MODULI OF SHTUKAS II

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1. Goal

We're going to start by stating the formula which is the goal of this work. We'll then spend most of the talk explaining the meaning of some parts of it. For $f \in \mathcal{H}$ the Hecke algebra, define

$$\mathbb{I}_r(f) := \langle \theta^\mu_*[\mathrm{Sht}^\mu_T], f * \theta^\mu_*[\mathrm{Sht}^\mu_T] \rangle_{\mathrm{Sht}'^r_G} \in \mathbf{Q}.$$

Here $\theta^\mu_*[\mathrm{Sht}^\mu_T] \in \mathrm{Ch}_{c,r}(\mathrm{Sht}'^r_G)_{\mathbf{Q}}$, a cycle in the "rational Chow group of dimension r cycles proper over k".

This Chow group has an action of ${}_c\mathrm{Ch}_{2r}(\mathrm{Sht}'^r_G \times \mathrm{Sht}'^r_G)_{\mathbf{Q}}$, the "rational Chow group of dimension $2r$ cycles proper over the first factor". This actually has an algebra structure.

The main goals for today are:

- Define a map $\mathcal{H} \to {}_c\mathrm{Ch}_{2r}(\mathrm{Sht}'^r_G \times \mathrm{Sht}'^r_G)$.
- Define a map $\theta^\mu \colon \mathrm{Sht}^\mu_T \to \mathrm{Sht}'^r_G$.

2. The Hecke algebra

Let $G = \mathrm{PGL}_2$, X/k, and $F = k(X)$. Write

$$K = \prod_{x \in |X|} K_x, \quad K_x = G(\mathcal{O}_x).$$

Definition 2.1. The *spherical Hecke algebra* is

$$\mathcal{H} = C_c^\infty(K \backslash G(\mathbf{A})/K, \mathbf{Q}) = {\bigotimes_{x \in |X|}}' C_c^\infty(K_x \backslash G(F_x)/K_x, \mathbf{Q}).$$

The algebra structure is by convolution.

Let $M_{x,n}$ be the subset of $\mathrm{Mat}_2(\mathcal{O}_x)$ with determinant n, viewed in $G(F_x)$. Thus

$$M_{x,0} = K_x$$
$$M_{x,1} = K_x \begin{pmatrix} \varpi_x & \\ & 1 \end{pmatrix} K_x$$
$$\vdots$$

Let $h_{nx} \in \mathcal{H}_x$ be the characteristic function of $M_{X,n}$. By Cartan decomposition, these form a $\mathbf{Q}$-basis of $\mathcal{H}_x$.

Let $D = \sum_{x \in |X|} n_x x$ be an effective divisor. Then $h_D = \otimes_{x \in X} h_{n,x} \in \mathcal{H}$ is a $\mathbf{Q}$-basis for $\mathcal{H}$.

3. Hecke correspondences

Let μ be an r-tuple with the same number of μ_+'s and μ_-'s. We define $\mathrm{Sht}_2^\mu(h_D)(S)$ to parametrize

- $(x_1, \dots, x_r)$ maps $S \to X$.
- a commutative diagram

$$\begin{array}{ccccccccc}
\mathcal{E}_0 & \dashrightarrow & \mathcal{E}_1 & \dashrightarrow & \dots & \dashrightarrow & \mathcal{E}_r & \xrightarrow{\sim} & {}^\tau\mathcal{E}_0 \\
\downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi_1} & & & & \downarrow{\scriptstyle \phi_r} & & \downarrow{\scriptstyle {}^\tau\phi_0} \\
\mathcal{E}'_0 & \dashrightarrow & \mathcal{E}'_1 & \dashrightarrow & \dots & \dashrightarrow & \mathcal{E}'_r & \xrightarrow{\sim} & {}^\tau\mathcal{E}'_0
\end{array}$$

  with top and bottom rows in Sht_2^μ, such that:
- the map $\det \phi_i$ has divisor $D \times S$.

For $G = \mathrm{PGL}_2$, we define

$$\mathrm{Sht}_G^\mu(h_D) := \mathrm{Sht}_2^\mu(h_D) / \mathrm{Pic}_X(k).$$

We have a commutative diagram

$$\begin{array}{ccccc}
 & & \mathrm{Sht}_G^r(h_D) & & \\
 & \swarrow{\scriptstyle p^\leftarrow} & & \searrow{\scriptstyle p^\rightarrow} & \\
\mathrm{Sht}_G^r & & & & \mathrm{Sht}_G^r \\
 & \searrow & & \swarrow & \\
 & & X^r & &
\end{array}$$

Lemma 3.1. *The maps $p^\leftarrow$ and $p^\rightarrow$ are representable and proper. The map $(p^\leftarrow, p^\rightarrow)$ is also representable and proper.*

Proof. The fibers of $p^\rightarrow$ are closed subschemes in a product of Quot schemes. For $p^\leftarrow$, dualize. For $(p^\leftarrow, p^\rightarrow)$, the fibers are closed in a product of Hom schemes. Properness follows from Sht being separable and $p^\leftarrow$ being proper. □

Lemma 3.2. *The geometric fibers of $\mathrm{Sht}_G^r(h_D) \to X^r$ have dimension r. Therefore $\dim \mathrm{Sht}_G^r(h_D) = 2r$.*

We now define

$$H \colon \mathcal{H} \to {}_c\mathrm{Ch}_{2r}(\mathrm{Sht}_G^r \times \mathrm{Sht}_G^r)_{\mathbf{Q}}$$

sending $h_D \mapsto (p^\leftarrow, p^\rightarrow)_*[\mathrm{Sht}_G^r(h_D)]$.

