Beilinson-Kato and Beilinson-Flach elements in p-adic families II

Notes by Tony Feng
for a talk by Henri Darmon

June 16, 2016

Again, this is a slides talk so I'm just jotting down some impressions.

1 Overview

1.1 Rubin's formula

Rubin proved a formula for CM elliptic curves, basically stating that if $L(E, 1) = 0$ then there is a global point P such that some Katz p-adic L-function equals a period times the square of the logarithm of P.

1.2 Perrin-Riou's conjecture

Perrin-Riou conjectured a generalization to all elliptic curves over $\mathbb{Q}$.

One key ingredient is a *Perrin-Riou L-function*, which is a p-adic L-function valued in de Rham cohomology (which is viewed as the Dieudonne module of the Galois representation).

1.3 Compare and contrast

- both formulas involve the square of the logarithm of a class,
- Rubin's formula involves a value whereas Perrin-Riou's formula involves a derivative
- Rubin's formula involves a value outside the classical range of interpolation, whereas Perrin-Riou's formula involves a derivative at a point in the classical range of interpretation.

The goals are

1. clarify the relationship between Rubin's original formula and Perrin-Riou's formula,
2. use the ideas emerging from this comparison to prove Perrin-Riou's formula in special cases.

1.4 Recap

- There is a Beilinson-Kato class $\kappa_{BK}(E,\chi_1,\chi_2) \in H^1(\mathbb{Q}, V_p(E)(\chi_1))$.
- There is a Perrin-Riou L-function $\mathcal{L}_p^{PR}(E,\chi_1,\chi_2)$ and a PR-Garret-Rankin L-function $\mathcal{L}_p^{PR}(f,\underline{g},\underline{h})$.
- The pair (χ_1,χ_2) gives rise to a genus character ψ, from which we get a Hida family of theta series specializing to θ_ψ.
- There is a Beilinson-Flach (cohomology) class $\kappa_{BF}(E,\theta_\psi)$.
- We are trying to relate
$$\frac{d}{ds}\mathcal{L}_p^{PR}(f,\chi_1,\chi_2)$$
to the square of the p-adic logarithm of some point. Perrin-Riou relates it to the BK logarithm of $\kappa_{bk}(f,\chi_1,\chi_2)$.
- Also there is a formula relating
$$\log_p \kappa_{BF}(f,\theta_\psi,E_1(1,\chi)) \leftrightarrow \frac{d}{ds}\mathcal{L}_p^{PR}(F,\underline{\theta}_\psi,\mathcal{E}(1,\chi)).$$
- There is also a Λ-adic Siegel-Weil formula relating Eisenstein series and the theta function. From this one recovers a relation between the Kato and BF classes:
$$\log_p \kappa_{BK}(f,\chi_1,\chi_2) = \log 2\kappa_{BF}(f,\theta_\psi,E_1(1,\chi)).$$
Why is this useful? Because the data of (K,ψ) gives rise to a Heegner point.

The BF class and the Heegner point can be compared; that is the subject of today's talk. The comparison proceeds via two *different* "Rubin-style formulas".

2 A broader setting: generalized Kato classes

What are the objects that play the role, for $\mathcal{L}_p^{PR}(f,\underline{g},\underline{h})$ of

- the Coates-Wiles classes $\kappa_{CW}(\psi)$ for $\mathcal{L}_p^{Katz}(\psi)$?
- the BK classes $\kappa_{BK}(f,\chi_1,\chi_2)$ for $\mathcal{L}_p^{PR}(f,\chi_1,\chi_2)$?

The *generalized Kato classes* are a natural extension suitable for studying $\mathcal{L}_p(f,\underline{g},\underline{h})$ for arbitrary Hida families $\underline{g}$ and $\underline{h}$.

These come in three "flavors".

2.1 Setup

Let f be a modular form on $\Gamma_0(N)$ with trivial nebentypus, and g, h modular forms with the same weight k but opposite nebentypus. Let V_f, V_g, V_h be the representations attached and V_{fgh} the Kummer self-dual Tate twist of the tensor product. We can associate a class κ_{fgh}.

Let's do the example $k = 2$. Let $c \in [1, 3]$ be the number of cuspidal forms among $\{f, g, h\}$.

Fact: we can construct an element $\gamma(f, g, h) \in H^{1+c}_{\text{ét}}(X_1(N)^c, \mathbb{Q}_p(2))$ associated to (f, g, h).

