Perfectoid Spaces

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1 Perfectoid Rings

Fix a prime p. We make references throughout to [12] Scholze's IHES paper on perfectoid spaces, [13] Scholze's MSRI lectures.

Recall that a *complete Tate ring* A is a complete topological ring A such that there exists a complete subring $A_0 \subset A$ which is open and a finitely generated ideal $I \subset A_0$ such that $\{I^n : n \in \mathbb{N}\}$ is a neighborhood basis of 0; also, there exists $\varpi \in A^\times$ such that $\varpi^n \to 0$ as $n \to \infty$. Such a ϖ is called a *pseudo-uniformizer*.

Definition 1.1. A subset $S \subset A$ is *bounded* if $S \subset \varpi^{-n} A_0$ for some n. We denote by A^0 the ring of *power-bounded elements* of A:

$$A^0 = \{a \in A \mid \{a^n\} \text{ is bounded}\}.$$

Definition 1.2. A *perfectoid ring* is a complete Tate ring A such that

- A^0 is bounded,
- there exists a pseudo-uniformizer $\varpi \in A$ such that $p/\varpi^p \in A^0$, and
- the map $\Phi \colon A^0/(\varpi) \mapsto A/(\varpi^p)$ sending $x \mapsto x^p$ is an isomorphism.

Example 1.3. Examples of perfectoid rings:

1. $\mathbb{Q}_p^{\mathrm{cyc}} := \mathbb{Q}_p(\zeta_{p^n} \forall n)^\wedge$ where ζ_{p^n} is a primitive p^nth root of unity.
2. $\mathbb{F}_p((T^{1/p^\infty})) := \left(\bigcup_n \mathbb{F}_p((T^{1/p^n}))\right)^\wedge$ with the T-adic topology.

Example 1.4. If A is a complete Tate ring $p = 0$ and A^0 bounded, then

$$A \text{ perfectoid} \iff A \text{ perfect}.$$

Reference: [12, Proposition 5.9]

Example 1.5. If $A = K$ is a non-archimedean field then K is perfectoid if and only if the valuation is non-discrete, $|p| < 1$, and $\Phi\colon \mathcal{O}_K/(p) \to \mathcal{O}_K/(p)$ is surjective. [13, Proposition 6.1.8, 6.1.9]

We saw earlier that if we have such a Tate ring, then we can form a Huber pair and then take its adic spectrum. However, it is not clear that this gives rise to an adic space because it is not clear that the structure presheaf will be a sheaf.

Theorem 1.6. *Let (A, A^+) be a Huber pair (i.e. $A^+ \subset A$ is an open subring which is also integrally closed) with A perfectoid. Then for all rational subdomains $U \subset X = \mathrm{Spa}(A, A^+)$ the ring $\mathcal{O}_X(U)$ is perfectoid, which implies that $\mathcal{O}_X(U)^0$ is bounded for all U, hence (A, A^+) is sheafy.*

Proof. The original proof was [12, Theorem 6.3iii], but the argument there is different. □

2 Tilting

Definition 2.1. Let A be a perfectoid ring. We define its *tilt* to be

$$A^\flat := \varprojlim_{x \mapsto x^p} A.$$

Its elements will be expressed as

$$(x^{(0)}, x^{(1)}, \ldots) \quad \text{such that } x^{(n)} = (x^{(n+1)})^p.$$

This is a ring under the operations

$$(x^{(n)})(y^{(n)}) = (x^{(n)} y^{(n)})$$

and

$$(x^{(n)}) + (y^{(n)}) = \left(\lim_{k \to \infty} (x^{(n+k)} + y^{(n+k)})^{p^k} \right).$$

Example 2.2. We have $(\mathbb{Q}_p^{\mathrm{cyc}})^\flat = \mathbb{F}_p((T^{1/p^\infty}))$ with

$$T = (1, \epsilon, \epsilon_2, \ldots) - (1, 1, \ldots).$$

where ϵ_n is a primitive p^nth root of unity. We take

$$T^{1/p^n} = (\epsilon_n, \epsilon_{n+1}, \ldots) - (1, 1, \ldots).$$

We have $\mathbb{Z}_p^* \xrightarrow{\sim} \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{cyc}}/\mathbb{Q}_p)$ via the cyclotomic character:

$$a \mapsto (\epsilon_n \mapsto \epsilon_n^{a \mod p^n}).$$

Since tilting is supposed to preserve Galois groups, we get as expected an action of $\mathbb{Z}_p^*$ on $\mathbb{F}_p((T^{1/p^\infty}))$ with

$$a \in \mathbb{Z}_p^*\colon (T^{1/p^n} \mapsto (1 + T^{1/p^n})^a - 1).$$

Lemma 2.3. *If A is a perfectoid ring, then*

1. *$A^\flat$ is a perfectoid ring with $p = 0$.*
2. *We have*
$$A^{\flat 0} = \varprojlim_{x \mapsto x^p} A^0 = \varprojlim_{x \mapsto x^p} A^0/(p).$$
3. *There exists a pseudo-uniformizer $\varpi \in A$ with $\varpi^{1/p^n} \in A$ for all n. Write*
$$\varpi^\flat = (\varpi, \varpi^{1/p}, \ldots).$$
Then $A^\flat = A^{\flat 0}[1/\varpi^\flat]$.
4. *There is a multiplicative (but not additive) map $A^\flat \to A$ sending $(x^{(n)}) \mapsto x^{(0)}$, denoted $x \mapsto x^\#$. It induces an isomorphism of rings*
$$A^{\flat 0}/\varpi^\flat \xrightarrow{\sim} A^0/\varpi, \quad \textit{where } \varpi = \varpi^{\flat \#}.$$
5. *Fixing A and $A^\flat$, the association $A^+ \rightsquigarrow A^{+\flat} = \varprojlim_{x \mapsto x^p} A^+$ gives a bijection between Huber pairs (A, A^+) and $(A^\flat, A^{\flat +})$*

