The Fargues-Fontaine Curve

Notes by Tony Feng
for a talk by Pierre Colmez

April 4, 2016

1 Preliminaries on Fontaine's Rings

1.1 Construction of $C^\flat$

We start with some (pre)historical remarks. We denote by C a complete algebraically closed field of characteristic 0; we can imagine $C = \mathbb{C}_p$. We associate to C the set

$$C^\flat = \{(x^{(n)}) \mid (x^{(n+1)})^p = x^{(n)} \text{ for all } n\}.$$

We can define on this set multiplication and addition operations making it into a commutative ring:

$$(x^{(n)})(y^{(n)}) := (x^{(n)}y^{(n)})$$

and

$$(x^{(n)}) + (y^{(n)}) := \left(\lim_{k\to\infty}(x^{(n+k)} + y^{(n+k)})^{p^k}\right).$$

For $x \in C$, we get $x^\flat = (x, x^{1/p}, x^{1/p^2}, \ldots) \in C^\flat$ which is well-defined *up to* $\epsilon^{\mathbb{Z}_p}$ where $\epsilon = (1, \zeta_p, \zeta_{p^2}, \ldots)$. Then we denote

$$x^\sharp := x^{(0)}.$$

Theorem 1.1. *$C^\flat$ is an algebraically closed field of characteristic p, complete for the valuation*

$$v_{C^\flat}(x) = v_p(x^\sharp)$$

and we have $k_{C^\flat} = k_C$.

1.2 Construction of A_{inf}

Definition 1.2. Let $A_{\text{inf}} = W(\mathcal{O}_{C^\flat})$. An element $x \in A_{\text{inf}}$ can be (uniquely) represented as

$$x = \sum_{k\in\mathbb{N}}[x_k]p^k, \quad x_k \in \mathcal{O}_{C^\flat}.$$

We have a Frobenius endomorphism φ on A_{inf} by

$$\varphi\left(\sum[x_k]p^k\right) = \sum[x_k^p]p^k.$$

We also have a map

$$\theta\colon A_{\mathrm{inf}} \twoheadrightarrow \mathcal{O}_C$$

sending

$$\sum [x_k]p^k = \sum x_k^{\sharp} p^k.$$

Proposition 1.3. *θ is a surjective ring homomorphism with kernel generated by $p - [p^{\flat}]$. We have $\mathcal{O}_C = A_{\mathrm{inf}}/(p - [p^{\flat}])$.*

1.3 Construction of B_{dR} and B_{cris}

Definition 1.4. We define

$$B_{\mathrm{dR}}^{+} := \varprojlim_{k} A_{\mathrm{inf}}[1/p]/(p - [p^{\flat}])^k$$

and the subring

$$A_{\mathrm{cris}} := A_{\mathrm{inf}}\left[\frac{(p - [p^{\flat}])^k}{k!}, k \in \mathbb{N}\right]^{\wedge}.$$

The ring A_{cris} has an element $t := \log[\epsilon]$. It is easy to see that $\varphi(t) = pt$. Define $B_{\mathrm{cris}} = A_{\mathrm{cris}}[1/t]$, which has an action of φ. We have $B_{\mathrm{cris}} \subset B_{\mathrm{dR}} := B_{\mathrm{dR}}^{+}[1/t]$. Finally, we define $B_e = B_{\mathrm{cris}}^{\varphi=1}$.

These rings are related by the "fundamental exact sequence"

$$0 \to \mathbb{Q}_p \to B_e \to B_{\mathrm{dR}}/B_{\mathrm{dR}}^{+} \to 0.$$

Note that this implies

$$\operatorname{Gr} B_e = \mathbb{Q}_p + \frac{1}{t} C[1/t].$$

Surprisingly, B_e is a PID. This is the starting point for everything.

2 The Fargues-Fontaine curve

2.1 Informal description

The p-adic comparison theorems for crystalline/de Rham/étale cohomology lead one to consider the category of pairs $(W_e, W_{\mathrm{dR}}^{+})$ where W_e is a free B_e-module and W_{dR}^{+} is a free B_{dR}^{+}-module such that

$$B_{\mathrm{dR}} \otimes_{B_e} W_e = B_{\mathrm{dR}} \otimes_{B_{\mathrm{dR}}^{+}} W_{\mathrm{dR}}^{+}.$$

(In comparison theorems W_e is the crystalline cohomology, W_{dR}^{+} is the de Rham cohomology, and the étale cohomology can be recovered from this setup.)

