New Euler Systems for Automorphic Galois Representations

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This is joint work with Lei, Loeffler, and Skinner.

1 General introduction to Euler systems

Definition 1.1. Let V be a p-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, unramified outside $\Sigma \ni p$. An *Euler System* for V is a collection of cohomology classes $(z_m)_{m\geq 1}$ with $z_m \in H^1(\mathbb{Q}(\mu_m), V^*(1))$ such that

- z_m takes values in a $\mathbb{G}_{\mathbb{Q}}$-stable lattice $T \subset V^*(1)$ which is independent of m.
- (Norm relations)

$$\mathrm{cores}^{\mathbb{Q}(\mu_{m\ell})}_{\mathbb{Q}(\mu_m)} z_{m\ell} = \begin{cases} z_m & \ell \mid m, \ell \in \Sigma \\ P_\ell(\sigma_\ell^{-1}) z_m & \text{otherwise} \end{cases}$$

  where

$$P_\ell(x) = \det(1 - \sigma_\ell^{-1} X|_V)$$

  where σ_ℓ^{-1} is the arithmetic Frobenius.

Remark 1.2. A similar definition can be made over number fields. In that case one needs classes for *all* abelian extensions, not just the cyclotomic ones.

Here are the known Euler systems over $\mathbb{Q}$:

1. cyclotomic units ($V = \mathbb{Q}_p$)
2. Kato's Euler system ($V = V_pE$ or V_pf, with f a modular form of weight ≥ 2)
3. Rankin-Selberg Euler system (LLZ, KLZ): these are comprised of Beilinson-Flach elements $(BF_m^{(f,g;j)})_m$ where f, g are modular forms of weights $k+2, k'+2 \geq 2$. Here $V = V_pf \otimes V_pg(1+j)$.

When we first constructed the Rankin-Selberg Euler system we were afraid that it might be 0. However, we know that is not the case because of:

Theorem 1.3 (BDR, KLZ)**.** *If p doesn't divide the levels of f and g, then*

$$\log(BF_1^{(f,g;j)} = (*)L_p(f, g, 1 + j)$$

where the explicit constant $(*)$ can be computed in examples.

2 The Rankin-Selberg Euler system

2.1 Variants

In [LLZ] we construct an Euler system for $V_p f$ (take one form to be CM).

Lamplugh studied the case when both forms have CM.

2.2 Outline of the construction

Fix $N \geq 1$ and $m \geq 1$. The geometric input is a Siegel unit

$$g_{1/m^2N} \in \mathcal{O}(Y_1(m^2N))^* = H^1_{mot}(Y_1(m^2N), \mathbb{Q}(1)).$$

We then consider an embedding

$$Y_1(m^2N) \to Y_1(N)^2$$

by $z \mapsto (z, z + 1/m)$.

The idea is then to push forward the Siegel unit along this embedding. On motivic cohomology we have

$$H^1_{mot}(Y_1(m^2N), \mathbb{Q}(1)) \xrightarrow{i_{m,N}} H^3_{mot}(Y_1(N)^2 \times \mu^0_M, \mathbb{Q}(2))$$

Motivic cohomology has a regulator map to etale cohomology, and then we use Hoschild-Serre to get a map to $H^1(\mathbb{Q}(\mu_m), H^2_{\text{ét}}(\overline{Y_1(N)}^2, \mathbb{Q}_p(2))$.

The Galois representation $V_p(f) \otimes V_p(g)(1+j)$ appears as a quotient of $H^2_{\text{ét}}(\overline{Y_1(N)}^2, \mathbb{Q}_p(2)$, so we can take a projector to $H^1(\mathbb{Q}(\mu_m), V_f^* \otimes V_g^*)$.

3 New Euler systems

What were the important ingredients? The key thing is obviously the embedding $i_{m,N}$. What are the important properties of it?

- it has the right relative dimension.
- Also, it's important that it "picks up" $\mathbb{Q}(\mu_m)$.

Note that $i_{m,N}$ is a perturbation of the diagonal embedding $Y_1(N) \hookrightarrow Y_1(N)^2$ which corresponds to $\mathrm{GL}_2 \hookrightarrow \mathrm{GL}_2 \times \mathrm{GL}_2$. Can we replace this with another group?

