infty^+) = \left(\varinjlim (A_i, A_i^+)\right)^\wedge$$

for a filtered system of perfectoid (A, A^+)-algebras (A_i, A_i^+) such that $\text{Spa}(A_i, A_i^+) \to \text{Spa}(A, A^+)$ is étale. Here the $\wedge$ means the ϖ-adic completion for some pseudo-uniformizer ϖ of A, which becomes a pseudo-uniformizer for A_∞ as well.

Definition 1.2. A morphism $f \colon X \to Y$ of perfectoid spaces is *pro-étale* if it is affinoid pro-étale locally on source and target.

Remark 1.3. This definition is reasonable because the property of being affinoid pro-étale is well-behaved under localization, so the property of being pro-étale is indeed local in the analytic topology.

The content of this assertion is that if $\text{Spa}(B, B^+) \to \text{Spa}(A_\infty, A_\infty^+)$ is a rational subset then $\text{Spa}(B, B^+) \to \text{Spa}(A, A^+)$ is affinoid pro-étale, and similarly if $\text{Spa}(B, B^+) \to \text{Spa}(A_\infty, A_\infty^+)$ is finite étale.

The key step to proving these results is to show that rational subdomains or finite étale morphisms come from some finite layer.

Remark 1.4. Pro-étale morphisms are *not necessarily open*. For example, the inclusion of a point in a profinite set (considered as an affinoid perfectoid space over some perfectoid field) is affinoid pro-étale. Indeed, you can consider the inverse limit over all open neighborhoods of the point.

Proposition 1.5. *Pro-étale morphisms are stable under base change and composition.*

1.2 The pro-étale topology

Definition 1.6. The *pro-étale topology* on Perf is the (pre)topology whose class of covers is generated by:

- all open covers in the analytic topology,
- all affinoid pro-étale maps $\mathrm{Spa}(A_\infty, A_\infty^+) \to \mathrm{Spa}(A, A^+)$ that are surjective (on points).

Warning 1.7. A family of pro-étale morphisms $f_i \colon X_i \to X$ that is jointly surjective is not necessarily a covering, for the same reason that a quasicompactness condition is necessary in the fpqc topology. For instance, the map from the disjoint union of singleton points of a profinite set to the profinite set is not a covering in this topology.

Just as with the fpqc topology, what one needs is an additional quasicompactness condition saying that every quasicompact open on the base is the image of some quasicompact open in the source.

Proposition 1.8. *The structure sheaf $X \mapsto \mathcal{O}_X(X)$ is a sheaf for the pro-étale topology, and moreover $H^i_{\text{pro-étale}}(X, \mathcal{O}_X) = 0$ for all $i > 0$ if X is affinoid perfectoid.*

Proof. It is part of the definition of a perfectoid space that $\mathcal{O}_X$ is a sheaf for the analytic topology. We need to know that if $\mathrm{Spa}(A_\infty, A_\infty^+) \to \mathrm{Spa}(A, A^+)$ is affinoid pro-étale and surjective, then the complex

$$0 \to A \to A_\infty \to A_\infty \widehat{\otimes}_A A_\infty \to \dots \tag{1}$$

is exact.

Write $(A_\infty, A_\infty^+) = \left(\varinjlim (A_i, A_i^+)\right)^\wedge$ as in the definition. Then consider

$$0 \to A^+/\varpi \to A_\infty^+/\varpi \to A_\infty^+/\varpi \otimes_{A^+/\varpi} A_\infty^+/\varpi \to \dots \tag{2}$$

This is the filtered direct limit of the complexes

$$0 \to A^+/\varpi \to A_i^+/\varpi \to A_i^+/\varpi \otimes_{A^+/\varpi} A_i^+/\varpi \to \dots \tag{3}$$

Since the map $\mathrm{Spa}(A_\infty, A_\infty^+) \to \mathrm{Spa}(A, A^+)$ is surjective, the same holds for $\mathrm{Spa}(A_i, A_i^+) \to \mathrm{Spa}(A, A^+)$.

We need to use the fact that

$$H^i_{\text{ét}}(X, \mathcal{O}_X^+/\varpi) \overset{a}{=} \begin{cases} A^+/\varpi & i = 0 \\ 0 & i > 0 \end{cases}$$

where $\overset{a}{=}$ means an equality at the almost level. This is a classical result of Tate for rigid analytic spaces. For perfectoid spaces, it is proved by using tilting to reduce to the case of characeristic p. Then you can reduce to rigid analytic spaces using noetherian approximation.

The point is that the fact implies that (3) is almost-exact. Hence (2) is also almost exact. As we saw yesterday, the perfectoid property allows one to upgrade this to an almost integral level, so

$$0 \to A^+ \to A^+_\infty \to A^+_\infty \otimes_{A^+} A^+_\infty \to \dots$$

is almost exact. □

Corollary 1.9. *The pro-étale topology is subcanonical (i.e. every representable functor is a sheaf).*

Proof sketch. You first show that you can glue morphisms in the analytic topology, then you show that you can glue morphisms in the pro-étale topology by reducing the preceding proposition. □

Warning 1.10. The property of being pro-étale is *not* local in the pro-étale topology.

1.3 Diamonds

If we consider a larger class of morphisms which are pro-étale locally in the pro-étale topology, then it doesn't really change our topology.