Important preliminary remark: if g is an Eisenstein series of weight 2, then there is a modular unit $u_g \in \mathcal{O}^*_{Y_1(N)}$ such that

$$d \log u_g = g.$$

- If $c = 1$ and g, h are Eisenstein then we consider the δ from Kummer theory and

$$\gamma(f, g, h) = \delta(u_g) \cup \delta(u_h).$$

- If $c = 2$ and h is Eisenstein, then we have $i\colon H_1(N) \hookrightarrow Y_1(N)^2$ and

$$i_*\colon H^1_{\text{ét}}(Y_1(N), \mathbb{Q}_p(1)) \to H^3_{\text{ét}}(Y_1(N)^2, \mathbb{Q}_p(2))$$

  and $\gamma(f, g, h) = i_*(\delta(u_h))$.

- If $c = 3$, then $\Delta \in CH^2(X_1(N)^3)$ be the diagonal and $\gamma(f, g, h)$ is the class of Δ.

In all cases there is a simple modification of these classes yielding a class in the cohomology of the completed curve.

The Serre spectral sequence from geometric to absolute cohomology gives a map from etale cohomology to $H^1(Q, H_{\text{ét}})$ and we consider the image of $\gamma(f, g, h)$. Then there is a formula which looks like tensoring with the representations corresponding to the cuspidal terms.

Assemble these classes into a Λ-class $\underline{\kappa}(f, \underline{g}, \underline{h})$. Then specialize to weight $k = 1$.

2.2 A general theme

Coates-Wiles says that if the L-function doesn't vanish, then the CW class is crystalline.

Kato and BDR proved analogous statements for the Kato and generalized Heegner classes.

From this, we get analogues of the Coates-Wiles results on "BSD in analytic rank 0".

One can use Euler systems to strengthen this to a bound on Selmer groups, hence Ш. This was done by Karl Rubin in the Coates-Wiles setting, in the BK setting ($c = 1$) by Kato, and in the Beilinson-Flach ($c = 2$) by Zerbes et al.

2.3 Importance of p-stabilizations

From f, g, h we get four classes

$$\kappa(f, g_{\alpha/\beta}, h_{\alpha/\beta})$$

where the α, β are distinct roots of some Hecke relation for g, h.

So we also get four logarithms, to the four summands of

$$H^1(\mathbb{Q}_p, V_f \otimes V_{gh}) = \bigoplus H^1(\mathbb{Q}_p, V_f \otimes V_{\alpha/\beta,\alpha/\beta}.$$

3 Results

Theorem 3.1. *For all* $(f, \underline{g}, \underline{h})$ *specializing to* (f, g_α, h_α) *in weights* $(2, 1, 1)$

1. $\log_{\alpha\alpha} \kappa(f, g_\alpha, h_\alpha) = 0$*,*
2. $\log_{\beta\beta} \kappa(f, g_\alpha, h_\alpha) \sim \frac{d}{ds}(\mathcal{L}_p(f, \underline{g}, \underline{h}))_{s=1}$ *modulo* Fil^0*. (This generalizes PR's formula.)*

We also need a Perrin-Riou style formula.

Theorem 3.2 (Lauder, Rotger, D)**.** *For all* $(f, \underline{g}, \underline{h})$ *specializing to* (f, g_α, h_α) *in weight* $(2, 1, 1)$

1. *If g is cuspidal, then* $\log_{\alpha\beta} \kappa(f, g_\alpha, h_\alpha) \sim \mathcal{L}_p^{g_\alpha}(f, g_\alpha, h_\alpha)$
2. *Analogous for h.*

There are more formulas. We need a formula of Waldspurger, which is for something like $L(f \otimes \theta_\psi, 1 + t)$ where ψ is a Hecke character.

$$L(f \otimes \theta_\psi, 1 + t) \sim \left(\sum_{\mathfrak{a}} \psi^{-1} \kappa(\mathfrak{a}) \delta^{t-1} f(\mathfrak{a}) \right)^2.$$

Using Waldspurger's formula one gets a Rubin-style Gross-Zagier formula.

Theorem 3.3.

$$\mathcal{L}_p^{Wald}(f, \psi) \sim \log_p^2(P_\psi).$$

The point is that the $d^{-1} f^{[p]}(\mathfrak{a}) \sim \log_p P_{E,\mathfrak{a}}$ because the LHS is the Coleman primitive.

4 Conclusion

The comparison involved two-different Rubin-style formulas.