MODULI OF SHTUKAS I

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1. Hecke stacks

Let X be a (smooth, projective, geometrically connected) curve over $\mathbf{F}_q$. Let $F = \mathbf{F}_q(X)$. Fix integers $n \geq 1$ and $r \geq 0$. Let $\mu = (\mu_1, \ldots, \mu_r)$ with each $\mu_i = \pm 1$. Usually we require that r is even, and moreover that $\sum \mu_i = 0$.

In the previous talk we met the Hecke stack Hk_n^μ, parametrizing the stack of modifications of type μ of rank n vector bundles. If S is an $\mathbf{F}_q$-scheme, then $\mathrm{Hk}_n^\mu(S)$ is the groupoid of

- vector bundles $(\mathcal{E}_0, \ldots, \mathcal{E}_r)$ on $X \times S$.
- If $\mu_i = +1$, a map $\phi_i \colon \mathcal{E}_{i-1} \hookrightarrow \mathcal{E}_i$ with cokernel an invertible sheaf supported on Γ_{x_i}. If $\mu_i = -1$, a map $\phi_i \colon \mathcal{E}_i \hookrightarrow \mathcal{E}_{i-1}$ with cokernel an invertible sheaf supported on Γ_{x_i}.

Example 1.1. If $\mu_i = +1$, demanding a map

$$\mathcal{E}_{i-1} \hookrightarrow \mathcal{E}_i \hookrightarrow \mathcal{E}_{i-1} \otimes \mathcal{O}(\Gamma_{x_i}).$$

amounts to specifying a line in an n-dimensional vector space.

We have a map

$$\mathrm{Hk}_n^\mu \to \mathrm{Bun}_n \times X^r$$

sending

$$(\underline{\mathcal{E}}, \underline{x}, \underline{\phi}) \to (\mathcal{E}_0, \underline{x}).$$

This map is smooth of fiber dimension $r(n-1)$, so Hk_n^μ is smooth of dimension $n^2(g-1) + nr$.

2. Moduli of shtukas for GL_n

2.1. Definition.

Definition 2.1. A *shtuka* of type μ and rank n is a "Hecke modification" plus a Frobenius structure. More precisely, $\mathrm{Sht}_n^\mu(S) = \{(\underline{\mathcal{E}}, \underline{x}, \underline{\phi})\}$ together with an isomorphism $\mathcal{E}_r \cong {}^\tau\mathcal{E}_0 := (\mathrm{Id}_X \times \mathrm{Frob}_S)^* \mathcal{E}_0$.

We have a cartesian diagram

$$\begin{array}{ccc} \mathrm{Sht}_n^\mu & \longrightarrow & \mathrm{Hk}_n^\mu \\ \downarrow & & \downarrow {\scriptstyle p_0 \times p_r} \\ \mathrm{Bun}_n & \xrightarrow{\mathrm{Frob} \times \mathrm{Id}} & \mathrm{Bun}_n \times \mathrm{Bun}_n \end{array}$$

Example 2.2. For $n = 1$, the choice of points x_i determines the higher $\mathcal{E}_i$ from $\mathcal{E}_0$, namely $\mathcal{E}_i = \mathcal{E}_{i-1} \otimes \mathcal{O}(x_i)$. So $\mathrm{Hk}_1^\mu \cong \mathrm{Pic}_X \times X^r$.

For a point of H_1^μ to be an element of Sht_1^μ, we also need $\mathcal{E}_r \cong {}^\tau\mathcal{E}_0$, i.e.

$$ {}^\tau\mathcal{E}_0 \otimes \mathcal{E}_0^{-1} \cong \mathcal{O}(\sum \mu_i x_i). $$

Thus Sht_1^μ is a familiar object, classically known as a "Lang torsor". It is a fiber of the Lang isogeny $\mathrm{Pic}_X \to \mathrm{Pic}_X$, hence a torsor for $\mathrm{Bun}_1(\mathbf{F}_q)$.

Example 2.3. For $r = 0$, $\mathrm{Sht}_n^\mu(S)$ is a vector bundle $\mathcal{E}$ on $X \times S$ and an isomorphism $\mathcal{E} \cong {}^\tau\mathcal{E}$. This looks like part of a descent datum. If $S = \mathrm{Spec}\ \overline{\mathbf{F}}_q$, then such $\mathcal{E}$ come from $\mathcal{E}$ on X itself via pullback.

