AUTOMORPHIC FORMS OVER FUNCTION FIELDS

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1. Cuspidal automorphic forms

1.1. **Goal.** Let X/k be a curve over a finite field and $F = k(X)$. Let $\mathbf{A} = \mathbf{A}_F$ and $\mathcal{O} = \prod_{x\in|X|} \mathcal{O}_x$, where $\mathcal{O}_x$ is the completed local ring of X at x.

Let $G = \mathrm{GL}_d$ and Z be the center of G. Let $G(\mathcal{O})$ be the maximal compact subgroup of $G(\mathbf{A})$.

Definition 1.1. A function $f\colon G(F)\backslash G(\mathbf{A}) \to \mathbf{C}$ is *smooth* if it factors through $G(F)\backslash G(\mathbf{A})/K$ for some open subgroup K of $G(\mathbf{A})$. It is *cuspidal* if for any proper standard parabolic $P \subset G$, with unipotent N, the *constant term*

$$\varphi_P(g) = \int_{N(F)\backslash N(\mathbf{A})} \varphi(ng)\, dn$$

vanishes.

The main goal of the talk is to prove:

Theorem 1.2 (Harder)**.** *For any compact open $K \subset G(\mathbf{A})$, all cuspidal functions φ acting on $G(F)\backslash G(\mathbf{A})/K \to \mathbf{C}$ have support uniformly finite modulo $Z(\mathbf{A})$.*

1.2. **Automorphic representations.**

Definition 1.3. A smooth function $\varphi\colon G(F)\backslash G(\mathbf{A}) \to \mathbf{C}$ is called *automorphic* if its space spanned by right translations $G(\mathbf{A})$ of φ is admissible. (A smooth representation is *admissible* if the fixed vectors under any compact subgroup are finite dimensional.)

Definition 1.4. A function $\varphi\colon G(\mathbf{A}) \to \mathbf{C}$ has *central character* χ if $\chi(zg) = \chi(z)\varphi(g)$ for all $z \in Z(\mathbf{A})$.

Remark 1.5. If φ is cuspidal automorphic form with a central character, after twisting by $\mu \circ \det$ for some idele character μ, we may view φ as a function on $G(F)\backslash G(\mathbf{A})/Ka^{\mathbf{Z}}$, where $a \in Z(\mathbf{A}) = \mathbf{A}^\times$ has $\deg a = 1$.

Harder's theorem implies that $\mathcal{A}_{\mathrm{cusp}}(G(F)\backslash G(\mathbf{A})/Ka^{\mathbf{Z}})$ is of finite dimension.

Definition 1.6. We define $\mathcal{A}_{G,\mathrm{cusp},\chi}$ to be the space of automorphic cuspidal forms of central character χ. This has an action $G(\mathbf{A})$ by right translation.

Theorem 1.7. *For any $\chi \in \chi_G$, $\mathcal{A}|_{G,\mathrm{cusp},\chi}$ is an admissible representation of $G(\mathbf{A})$. Moreover, it has a countable direct sum decomposition*

$$\mathcal{A}_{G,\mathrm{cusp},\chi} = \bigoplus_{\pi\in\Pi_{G,\mathrm{cusp},\chi}} \pi.$$

Here $\Pi_{G,\mathrm{cusp},\chi}$ is the set of equivalence classes of irreducible automorphic cuspidal representations of central character χ.

What is the content of this statement? It's obvious that π occurs as a subquotient. The theorem says that it actually occur as an honest subrepresentation, and also asserts a multiplicity one statement.

Proof. Admissibility follows from Harder.

Semisimplicity: after twisting $\mathcal{A}_{G,\mathrm{cusp},\chi} \otimes (\mu \circ \det)$, we can assume that is χ is unitary. Then

$$\langle \varphi_1, \varphi_2 \rangle := \int_{G(F)Z(\mathbf{A})\backslash G(\mathbf{A})} \overline{\varphi_1} \varphi_2 \, dg$$

defines a $G(\mathbf{A})$-invariant positive definite Hermitian scalar product on $\mathcal{A}_{G,\mathrm{cusp},\chi}$. Since $G(\mathbf{A})$ has a countable open basis at e, this implies

$$\mathcal{A}_{G,\mathrm{cusp},\chi} = \bigoplus \pi^{m(\pi)}$$

with $m(\pi) = \dim \mathrm{Hom}_{G(\mathbf{A})}(\pi, \mathcal{A}) \geq 1$.