We can apply this strategy in the following cases.

1. $\mathrm{GL}_2 \hookrightarrow \mathrm{Res}_{\mathbb{Q}}^F \mathrm{GL}_2$. ($F$ a real quadratic field)
2. $\mathrm{GL}_2 \times \mathrm{GL}_2 \hookrightarrow GSP_4$.
3. $\mathrm{GL}_2 \hookrightarrow U(2,1)$

In all of these cases we can repeat the strategy to obtain new Euler systems.

3.1 Euler system for Hilbert modular forms

This corresponds to the first case above.

Let F be a real quadratic field of narrow class number 1. Let $G = \mathrm{Res}_{\mathbb{Q}}^F \mathrm{GL}_2$.

Let $\sigma_1, \sigma_2 \colon \mathcal{F} \hookrightarrow \mathbb{R}$ be the two embeddings and $\mathrm{GL}(\mathbb{Z})$ act on $\mathbb{H} \times \mathbb{H}$ through the two embeddings.

Definition 3.1. Let $\mathcal{N}$ be a non-zero ideal of $\mathcal{O}_F$. Let

$$U_1(\mathcal{N}) = \left\{ \gamma \in \mathrm{GL}_2(\mathcal{O}_F) \mid \gamma = \begin{pmatrix} * & * \\ & 1 \end{pmatrix} \mod N \right\}.$$

Let $\mathcal{F}$ be a Hilbert cuspidal eigenform of level $\mu_1(M)$ and parallel weight 2.

Theorem 3.2 (Blasius, Rogawski, Taylor)**.** *There exists a 2-dimensional p-adic representation $V_{\mathcal{F}}$ of G_F attached to $\mathcal{F}$.*

We want a representation of $G_{\mathbb{Q}}$, so we take the tensor induction. Call this $V_{\mathcal{F}}^{Asai}$. This is a 4-dimensional representation of $G_{\mathbb{Q}}$, with the property that

$$V_{\mathcal{F}}^{Asai}|_{G_F} = V_{\mathcal{F}} \otimes V_{\mathcal{F}^\sigma}.$$

We define L_{Asai} to be the associated L-function.

Theorem 3.3 (Brylinski-Labesse)**.** *$V_{\mathcal{F}}^{Asai,*}$ appears as a quotient of $H^2_{\acute{e}t}(\overline{\mathcal{Y}_1(\mathcal{N})}, \mathbb{Q}_p(2))$.*

Here $\mathcal{Y}_1(\mathcal{N})$ is the Hilbert modular surface attached to $U_1(\mathcal{N})$. (For instance, its complex points are $\mathbb{H} \times \mathbb{H}/U_1(\mathcal{N})$.)

Let $N = \mathcal{N} \cap \mathbb{Z}$. Then we get $Y_1(N) \to \mathcal{Y}_1(\mathcal{N})$. The perturbation of it is as folows. Let a be a generator of $\mathcal{O}_F/\mathbb{Z}$. Then we take

$$i_{m,N} \colon Y_1(m^2N) \to \mathcal{Y}_1(\mathcal{N}) \times \mu_m^0$$

by

$$z \mapsto \left(z + \frac{\sigma_1(a)}{m}, z + \frac{\sigma_2(a)}{m} \right).$$

Again, we have induced maps on motivic cohomology, then the étale regulator, then the Hochschild-Serre spectral sequence to get a class in

$$H^1(\mathbb{Q}(\mu_m), H^2_{\text{ét}}(\overline{\mathcal{Y}_1(\mathcal{N})}, \mathbb{Q}_p(2))$$

and finally we use the Brylinski-Labesse theorem to project to $H^1(\mathbb{Q}(\mu_m), V_{\mathcal{F}}^{Asai,*})$.

Theorem 3.4 (LLZ)**.** *The $z_m^{\mathcal{F}}$ form an Euler system.*

Is this question the trivial Euler system?

The Asai-Flach element (the thing in $H^3_{mot}(\mathcal{Y}(\mathcal{N}), \mathbb{Q}(2)))$ is non-zero. To show this, we evaluate under Deligne's regulator to Deligne cohomology (?). What she wrote down is $(\mathrm{Fil}^1 H^2_{\mathrm{dR}}(X/\mathbb{C}))^\vee$.