More generally, in this case

$$ \mathrm{Sht}_n^\mu = \coprod_{\mathcal{E}} [\mathrm{Spec}\ \mathbf{F}_q / \mathrm{Aut}\, \mathcal{E}]. $$

What exactly does this mean? Concretely, an element of Sht_n^μ is an $\mathrm{Aut}(\mathcal{E})$-torsor on S, which we can think of as a twisted form of $p_X^*(\mathcal{E})$ on $X \times S$.

2.2. Basic geometric facts about Sht_n^μ.

(1) Sht_n^μ is a Deligne-Mumford stack, smooth and locally of finite type.
(2) There is a morphism

$$ \mathrm{Sht}_n^\mu \to X^r $$

which is separated, smooth, and of relative dimension $r(n-1)$.

2.3. Level structure.

Definition 2.4. Let $D \subset X$ be a finite closed subscheme (in this case, just a finite collection of points with multiplicities). A *level D structure* on $(\underline{\mathcal{E}}, \underline{x}, \underline{\phi})$ is an isomorphism

$$ \mathcal{E}_0|_{D\times S} \xrightarrow{\sim} \mathcal{O}_{D\times S}^{\oplus n} $$

such that $|D| \cap \{x_1, \ldots, x_r\} = \emptyset$, which is compatible with Frobenius in the sense that the following diagram commutes:

$$ \begin{array}{ccc} \mathcal{E}_0|_{D\times S} & \xrightarrow{\sim} & \mathcal{O}_{D\times S}^{\oplus n} \\ \downarrow{\scriptstyle\sim} & & \| \\ {}^\tau\mathcal{E}_0|_{D\times S} & \xrightarrow{\sim} & {}^\tau\mathcal{O}_{D\times S}^{\oplus n} \end{array} $$

Note that there is an action of $\mathrm{GL}_n(\mathcal{O}_D)$ on the set of level structures.

In practice, we'll introduce level structure in order to rigidify the objects.

2.4. **Stability conditions.** The components of Bun_n are indexed by $\mathbf{Z}$, via

$$\mathcal{E} \mapsto \deg \det \mathcal{E}.$$

We need to fix this to get something of finite type. But that still won't be enough, since we have things like $\mathcal{O}(ap) \oplus \mathcal{O}(-ap)$. For a vector bundle $\mathcal{E}$, let

$$M(\mathcal{E}) := \max\{\deg \mathcal{L} \mid \mathcal{L} \hookrightarrow \mathcal{E}\}.$$

This is enough to cut down to something of finite type.

Definition 2.5. Define $\mathrm{Sht}^{\mu}_{n,D,d,m}$ to be the stack whose S-points are

- $(\underline{\mathcal{E}}, \underline{x}, \underline{\phi})$, $\mathcal{E}_r \xrightarrow{\sim} {}^{\tau}\mathcal{E}_0$
- A level D structure,
- $\deg(\det \mathcal{E}_i) = d$, $M(\mathcal{E}_0) \leq m$.

Facts:

(1) If $D \gg 0$ (with respect to n, m, d) then $\mathrm{Sht}^{\mu}_{n,D,d,m}$ is represented by a quasi-projective variety.
(2) The map $[\mathrm{Sht}^{\mu}_{n,D,d,m} / \mathrm{GL}_n(\mathcal{O}_D)] \hookrightarrow \mathrm{Sht}^{\mu}_n$ is an open embedding.
(3) Sht^{μ}_n is the union of these substacks for varying d, m.

This is enough to check that Sht^{μ}_n is a DM stack locally of finite type over $\mathbf{F}_q$.

2.5. **Smoothness.** Recall the cartesian square

$$\begin{array}{ccccc} \mathrm{Sht}^{\mu}_n & \longrightarrow & \mathrm{Hk}^{\mu}_n & \longrightarrow & X^r \\ \downarrow & & \downarrow{\scriptstyle p_0 \times p_r} & & \\ \mathrm{Bun}_n & \xrightarrow{\mathrm{Frob} \times \mathrm{Id}} & \mathrm{Bun}_n \times \mathrm{Bun}_n & & \end{array}$$

Note that $d\,\mathrm{Frob} = \mathrm{Frob}_* = 0$, and $\mathrm{Id}_* = \mathrm{Id}$. On the other hand, p_{0*} and p_{r*} are both surjections.