Fargues and Fontaine were looking for a geometric object that would explain why this category has good properties. Roughly speaking, they constructed a curve X from Spec B_e

completed by adding a point at ∞ (corresponding to the valuation given by the grading; in general you should imagine that you can add a "point at ∞" whenever one has a filtered Dedekind domain).

So this curve X has the properties that $B_e = \mathcal{O}(X - \{\infty\})$, and $B_{dR}^+ = \mathcal{O}_{X,\infty}$. Then the fundamental exact sequence can be interpreted as follows. The fact that

$$B_e \twoheadrightarrow B_{dR}/B_{dR}^+$$

is a surjection is saying that we can find find a function which has any particular polar part at ∞. The short exact sequence tells us that the global sections are $\mathbb{Q}_p$, which makes us imagine that the curve is "proper". (Note however that the residue field at ∞ is C, which is weird since it's infinite-dimensional over $\mathbb{Q}_p$.)

In these terms the category of pairs (W_e, W_{dR}^+) corresponds to the category of vector bundles over $X = \operatorname{Spec} B_e \coprod$ (formal neighborhood around ∞) by the Beauville-Laszlo interpretation. The comparison isomorphism is what you need to glue two vector bundles.

What is the meaning of $B_e = (B_{\text{cris}})^{\varphi=1}$? It suggests that our X should be obtained by taking the quotient of some bigger space by φ. Indeed, we have

$$X^{\text{ad}} = Y^{\text{ad}}/\varphi^{\mathbb{Z}}$$

where $Y^{\text{ad}} = \operatorname{Spa}(A_{\text{inf}}) - (p[p^\flat])$.

Remark 2.1. One might wonder why we don't build Y using $\operatorname{Spa}(B_{\text{cris}})$, in light of $B_e = (B_{\text{cris}})^{\varphi=1}$. This is bad because φ is not an automorphism of B_{cris}; we should only quotient by automorphisms. If we were to replace B_{cris} by the largest subring on which φ is an isomorphism, then one does indeed arrive at the same Y.

2.2 First construction

Definition 2.2. Let $[E : \mathbb{Q}_p] < \infty$. Define $A_{\text{inf},E} = \mathcal{O}_E \otimes_{W(k_E)} A_{\text{inf}}$ where ϖ is a uniformizer of E. For $x \in A_{\text{inf},E}$ we can write

$$x = \sum [x_k]\varpi^k$$

which admits an action of $\varphi_E = 1 \otimes \varphi^f$ where $q = |k_E| = p^f$. Then

$$\varphi_E(\sum [x_k]\varpi^k) = \sum [x_k^q]\varpi^k.$$

The expression suggests that $A_{\text{inf},E}$ is similar to $\mathcal{O}_C[[T]]$.

This suggests defining the Newton polygon

$$\text{NP}_x := \text{convex hull } \{(k, v_{C^\flat}(x_k)\}.$$

Remark 2.3. The theory of Newton polygons is a little subtler than usual because there are infinitely many coefficients, but it is a theorem that things work out.

Theorem 2.4. *If $\lambda < 0$ is a slope of NP_x with multiplicity d then there exist $a_1, \ldots, a_d \in \mathcal{O}_{C^\flat}$ such that $\frac{v_{C^\flat}(a_i)}{v_p(\varpi)} = -\lambda$ for each i and*

$$(\varpi - [a_1]) \ldots (\varpi - [a_k]) \mid x.$$

Remark 2.5. This is the result we would have expected if we were working instead with $\mathbb{C}[[t]]$. In the case $\mathcal{O}_C[[T]]$ or $\mathcal{O}_{C^\flat}[[T]]$ the a_i would be unique, but they are *not* unique here.