If we pair with a suitable element then we'll get an L-value; the challenge is then to show that this is non-zero.

Let $\eta \in \mathcal{O}_F^*$ be a norm-1 element with $\sigma_1(\eta) < 0$ and $\sigma_2(\eta) > 0$. Let $\omega_{\mathcal{F}}$ be a differential on $\mathcal{Y}_1(\mathcal{N})$ such that its pullback to $\mathbb{H} \times \mathbb{H}$ is a certain thing in $\mathrm{Fil}^1 H^2_{\mathrm{dR}}$, then it is a theorem of Kings that the pairing is some explicit constant times $L'_{Asai}(\mathcal{F}, 1)$. In particular, the punchline is that we can show that the image under the Deligne regulator is non-zero by showing that a certain L-value is non-zero.

So does the image of the Asai-Flach element vanish in étale cohomology? It is conjectured that the etale regulator is not injective, but we do not know it. So we do not know in general that $z_1^{\mathcal{F}}$ is non-zero. The *expectation* is that the Bloch-Kato logarithm of the class should be related to a non-critical value of a p-adic Asai L-function. But this is not known to exist! The construction of this p-adic L-function, and the verification of non-vanishing for the Euler system, is ongoing work with Loeffler/Skinner.

However, we do know in an explicit example that the Euler system is non-zero. Let $\mathcal{F}$ be a Hilbert modular form defined over $\mathbb{Q}(\sqrt{37})$ of weight $(4, 2)$ and level 1. We computed that $\log_{BK}(z_1^{\mathcal{F}}) \neq 0$.

3.2 An Euler system for GSp_4

Let $J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$. Then

$$\mathrm{GSp}_4(R) = \{g \in \mathrm{GL}_4(R) \colon g^t J g = \mu J\}.$$

We have a map $i\colon \mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathrm{GSp}_4$ via

$$\begin{pmatrix} a & b \\ c & a \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \mapsto \begin{pmatrix} a & & b & \\ & a' & & b' \\ c & & d & \\ & c' & & d' \end{pmatrix}$$

Let

$$\widetilde{\Gamma_1(N)} = \left\{ g \in \mathrm{GSp}_4(\mathbb{Z}) \colon g \equiv \begin{pmatrix} * & * \\ 0 & I_2 \end{pmatrix} \mod N \right\} s$$

Let $\widetilde{Y_1(N)}$ be the associated Siegel 3-fold. Then we get

$$i \colon Y_1(N) \times Y_1(N) \to \widetilde{Y_1(N)}.$$

We need a perturbation:

$$i_{m,N} \colon Y_1(m^2N)^2 \hookrightarrow Y_1\widetilde{(m^2N)} \xrightarrow{\begin{pmatrix} 1 & & & 1/m \\ & 1 & 1/m & \\ & & 1 & \\ & & & 1 \end{pmatrix}} \widetilde{Y_1(N)} \times \mu_m^0.$$

As before, take $g_{1/m^2N} \cup g_{1/m^2N}$ and push forward to motivic cohomology, then take an étale regulator and Hochschild-Serre spectral sequence.

Let $\mathcal{F}$ be a Siegel cuspidal eigenform of weight $(3, 3)$ and level $\widetilde{\Gamma_1(N)}$. There is an associated spin representation which is a quotient of the relevant étale cohomology group.

Theorem 3.5 (LSZ)**.** *The classes $z_m^{\mathcal{F}}$ form an Euler system.*

Again, the issue is that we cannot construct spin p-adic L-functions.

Remark 3.6. Let me record something that David Loeffler told me afterwards when I asked what these perturbations were. The point is that you perturb by something in the Borel of the bigger group which is *not* in the Borel in the smaller group. In fact, you can view it as the image of $1/m$ under a coroot for the bigger group which doesn't come from a coroot of the smaller group.

As for why this is necessary, the perturbation matches the translations which are involved in expressing twists of the L-values in terms of modular symbols, as in Mazur-Tate-Teitelbaum.