On the theory of higher rank Euler and Kolyvagin systems

Notes by Tony Feng
for a talk by David Burns

June 15, 2016

1 Overview

Let k be a number field and $p \neq 2$ a prime. Fix a set of places S, which contains all places dividing p or ∞.

Let T be a finitely generated free $\mathbb{Z}_p$-module, with a continuous $\mathbb{Z}_p$-linear action of $G_k := \mathrm{Gal}(k^s/k)$, unramified outside S. Let M be a fixed power of p; we consider the modular representation $A := T/M$.

Let K/k be a large pro-p abelian extension; we'll be interested in the collection of subfield extensions

$$\Omega := \{F : k \subset F \subset K\}$$

For $F \in \Omega$, we set $G_F = \mathrm{Gal}(F/k)$ and $\mathcal{O}_F := \mathcal{O}_{F, S \cup S_{\mathrm{ram}}(F/k)}$. (Apologies for this notation.)

Theorem 1.1 (Mazur, Rubin)**.** *Under the "standard hypotheses on T and K" there is a canonical homomorphism from the module of Euler Systems to the module of Kolyvagin Systems, given by the "Kolyvagin derivative"*

$$ES(T) \xrightarrow{D_A} KS(A).$$

An *Euler system* is a collection of cohomology classes $c = \{(c_f)_{f \in \Omega} : c \in H^1(\mathcal{O}_F, T)\}$ which are "compatible" in the sense that their restrictions agree up to Euler factors.

A *Kolyvagin system* consists of $\{(\kappa_n)_{n \in \mathcal{N}} : \kappa_n \in H^1_n(k, A)\}$ where $\mathcal{N}$ is something like squarefree ideal classes. As n varies, the cohomology classes are linked by homomorphisms which are "finite/singular comparison maps".

The importance of the theorem is that the left hand side has links to L-values, while the right hand side is a machine that controls Selmer groups.

Let

$$Y_T := \bigoplus_{v \in S_\infty(k)} H^0(k_v, T^*(1))$$

and $r = r_T := \mathrm{rank}_{\mathbb{Z}_p}(Y_T)$ be the "rank" of the Euler system. The situation of interest is $r > 1$.

2 Higher rank Euler systems

Perrin-Riou defined a notion of higher rank Euler systems $ES_r(T)$ parametrizing classes $c_F \in \bigwedge^r_{\mathbb{Z}_p[G_F]} H^1(O_F, T)$. Rubin had the insight that it was "too optimistic" for higher rank Euler systems not to have denominators (he considered $T = \mathbb{Z}_p(1)$); he instead suggested how to bound the denominators that should appear. This answer is given in terms of a "Rubin lattice" which will appear later.

On the other side, $KS_r(A)$ parametrizes classes $\kappa_n \in \bigwedge^r_{\mathbb{Z}/M} H^1_n(k, A)$.

Another type of system was discovered: the *Stark Systems* (so called because examples included systems arising from Stark units) $\kappa'_n \in \bigwedge^{r+\nu(n)}_{\mathbb{Z}/M} H^1_n(k, A)$ where $\nu(n)$ is the number of prime divisors of n.

There is a map

$$\pi_A \colon SS_r(A) \to KS_r(A).$$

It turns out that the module of Stark systems is easier to study. They also control Selmer groups. For Kolyvagin systems, the relation between the different classes is more complicated. (The map π_A is not surjective for $r > 1$.)

However, the Kolyvagin systems have some advantages. The relation between classes is functorial. Also, it is harder to see how to get Stark systems from Euler systems (for instance, it is mystifying that the exponent of the exterior power varies so much).

It is expected that there is a generalization of the Kolyvagin derivative

$$D_r \colon ES_r(T) \dashrightarrow KS_r(A)$$

Problems.

1. What is D_r?
2. Mazur-Rubin-Perrin-Riou have results for the case $\mathbb{Z}_p(1)$, but can one do it for more general coefficients for T, A?