Corollary 2.6. *The closed prime ideals of $A_{\mathrm{inf},E}$ are*

- (0)*, with residue field* $\mathrm{Frac}(A_{\mathrm{inf},E})$*,*
- *maximal ideals, with residue fields $k_{C^\flat} = k_C$.*
- (ϖ)*, with residue field $C^\flat$.*
- $(\varpi - [a])$*, up to some equivalence relation, with residue field K_a which is algebraically closed and complete for v_p, and has $K_a^\flat \cong C^\flat$.*
- *$W(\mathfrak{m}_{C^\flat})$"$= [\varpi^\flat]$" with residue field is $\widetilde{E} := E \otimes W(k_{C^\flat})$.*

Now we define the curve $Y_{E,C^\flat} =: Y_E := \mathrm{Spec}'\, A_{\mathrm{inf},E} - \{\varpi[\varpi^\flat] = 0\}$, where Spec' means that we take only the *closed* prime ideals.

Proposition 2.7. *The points of Y_E correspond to E-untilts of $C^\flat$.*

Proof. For $y \in Y_E$ of the form $y = (\varpi - [a])$ the residue field K_y is an "E-untilt" of $C^\flat$. An E-untilt is a pair $(K \supset E, \iota\colon K^\flat \cong C^\flat)$. Given an E-untilt, we produce a point $(\varpi - [\iota(\varpi^\flat)]) \in Y_E$. This shows: □

2.3 Lubin-Tate theory

The description of the points of Y_E above has some problems; for instance there is no easy description for when $(\varpi - [a]) = (\varpi - [b])$. We can get a better parametrization of Y_E via Lubin-Tate theory. Associated to (E, ϖ), for $\alpha \in \mathcal{O}_E$ we have $\sigma_\alpha \in \alpha T + T^2\mathcal{O}_E[[T]]$ such that

$$\sigma_\alpha(X \oplus Y) = \sigma_\alpha(X) + \sigma_\alpha(Y)$$

and $\sigma_\varpi \equiv T^q \mod \varpi$. Then we can define

$$\sigma_{\alpha/\varpi^n}(T) = \sigma_\alpha(T^{q^{-n}}).$$

This gives an action of E on $\mathfrak{m}_{C^\flat}$, with α acting by the reduction of σ_α modulo p.

Theorem 2.8. *If $x \in \mathfrak{m}_{C^\flat}$, then*

1. $[x]_\varpi := \lim_{n\to+\infty}\sigma_{\varpi^m}([x^{q^{-n}}])$ *is the unique lift of* x *in* $A_{\mathrm{inf},E}$ *such that*

$$\varphi_E([x]_\varpi) = \sigma_\varpi([x]_\varpi).$$

2. We have $\sigma_\alpha([x]_\varpi) = [\sigma_a(x)]_\varpi$

3. The map $x \mapsto \xi_x := \frac{[x]_\varpi}{[x^{1/q}]_\varpi}$ *gives a bijection between*

$$(\mathfrak{m}_{C^\flat} - \{0\})/\mathcal{O}_E^* \cong Y_E.$$

3 The analytic curve

3.1 Construction

As we have just seen, Y_E is "the punctured open ball over $C^\flat$ modulo $\mathcal{O}_E^*$". So we would like to say:

$$Y_E = \widetilde{D}^*_{C^\flat}/\mathcal{O}_E^* = \widetilde{D}^*_C/\mathcal{O}_E^*$$

where D is the open unit ball. To make sense of this we need diamonds; indeed, giving rigorous meaning to this expression was one of the motivations for Scholze's theory of diamonds.

The *adic Fargues-Fontaine curve* Y_E^{ad} is defined to be

$$Y_E^{\mathrm{ad}} := \mathrm{Spa}(A_{\mathrm{inf},E}) - \{\varpi[\varpi^\flat] = 0\}.$$

We will eventually define

$$X_E^{\mathrm{ad}} := Y_E^{\mathrm{ad}}/\varphi^{\mathbb{Z}}$$

after we show that this makes sense.

Remark 3.1. Proving that these really are adic spaces, i.e. the structure sheaves are sheafy, is quite nontrivial.