To see that $m(\pi) = 1$, we use that the Whittaker spaces are 1-dimensional. If $\psi \colon F\backslash \mathbf{A}_F \to \mathbf{C}^\times$ is a non-trivial unitary character, and U is the unipotent radical of the Borel, then we have

$$\mathrm{Hom}_{U(\mathbf{A})}(\pi, \psi) = \mathrm{Hom}_{G(\mathbf{A})}(\pi, \mathrm{Ind}_{U(\mathbf{A})}^{G(\mathbf{A})} \psi)$$

The latter is one-dimensional, which we can prove by passing to the local Whittaker model.

If $\xi \colon \pi \hookrightarrow \mathcal{A}_{G,\mathrm{cusp},\chi}$ then we get a map $\pi \to W_\xi$, sending

$$\varphi \mapsto W_{\xi(\varphi)}(g) := \int_{U(F)\backslash U(\mathbf{A})} \xi(\varphi)(ng)\psi(n)^{-1} dn.$$

From this we can "recover"

$$\xi(\varphi)(g) = \sum_{\gamma \in U_{d-1}(F)\backslash G_{d-1}(F)} W_{\xi(\varphi)}[\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g]$$

so the 1-dimensionality of the Whittaker model for π implies $m(\pi) = 1$. □

2. Reduction theory on Bun_G

Definition 2.1. The slope of $\mathcal{E}$ is defined to be $\mu(\mathcal{E}) := \frac{\deg \mathcal{E}}{\mathrm{rank}\, \mathcal{E}}$. We have $\deg \mathcal{E} = \deg \det \mathcal{E}$.

By Riemann-Roch,

$$\chi(\mathcal{E}) = \deg \mathcal{E} + \mathrm{rank}\, \mathcal{E}(1 - g_X).$$

Definition 2.2. A (non-zero) vector bundle $\mathcal{E}$ over X is said to be *semistable* if for all sub-bundles

$$0 \subsetneq \mathcal{F} \subsetneq \mathcal{E},$$

we have $\mu(\mathcal{F}) \leq \mu(\mathcal{G})$. There is an equivalent formulation in terms of quotients.

Definition 2.3. A filtration of a vector bundle $\mathcal{E}$ on X

$$0 = F_0\mathcal{E} \subset F_1\mathcal{E} \subset \ldots \subset F_s\mathcal{E} = \mathcal{E}$$

is a *Harder-Narasimhan (HN) filtration* if $F_j\mathcal{E}/F_{j-1}\mathcal{E}$ are semistable with slopes μ_j satisfying

$$\mu_1 > \mu_2 > \ldots > \mu_j.$$

Example 2.4. Let $X = \mathbf{P}^1/k$. Then $\mathcal{E} = \bigoplus_{i=1}^s \mathcal{O}(n_i)^{r_i}$ with $n_1 > n_2 > \ldots > n_s$ integers. Then the HN filtration is

$$0 \subset \mathcal{O}(n_1)^{r_1} \subset \mathcal{O}(n_1)^{r_1} \oplus \mathcal{O}(n_2)^{r_2} \subset \ldots \subset$$

Theorem 2.5 (Harder-Narasimhan)**.** *Any non-zero vector bundle over X admits a unique HN filtration.*

Proof. Let μ_1 be the maximal slope of a sub-bundle $\mathcal{F} \subset \mathcal{E}$. By Riemann-Roch, we know this to be $< \infty$. We claim that in any HN filtration, $F_1\mathcal{E}$ is the maximal subbundle $\mathcal{E}_1$ with $\mu(\mathcal{E}_1) = \mu_1$. (The result would then follow by induction.)