I will talk about joint work with Takamichi Sano. The punchline is that we can do 2 for satisfactorially general coefficients. However, 1 is still only partially answered.

3 Another overview

Since this work is so technical, we give an overview first.

Let $r = r_T = \operatorname{rank}_{\mathbb{Z}_p}(Y_T)$.

Remark 3.1. In terms of the notation of Mazur-Rubin-Perrin-Riou's paper, this coincides with their "core rank of $(T, \mathcal{T}_{\mathrm{can}})$" under mild hypothesis.

Let

$$\mathcal{E}(T) = ES_r(T)$$
$$\mathcal{K}(T) = KS_r(T)$$
$$\mathcal{S}(T) = SS_r(T)$$

For $F \in \Omega$, let $R_F = \mathbb{Z}_p[G_F]$ and $A_F = A \otimes_{\mathbb{Z}_p} R_F$, which has an action of $R_F/M = \mathbb{Z}/M[G_F]$.

I will try to flesh out the following notion: for each $F \in \Omega$, one can define

- $\mathcal{K}(A_F), \mathcal{S}(A_F)$ with actions of $\overline{R_F}$
- a canonical homomorphism over $\mathbb{Z}_p[[G_k]]$

$$\mathcal{E}(T) \xrightarrow{D_r}$$

  but we don't a priori know that it lands in the right place. Kolyvagin systems parametrize classes which are "closely related" as n varies. Instead, the target here might be called "Kolyvagin collections" since we don't know that the classes are "closely related" as desired.

$$\mathcal{E}(T) \xrightarrow{D_r} KC_r(A_F) \supset \mathcal{K}(A_F).$$

Definition 3.2. Let $k \subset L \subset K$. We define the "Kolyvagin-derivable at level L systems"

$$\mathcal{E}_L^{\mathrm{der}}(T) := \{c \in \mathcal{E}(T) \colon D_{r,F}(c) \in \mathcal{K}(A_F) \forall F \in \Omega, F \supset L\}$$

In this context, the theorem of Mazur and Rubin can be stated as follows.

Example 3.3. For $r = 1$, $\mathcal{E}_L^{\mathrm{der}}(T) = \mathcal{E}(T)$.

I expect that it will be very difficult to find an example of a non-derivable Euler system, but I don't have evidence to conjecture that every one will be derivable.

So what *can* we say about the submodule of Kolyvagin derivable Euler systems (henceforth denoted $\mathcal{E}^*(T)$)?

First, it should be big.

Theorem 3.4. *Assume the "Standard hypotheses on T, K" and for all $F \in \Omega$, $H^1(\mathcal{O}_F, T)$ is $\mathbb{Z}_p$-free and $H^0(F, T) = 0$. (The first condition might seem strong, but we can reduce to this case via standard tricks of increasing the set.)*

Then there exists a canonical $\mathbb{Z}_p[[G_k]]$-submodule of $\mathcal{E}_F^{\mathrm{der}}(T)$ with the following properties.

1. *("Strongly derivable") There is a commutative diagram*

$$\begin{array}{ccc} \mathcal{E}_F^*(T) & \xrightarrow{D_F} & \mathcal{K}(A_F) \\ & \searrow^{\rho_F} \quad \nearrow_{\pi_F} & \\ & \mathcal{S}(A_F) & \end{array}$$

2. Im ρ_f *determines every Fitting ideal*

$$Fit^i_{\overline{R_F}}(Ш^2_S(F,A)).$$

3. *If k_∞/k is a $\mathbb{Z}_p$-tower, $k_\infty \subset K$, then $\mathcal{E}^*_F(T)$ determines whether or not $H^2(\mathcal{O}_{k_\infty}, T)$ is torsion over* $\mathbb{Z}_p[[\mathrm{Gal}(k_\infty/k)]]$.