3.2 Some properties

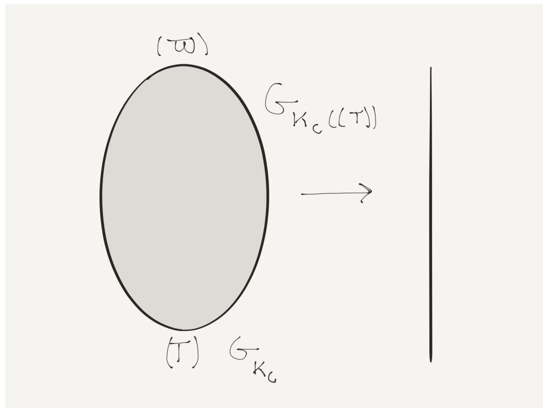
For any $u \in \mathfrak{m}_{C^\flat} - \{0\}$, we get

$$\mathcal{O}_{\widetilde{E}}[[T^{q^{-\infty}}]] \hookrightarrow A_{\mathrm{inf},E}$$

by sending $T \mapsto [u]$.

Theorem 3.2. $A_{\mathrm{inf},E}$ *is the* (ϖ, T)*-completion of the maximal extension of* $\mathcal{O}_{\widetilde{E}}[[T^{q^{-\infty}}]]$ *unramified outside* $T = 0$.

We have $\mathcal{O}_{\widetilde{E}}[[T]] \to \mathcal{O}_{\widetilde{E}}[[T^{q^{-\infty}}]] \to A_{\mathrm{inf},E}$. Therefore, we can view $\mathrm{Spa}(A_{\mathrm{inf},E})$ as a cover of $\mathrm{Spa}\,\mathcal{O}_{\widetilde{E}}[[T]]$, which we can think of as a "unit disk".



That is, $\mathrm{Spa}(A_{\mathrm{inf},E})$ is a profinite covering ramified only over the point (T), with fiber $G_{k_C((T))}$ over all points except (T), where it has fiber G_{k_C}.

There is a map $\delta\colon\ \mathrm{Spa}\,\mathcal{O}_E[[T]] \to [0,\infty]$ defined by

$$\delta(\overline{x}) := \frac{v_{\overline{x}}(\varpi)}{v_{\overline{x}}([\varpi^\flat])}.$$

(In terms of absolute values, this would be the "radius function" on the unit disk.) Composing this with the map from $\mathrm{Spa}\,A_{\mathrm{inf},E}$, we obtain a map

$$\mathrm{Spa}\,A_{\mathrm{inf},E} \to [0,+\infty]$$

which sends

$$v_x \mapsto v_{\overline{x}} \mapsto \delta(\overline{x})$$

and $Y_E^{\mathrm{ad}} = \delta^{-1}((0,\infty)) \subset \mathrm{Spa}\,A_{\mathrm{inf},E}$.

Definition 3.3. We define $Y_I := \delta^{-1}(I)$, with $\mathcal{O}(Y_I)$ being $A_{\mathrm{inf},E}[1/\varpi, 1/[\varpi^\flat]]$ completed with respect to the family of valuations v_r for $r \in I$, where

$$v_r\Big(\sum [x_k]\varpi^k\Big) = \begin{cases} \inf(v_{C^\flat}(x_k) + krv_p(\varpi)) & r \geq 1, \\ \inf(\frac{1}{r}v_{C^\flat}(x_k) + kv_p(\varpi)) & r \leq 1. \end{cases}$$

Proposition 3.4. *Y_I enjoys the following properties:*

1. *$\mathcal{O}(Y_I)$ is a Fréchet algebra, and is Banach if I is compact.*

2. *If $\min I \neq 0$ then $\mathcal{O}(Y_I)$ is Bézout, and is even a PID if I is compact.*

We have

$$\delta(\varphi(x)) = \frac{1}{q}\delta(x),$$

so φ acts properly on Y_E. This implies that $X_E^{\mathrm{ad}} := Y_E^{\mathrm{ad}}/\varphi^{\mathbb{Z}}$ is compact, since $Y_{[1,q]}$ covers it.

Theorem 3.5. $X_E^{\mathrm{ad}} = (X_E)^{\mathrm{ad}}$ *for some scheme* X_E.

If $[E' : E] = +\infty$ then

$$X_{E'} = E' \otimes_E X_E.$$

The content here is that

$$\mathcal{M}(Y_{E'}^{\mathrm{ad}})^{\varphi_{E'}=1} = E' \otimes_E \mathcal{M}(Y_E^{\mathrm{ad}})^{\varphi_E=1}.$$

where $\mathcal{M}$ denotes meromorphic functions.

Theorem 3.6. *All finite étale coverings of* X_E *are of this shape, so* $\pi_1(X_E) = \mathrm{Gal}(\overline{E}/E)$.