Definition 1.11. Such a morphism is called *locally quasi-profinite*.

This condition can be checked at the level of geometric fibers:

Proposition 1.12. *A morphism is locally quasi-profinite if and only if for all geometric points* $\mathrm{Spa}(C, C^+) \to Y$*, the fiber* $X \times_Y \mathrm{Spa}(C, C^+) \to \mathrm{Spa}(C, C^+)$ *is pro-étale, which is equivalent (in this case) to being a profinite set with points having residue field* C*.*

Definition 1.13. A *diamond* is a sheaf $\mathcal{F}$ on the pro-étale toplogy on perfectoid spaces in characteristic p such that there exists a map $h_Y \to \mathcal{F}$ for some representable functor h_Y which is surjective, relatively representable and locally quasi-profinite.

Remark 1.14. This relation between this definition to perfectoid spaces is analogous to the relation between algebraic spaces and schemes.

Remark 1.15. We could *not* have made this definition with "locally quasi-profinite" replaced by "pro-étale".

2 The v-topology

We just saw that a sheaf for pro-étale covers is the same as a sheaf for locally quasi-profinite covers. Note that there is no flatness assumption here: Proposition 1.12 implies that quasi-profiniteness is purely a statement about fibers. This suggests that we can define a topology analogous to the fpqc topology without the "finite presentation" assumption; for this reason the topology was originally named "faithful" but was subsequently renamed.

Definition 2.1. The v-topology on Perf is the (pre)topology whose covers are generated by

- all open covers in the analytic topology,
- all surjective $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$.

Remark 2.2. A good analogy is the category of compact Hausdorff spaces with covers being *all* (not necessarily continuous) surjective maps.

Proposition 2.3. *The structure sheaf is a sheaf for the v-topology, and moreover if X is affinoid then $H^i_v(X, \mathcal{O}_X) = 0$ for all $i > 0$.*

Proof. Write $\mathcal{F} = \mathcal{O}_X^+/\varpi$. Given a surjective morphism

$$X' := \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+) =: X,$$

we need to show that

$$0 \to \mathcal{F}(X) \to \mathcal{F}(X') \to \mathcal{F}(X' \times_X X') \to \dots \tag{4}$$

is almost exact. The idea is to split the statement into two cases:

1. $X' \to X$ is a "w-localization" (in the sense of Bhatt-Scholze),
2. X' is arbitrary but X is "w-local".

Even though we haven't defined these notions yet, it hopefully seems plausible that given such a notion we should be able to reduce to these two cases.

Definition 2.4. A spectral space X is called *w-local* if every connected component has a unique closed point and the set X^c of closed points is closed in X. (This implies that

$$|X| \hookrightarrow X \to \pi_0(X)$$

is a homeomorphism of profinite sets).

Fact. For every affinoid perfectoid space X there exists a morphism $X^z \to X$ where X^z is affinoid perfectoid and w-local, which is "universal for morphisms from w-local spaces to X". This X^z is called the "w-localization". It is basically some profinite disjoint union of the localizations of all points:

$$X^z \to X = \underset{\substack{\longleftarrow \\ \text{finite open cover}}}{\lim} \left(\coprod U_i \to X\right).$$

We now return to the exactness of (4).

Case 1. A w-localization is pro-étale, in which case we already know the result.

Case 2. We assume that $X = \mathrm{Spa}(A, A^+)$ which is w-local. We want to prove the exactness of (4), which amounts to showing (by the usual story for faithfully flat descent) that B^+/ϖ is faithfully flat over A^+/ϖ. Therefore, a reformulation of the statement we want to show is that if $f\colon \operatorname{Spa}(B, B^+) \to X$ is any morphism, then B^+/ϖ is flat over A^+/ϖ and faithfully flat if f is surjective. (The point is that everything is flat over w-local spaces, as when dealing with valuative spaces.)

Consider the composition

$$h\colon Y \xrightarrow{f} X \xrightarrow{g} T := \pi_0(X)$$

Define the sheaves $\mathcal{A} := g_* O_X^+/\varpi$ and $\mathcal{M} := h_* O_Y^+/\varpi$. Then $H^0(T, \mathcal{M})$ is flat over $H^0(T, \mathcal{A})$ if and only if for all $y \in T$, $\mathcal{M}_y$ is flat over $\mathcal{A}_y$. (This is just a general statement about sheaves of rings on profinite sets.)

Now we use the key property of w-locality: y is the same as an inclusion of a closed point $\operatorname{Spa}(K, K^+) \xrightarrow{y} X$ where $K = K(y)$ is a valued field and K^+ is the valuation ring. You then check that $\mathcal{A}_y = K^+/\varpi$ and $\mathcal{M}_y = B_y^+/\varpi$. But flatness over K^+ is the same as torsion-freeness, so B_y^+ is flat over K^+, and so the same holds after modding out by ϖ.

In summary, the trick is that "you can pass to the fibers rather than the stalks" thanks to the w-locality. $\square$

Corollary 2.5. *The v-topology is subcanonical.*

We want to discuss gluing vector bundles in the v-topology.

Theorem 2.6. *The groupoid of vector bundles is a stack for the v-topology.*

For the analytic topology this was proved by Kedlaya-Liu. What we need to do additionally here is to establish descent of vector bundles for surjective affinoid maps. The trick is to use an approximation argument to reduce to the case of a point.