Corollary 2.6. *The maps* $(\mathrm{Frob}, \mathrm{Id})\colon \mathrm{Bun}_n \to \mathrm{Bun}_n \times \mathrm{Bun}_n$ *and* $(p_0, p_r)\colon \mathrm{Hk}^{\mu}_n \to \mathrm{Bun}_n \times \mathrm{Bun}_n$ *are transverse.*

Corollary 2.7. *The map* $\mathrm{Sht}^{\mu}_n \to X^r$ *is smooth, and so has relative dimension* $(n-1)r$.

2.6. **Summary.** Sht^r_G is a DM stack locally of finite type, with a smooth separated morphism $\mathrm{Sht}^r_G \to X^r$ of relative dimension r.

3. Moduli of Shtukas for PGL_2

Let $G = \mathrm{PGL}_2 = \mathrm{GL}_2 / \mathbf{G}_m$, and let Bun_G be the stack of G-torsors on X, which is isomorphic to $\mathrm{Bun}_2 / \mathrm{Bun}_1$, with the action being $\otimes$. This action lifts to Hk^{μ}_2, by

$$(\underline{\mathcal{E}}, \underline{x}, \underline{f}) \mapsto (\underline{\mathcal{E}} \otimes \mathcal{L}, \underline{x}, \underline{f} \otimes \mathrm{Id}).$$

This action doesn't restrict to Sht_2^μ unless $\mathcal{L} \cong {}^\tau\mathcal{L}$. Therefore, only the subgroup $\mathrm{Pic}_X(k)$ acts on Sht_2^μ. We have cartesian diagrams

$$\begin{array}{ccc} \mathrm{Pic}_X(\mathbf{F}_q) & \longrightarrow & \mathrm{Pic}_X \\ \downarrow & & \downarrow \\ \mathrm{Pic}_X & \xrightarrow{\mathrm{Frob}\times\mathrm{Id}} & \mathrm{Pic}_X \times \mathrm{Pic}_X\,. \end{array} \tag{3.1}$$

and

$$\begin{array}{ccc} \mathrm{Sht}_n^\mu & \longrightarrow & \mathrm{Hk}_n^\mu \\ \downarrow & & \downarrow{\scriptstyle p_0\times p_r} \\ \mathrm{Bun}_n & \xrightarrow{\mathrm{Frob}\times\mathrm{Id}} & \mathrm{Bun}_n \times \mathrm{Bun}_n \end{array} \tag{3.2}$$

and the objects for $G = \mathrm{PGL}_2$ are obtained by quotienting the second diagram (3.2) by the action of the corresponding groups in the first diagram (3.1).

3.1. **Independence of signs when $n = 2$.** If μ, μ' are r-tuples of signs and $n = 2$, then there is a canonical isomorphism $\mathrm{Sht}_G^\mu \xrightarrow{\sim} \mathrm{Sht}_G^{\mu'}$. We'll show this by giving an explicit isomorphism between Sht_G^μ, for any μ, and $\mathrm{Sht}^{\mu'}$ where $\mu' = (+1, \dots, +1)$.

Suppose we are given $(\underline{\mathcal{E}}, \underline{x}, \underline{\phi}, \iota) \in \mathrm{Sht}_G^\mu$. The key idea is that we can transform an injection $\mathcal{E}_{i-1} \hookleftarrow \mathcal{E}_i$ with deg 1 cokernel into $\mathcal{E}_{i-1} \hookrightarrow \mathcal{E}_i \otimes \mathcal{O}(x_i)$. So we take every instance of $\mathcal{E}_{i-1} \hookleftarrow \mathcal{E}_i$, which is a modification of type μ_-, into $\mathcal{E}_{i-1} \hookrightarrow \mathcal{E}_i \otimes \mathcal{O}(x_i)$, which is a modification of type μ_+. Given $(\underline{\mathcal{E}}, \underline{x}, \underline{\phi})$ let

$$D_i := \sum_{\substack{1\le j\le i \\ \mu_j = \mu_-}} \Gamma_{x_i}.$$

Let $\mathcal{E}_i' = \mathcal{E}_i \otimes \mathcal{O}_{X\times S}(D)$, and note that

$$\mathcal{E}_0' \hookrightarrow \mathcal{E}_1' \hookrightarrow \dots \hookrightarrow \mathcal{E}_r'$$

is an element of $\mathrm{Sht}_G^{\mu'}$.