To see that $\mathcal{E}_1$ exists, suppose you have $\mathcal{E}_1', \mathcal{E}_1''$ which both have r_1 with slope μ_1. Consider $\mathcal{F} := \langle \mathcal{E}_1' + \mathcal{E}_1'' \rangle$, the saturation of the subsheaf of $\mathcal{E}$ spanned by $\mathcal{E}_1'$ and $\mathcal{E}_1''$. Then

$$\deg \mathcal{F} \geq 2r_1\mu_1 - \deg(\mathcal{E}_1' \cap \mathcal{E}_1'')$$

(the inequality comes from the saturation) while

$$\operatorname{rank} \mathcal{F} = 2r_1 - \operatorname{rank}(\mathcal{E}_1' \cap \mathcal{E}_1'') > r_1.$$

So $\mu(\mathcal{F}) \geq \mu_1$ and dominates both $\mathcal{E}_1'$ and $\mathcal{E}_1''$.

To see that $F_1\mathcal{E}$, must be defined in this way, note that the definition of $\mathcal{E}_1$ forces it to be semistable. Therefore, its image in $F_i\mathcal{E}/F_{i+1}\mathcal{E}$ has slope at least $\mu(\mathcal{E}_1) \geq \mu_1$, so this image must be 0.

□

Write $B = TU$. By Weil's adelic uniformization, we can interpret

$$B(F)\backslash B(\mathbf{A})/B(\mathcal{O}) \leftrightarrow \text{ isomorphism classes of flags of rank } (1, \ldots, 1).$$

Let Δ be the set of simple roots of G.

Theorem 2.6 (Siegel Domain)**.** *Let $c_2 \geq 2g$ be an integer. Then*

$$G(\mathbf{A}) = G(F)U(\mathbf{A})T(\mathbf{A})^{\Delta}_{c_2}G(\mathcal{O})$$

where $T(\mathbf{A})^{\Delta}_{c_2} = \{t \in T(\mathbf{A})\colon \deg \alpha(t) \leq c_2 \forall \alpha \in \Delta\}$. In other words (by Iwasawa decomposition), for every $\mathcal{E}$ of rank d over X, there is at least one flag

$$0 \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_d = \mathcal{E}$$

such that $\deg(\mathcal{E}_{j+1}/\mathcal{E}_j) - \deg(\mathcal{E}_j/\mathcal{E}_{j-1}) \leq c_2$ for all j.

Proof. Take a subline bundle $\mathcal{L} \subset \mathcal{E}$ with $\mathcal{E}_1 = \langle \mathcal{L} \rangle$ (the saturation) such that

$$1 \leq \deg \mathcal{E} - d \deg \mathcal{L} + d(1-g) \leq d. \tag{2.1}$$

Why is this possible? The lower bound comes from Riemann-Roch applied to $H^0(\mathcal{E}\otimes \mathcal{L}^\vee)$, which is non-zero as soon as there exists $\mathcal{L} \hookrightarrow \mathcal{E}$. The upper bound comes from the inequality

$$\deg \mathcal{E}_1 \geq \det \mathcal{L} \geq \frac{\deg \mathcal{E}}{d} - g$$

which comes from the non-existence of extensions with too large separation of degree (by Serre duality).

By induction, we can lift a filtration with this property $\mathcal{E}/\mathcal{E}_1$. The only question is to check the desired inequality for $i = 1$. If $\mathcal{E}$ is semistable we can conclude as follows: the analog of (2.1) holds to tell us that

$$\mathcal{E}_2 - d \deg \mathcal{E}_1 + 2(1-g) \leq 2$$

so

$$\begin{aligned} \deg \mathcal{E}_2/\mathcal{E}_1 - \deg \mathcal{E}_1 &= \deg \mathcal{E}_2 - 2\deg \mathcal{E}_1 \\ &\leq 2 - 2(1-g) = 2g. \end{aligned}$$

If $\mathcal{E}$ is not semistable, take an HN filtration, whose associated subquotients are semistable by definition. We apply the conclusion from the semistable case to each subquotient. The only issue is to check that the inequality still holds at the endpoints. The desired inequalities end up following from the semistability.

□

Theorem 2.7. *Let $K \subset G(\mathcal{O})$ be a compact open subgroup. There exists an open subset $C_K \subset G(\mathbf{A})$ satisfying*

(1) $Z(\mathbf{A})G(F)C_K = C_K$, i.e. C_K is invariant under $Z(\mathbf{A})G(F)$, and
(2) $Z(\mathbf{A})G(F)\backslash C_K/K$ is finite.

Moreover, $\operatorname{supp}\varphi \subset C_K$ *for all cuspidal φ.*