Proof. See [13, Lemmas 6.2.2 and 6.2.4]. □

Theorem 2.4. 1. *There is a homeomorphism $X := \mathrm{Spa}(A, A^+) \to X^\flat := \mathrm{Spa}(A^\flat, A^{+\flat})$ sending $x = |\cdot|_x$ to $x^\flat = |\cdot|_{x^\flat}$ with*
$$|f|_{x^\flat} := |f^\#|_x.$$
Furthermore, it preserves rational subsets.

2. *If $U \subset X$ is rational then $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is perfectoid with tilt $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}(U^\flat))$.*

Proof. See [12, Theorem 6.3 i,ii]. □

Idea of proof of Theorem 1.6. If $p = 0$ in A, then we can write
$$(A, A^+) = \left(\varinjlim_i (A_i, A_i^+)^{\mathrm{perf}, \wedge} \right)^\wedge$$
because (A_i, A_i^+) reduced of topologically finite type over a perfectoid field. The question of sheafiness for (A, A^+) can in this way be reduced to that for (A_i, A_i^+), and then it follows from a result of Huber (basically by Noetherian approximation).

If $p \neq 0$ in A, then we can use tilting. The theorem having been established in positive characteristic, we can deduce the result for X from that for $X^\flat$ via
$$\mathcal{O}_{X^\flat}^+(U^\flat)/\varpi^\flat \cong \mathcal{O}_X^+(U)/\varpi.$$
Indeed, being a sheaf means that the first Cech cohomology group of this vanishes. The point is basically that if something is true modulo ϖ^n, then it is true for $\varpi^{n/p}$ by inverting Frobenius. Thus the point is that it allows you to automatically improve bounded results to arbitrarily fine results. □

3 Perfectoid Spaces

3.1 Étale morphisms

Definition 3.1. A *perfectoid space* is an adic space covered by $\mathrm{Spa}(A, A^+)$ with A perfectoid.

Theorem 2.4 implies that tilting glues to give a functor $X \mapsto X^\flat$.

We want to discuss the "étale site" of a perfectoid space.

Theorem 3.2. *Let A be a perfectoid ring with tilt $A^\flat$.*

1. *Any finite étale A-algebra is perfectoid.*
2. *The functor $B \mapsto B^\flat$ is an equivalence between*
   - *perfectoid A-algebras and $A^\flat$-algebras*
   - *finite étale A-algebras and finite étale $A^\flat$-algebras.*

Proof. See [12, Theorem 5.2]. □

Definition 3.3. A morphism $f\colon Y \to X$ of perfectoid spaces is called

1. *finite étale* if for all open affinoids $U = \mathrm{Spa}(A, A^+) \subset X$ the pre-image $f^{-1}(U) = \mathrm{Spa}(B, B^+)$ is also affinoid, with B finite étale over A and B^+ the integral closure of A^+ in B.
2. *étale* if for every $y \in Y$, there exists an open neighborhood $U \subset Y$ containing y and $V \subset X$ an open subset with $f(U) \subset V$, and a diagram

$$\begin{tikzcd} U \arrow[r, hook, "\text{open}"] & W \arrow[d, "\text{finite étale}"] \\ & V \end{tikzcd}$$

Remark 3.4. Why this definition of étale? One issue is that all perfectoid spaces are reduced, so there's no hope of formulating an infinitesimal lifting criterion. Also, things are never of finite type.

Remark 3.5. The analogous result for schemes is *false*. (For instance, you can take the étale locus of a branched cover of $\mathbb{P}^1$ of degree at least 2.) However, the analogous result for rigid analytic spaces is true.

Proof. See [12 Lemma 7.3 and Corollary 7.8]. □

3.2 The étale site

Proposition 3.6. *Étale morphisms of perfectoid spaces enjoy the following properties:*

1. *Finite étale morphisms of perfectoid spaces are stable under compositions and base change. (In particular, there exist fiber products in the category of perfectoid spaces; this is not the case for general adic spaces.)*

2. *étale morphisms of perfectoid spaces are open.*

3. *$f : X \to Y$ is étale if and only if $f^\flat : X^\flat \to Y^\flat$ is étale.*

Definition 3.7. The *étale site* $X_{\text{ét}}$ of X is the category of perfectoid spaces étale over X with topological coverings.

If we apply this to a perfectoid *field*, then we get an identification of absolute Galois groups, recovering a "classical" result of Fontaine-Wintenberger.

3.3 The philosophy of tilting

The topological properties of a perfectoid space X and its tilt then all *topological* information, such as $|X|, X_{\text{ét}}$ can be recovered from $X^\flat$. However, a perfectoid space over $\mathbb{Q}_p$ has a structure morphism to $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ which is "forgotten" by tilting.