Lemma 3.3. *The map H is a ring homomorphism.*

Idea of the proof: we need to show that $H(h_D * h_{D'}) = H(h_D) \cdot H(h_{D'})$. We can reduce to checking this over U^r, where we can see it directly.

Remark 3.4. For $g = (g_x) \in G(\mathbf{A})$, one usually defines a self-correspondence $\Gamma(g)$ of $\mathrm{Sht}_G^r|_{(X\setminus S)^r}$ where $S = \{x\colon g_x \notin K_x\}$. Then $\mathbf{1}_{KgK}|_{(X\setminus S)^r}$ is the same cycle as $\Gamma(g)$. However, in this case the total Hecke algebra only acts on the generic fiber. In the paper, the Hecke action is defined over all of X, using that $\mathrm{Sht}_G^r(h_D)$ is defined over all of X.

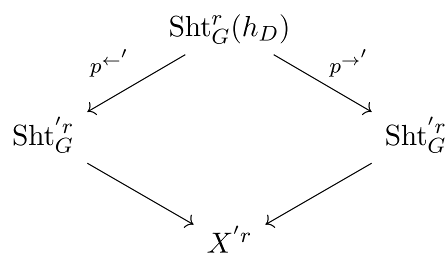
Here is a variant: let $v\colon X' \to X$ be an étale cover of degree 2, and X' and X geometrically connected. Define

$$\mathrm{Sht}'^r_G := (X')^r \times_{X^r} \mathrm{Sht}_G^r$$

and

$$\mathrm{Sht}'^r_G(h_D) := (X')^r \times_{X^r} \mathrm{Sht}_G^r(h_D).$$

Base changing the maps from earlier, we obtain the commutative diagram



This induces a map

$$H'\colon \mathcal{H} \to {}_c\mathrm{Ch}_{2r}(\mathrm{Sht}'^r_G \times \mathrm{Sht}^r_{G'})$$

which is again a ring homomorphism.

Definition 3.5. For $f \in \mathcal{H}$, we have an operator $f * (-) := H'(f)$ acting on $\mathrm{Ch}_{c,*}(\mathrm{Sht}'^r_G)_{\mathbf{Q}}$.

4. The Heegner-Drinfeld cycle

Let μ be a balanced r-tuple. Let $\widetilde{T} := \mathrm{Res}_{X'/X}\,\mathbf{G}_m$, and $T := \widetilde{T}/\mathbf{G}_m$.

We have an action of $\mathrm{Pic}_{X'}(k)$ on $\mathrm{Sht}^\mu_{\widetilde{T}}$. In particular, $\mathrm{Pic}_X(k)$ acts through its embedding into $\mathrm{Pic}_{X'}(k)$. Then $\mathrm{Sht}^\mu_T := \mathrm{Sht}^\mu_{\widetilde{T}} / \mathrm{Pic}_X(k)$ has a map

$$\pi^\mu_T\colon \mathrm{Sht}^\mu_T \to (X')^r,$$

which is a $\mathrm{Pic}_{X'}(k)/\mathrm{Pic}_X(k)$-torsor. Thus Sht^μ_T is proper smooth of dimension r over k. The spaces Sht^μ_T are canonically independent of μ, and so is the structure map to X^r (but not the one to X'^r).

Let $\mathcal{L}$ be a line bundle on $X' \times S$. Then we get a vector bundle $\nu_*\mathcal{L}$ of rank 2 on $X \times S$. This induces

$$\mathrm{Sht}^\mu_T \to \mathrm{Sht}^r_G$$

sending $(\underline{x}', \underline{\mathcal{L}}, \underline{f}, \iota) \mapsto (\nu(\underline{x}'), \nu_*\underline{\mathcal{L}}, \nu_* f, \nu_*\iota)$. Thus we get

$$\mathrm{Sht}^\mu_T \to \mathrm{Sht}'^r_G = (X')^r \times_{X^r} \mathrm{Sht}^r_G.$$

This is a finite étale morphism.

Definition 4.1. We define the *Heegner-Drinfeld cycle* $\theta^\mu_*[\mathrm{Sht}^\mu_T] \in \mathrm{Ch}_{c,r}(\mathrm{Sht}'^r_G)_{\mathbf{Q}}$. Then can define

$$\langle \theta^\mu_*[\mathrm{Sht}^\mu_T], f * \theta^\mu_*[\mathrm{Sht}^\mu_T] \rangle_{\mathrm{Sht}'^r_G}.$$

Lemma 4.2. *This pairing is independent of μ.*

The main result is the following:

Theorem 4.3. *If π is an everywhere unramified automorphic representation of G, then*

$$\mathcal{L}^{(r)}(\pi_{F'}, 1/2) \sim ([\mathrm{Sht}^\mu_T]_\pi, [\mathrm{Sht}^\mu_T]_\pi)_\pi.$$