Properties of Compatible Systems

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1 Introduction

Let F, E be number fields. We will consider Galois representations of $G_F = \mathrm{Gal}(\overline{F}/F)$. Let S be a finite set of places of F and Σ a set of places of E. Let n be an integer.

A *compatible system* is a set of representations $\{\rho_v\}_{v \in \Sigma}$ where each

$$\rho_v \colon G_F \to \mathrm{GL}_n(E_v)$$

such that

- For $w \notin S$ with residue characteristic distinct from that of v, then the characteristic polynomial of Frob_w has coefficients in E and is independent of v.

This notion was introduced by Serre. For simplicity, we assume that $E = \mathbb{Q}$ for the rest of the talk.

- Geometrically, such systems show up in $H^i_{\text{ét}}(X, \mathbb{Q}_p)$.
- Examples which do not come from geometry show up by interpolating GL_2 / F where F is real or imaginary quadratic.

Theorem 1.1 (Serre 1972)**.** *If E is an elliptic curve without CM, then ρ_ℓ has image an open subgroup of $\mathrm{GL}_2(\mathbb{Z}_\ell)$ for all ℓ, and is actually isomorphic to $\mathrm{GL}_2(\mathbb{Z}_\ell)$ for all but finitely many primes ℓ.*

Remark 1.2. In terms of above notation, $\ell \leftrightarrow v$.

Conjecture 1.3. *Let $\mathfrak{g}_\ell$ be the ℓ-adic Lie algebra of $\mathrm{Im}\, \rho_\ell$. Then there exists $\mathfrak{g}_{\mathbb{Q}}$ such that $\mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{Q}_\ell = \mathfrak{g}_\ell$ for all ℓ.*

Remark 1.4. This reflects a sense in which ρ is "independent of ℓ". One can describe this $\mathfrak{g}_{\mathbb{Q}}$ in terms of the Mumford-Tate group.

Conjecture 1.5. *If ρ_ℓ is irreducible at one ℓ, then it is at all ℓ.*

Larsen and Pink have written much on these conjectures.

Theorem 1.6 (Serre)**.** *If A is an abelian variety with $\dim A$ odd, and $\mathrm{End}_{\overline{k}}(A) = \mathbb{Z}$ then $\mathfrak{g}_\ell = \mathfrak{sp}_{2d} \oplus \mathbb{Q}_\ell \cdot 1d$.*

Theorem 1.7 (Zarhin '1985)**.** $\mathrm{End}_k(A) \otimes \mathbb{Z}/\ell\mathbb{Z} = \mathrm{End}_k(A[\ell])$.

This uses Faltings's results, but it requires some consideration of polarizations. In particular, this implies that if $\mathrm{End}_{\overline{k}}(A) = \mathbb{Z}$ then the Galois representation on $A[\ell]$ is irreducible for all but finitely many ℓ.

This irreducibility property is crucial for modularity lifting theorems.

2 Statement of Results

Theorem 2.1 (Snowden-W)**.** *Let F be a number field. Let Σ be a set of primes of (Dirichlet) density one, and for each $\ell \in \Sigma$ a (continuous) Galois representation*

$$\rho_\ell \colon G_F \to \mathrm{GL}_n(\mathbb{Q}_\ell).$$

Assume that they form a compatible system and that they have the property that they remain irreducible over any finite extension (=Lie irreducible). Then there is a subset $\Sigma' \subset \Sigma$ of density 1 such that $\overline{\rho}_\ell$ remain irreducible.

Remark 2.2. Taylor-Yoshida proved this kind of Lie irreducibility for representations which are automorphic, which are Steinberg at one place and coming from a unitary group. The point is that there are many conditions for which this hypothesis holds.

Remark 2.3. Patrikis showed that every irreducible ℓ-adic representation $\rho_\ell \colon G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ is of the form

$$\rho_\ell = \mathrm{Ind}_K^F(\rho'_\ell \otimes \Psi)$$

where Ψ is an Artin representation and ρ'_ℓ is Lie irreducible. This suggests that given a compatible system ρ_ℓ, one should try to show that ρ'_ℓ and Ψ also form a compatible system, and thus break the problem into the Artin case and the Lie-irreducible case. However, we don't know how to show this.

Assume also that

- If $w \notin S$ is a place of F with residue characteristic equal to $\ell \in \Sigma$, then ρ_ℓ is crystalline at w with Hodge-Tate weights independent of ℓ.

Theorem 2.4 (Patrikis, Snowden, W)**.** *Let $\{\rho_\ell\}_{\ell\in\Sigma}$ be a compatible system with Σ of density one. Assume $\rho'_\ell s$ are semisimple and let*

$$\rho_\ell \otimes \overline{\mathbb{Q}}_\ell = \bigoplus_{i=1}^{r} \rho_{\ell,i}^{m_{\ell,i}}$$

Then there exists a density one subset $\Sigma' \subset \Sigma$ such that $\overline{\rho}_\ell = \bigoplus \overline{\rho}_{\ell,i}^{m_{\ell,i}}$ with each $\overline{\rho}_{\ell,i}$ still distinct and irreducible.

In [BLGGT] there is a similar theorem, but which assumes that the Hodge-Tate weights are regular, but this excludes many geometric cases of interest.

Remark 2.5. We can strengthen the Snowden-Wiles theorem. Assuming Lie irreducibility, we can show that for any integer d there exists a subset Σ_d of density one such that for all $\ell \in \Sigma_d$, $\overline{\rho}_\ell|_{G_L}$ remains irreducible for all extensions L/F with $[L : F] \leq d$.

3 Proofs

This subject is a little counterintuitive; what ought to be hard is easy and what ought to be easy is hard.

Look at $G_\ell :=$ the Zariski closure of Im ρ_ℓ. Assume ρ_ℓ is semisimple; otherwise you need to divide by a radical. The connected component $G_\ell \supset G_\ell^0$ is of finite index, and we have a short exact sequence

$$1 \to G_\ell^{\mathrm{der}} \to G_\ell^0 \to G_\ell^{\mathrm{tor}} \to 1.$$

Theorem 3.1 (Serre)**.** *The index* $[G_\ell : G_\ell^0]$ *is independent of* ℓ*.*

It is hard to find the proof; I eventually tracked it down in a letter to Ribet in 1981. It is fairly elementary; the idea is that the index has to do with roots of unity, which you can detect in a compatible system.

Theorem 3.2 (Larsen)**.** *On a set of density one* Im $\rho_\ell \cap G_\ell^{\mathrm{der}}$ *is "large".*

By "large" we mean the analogue of Serre's theorem on elliptic curves. However, the methods cannot handle the toric part. This is quite hard, and uses group theory (of algebraic groups, p-adic groups, and also finite groups) extensively.