4 Some details about the diagram

In the rest of the talk I'm going to talk about the diagram

$$\begin{array}{ccc} \mathcal{E}^*_F(T) & \xrightarrow{D_F} & \mathcal{K}(A_F) \\ & {\scriptstyle\rho_F}\searrow \quad \nearrow{\scriptstyle\pi_F} & \\ & \mathcal{S}(A_F) & \end{array}$$

Remark 4.1. If $r = 1$, $k/\mathbb{Q}$ is abelian then by a theorem of Kato, under mild conditions on T, $\mathcal{E}(T)/\mathcal{E}^*_k(T)$ is small and simple.

For $r > 1$ and $T = \mathbb{Z}_p(1)$, $k/\mathbb{Q}$ real and abelian of degree r, we get a cyclotomic Euler system. We can prove that it belongs to $\mathcal{E}^*_k(T)$.

4.1 Exterior algebra

Let R be a commutative ring and X a finitely generated R-module. For $d \geq 0$, we define

$$\bigcap_R^d X := \left(\bigwedge_R^d X^*\right)^*$$

where $(-)^* = \mathrm{Hom}_R(-, R)$. We have a map

$$\bigwedge_R^d X \xrightarrow{\theta^d_X} \bigcap_R^d X$$

which is an isomorphism if X happens to be projective, but neither injective nor surjective in general.

Example 4.2. Let $R = \mathbb{Z}_p[G]$ for G a finite abelian group. Then

$$\bigcap_R^d X \overset{\theta^d_X}{\cong} \{a \in \mathbb{Q}_p \cdot \bigwedge_R^d X \mid \Phi(a) \in R \text{ for all } \Phi \in \bigwedge_R^d X^*\}.$$

This is the "Rubin lattice", which bounds the denominators of higher rank Euler systems.

In this setting one can define

$$\mathcal{E}(T) = \{(c_F)_{F\in\Omega} \colon c_F \in \bigcap_{R_F}^{d} H^1(\mathcal{O}_F, T)\}$$

and

$$\mathcal{E}(T) = \{(\kappa_n)_n \colon \kappa_n \in \bigcap_{\overline{R_F}}^{d} H^1_n(k, A_F)\}.$$

The point is that the bidual behaves much better than the exterior power in terms of constructing homomorphisms.

4.2 Construction of $\mathcal{E}^*_F(T)$

Let $F \in \Omega$. Consider

$$\det{}^{-1}_{R_F} R\Gamma_c(\mathcal{O}_F, T^*(1)).$$

Under Artin-Verdier duality, we have

$$\det{}^{-1}_{R_F} R\Gamma_c(\mathcal{O}_F, T^*(1)) \cong \det{}^{-1}_{R_F} R\Gamma(\mathcal{O}_F, T) \otimes_{R_F} \det{}^{-1}_{R_F}(Y_T \otimes R_F).$$

Picking a basis of Y_T, we can contract the last factor to get a map to

$$\to \det R_F{}^{-1} R\Gamma(\mathcal{O}_F, T).$$

Under our hypotheses, this has cohomology only in degrees 1 and 2. Now you can project to the component of $\mathbb{Q}_p \cdot R_F$ that kills $H^2(\mathcal{O}_F, T)$. Then by Artin-Verdier duality the thing has to be free of rank 1. So you unwind to

$$\to \mathbb{Q}_p \cdot \bigwedge_{R_F}^{r} H^1(\mathcal{O}_F, T).$$

Call the composite map Θ_F:

$$\Xi_F := \det{}^{-1}_{R_F} R\Gamma_c(\mathcal{O}_F, T^*(1)) \xrightarrow{\Theta_F} \mathbb{Q}_p \cdot \bigwedge_{R_F}^{1} H^1(\mathcal{O}_F, T)$$

Theorem 4.3. *We have*

1. Im $\Theta_F \subset \bigcap^r_{R_F} H^1(\mathcal{O}_F, T)$,
2. *there exists a surjective transition morphism* $\Xi_F \to \Xi_{F'}$ *for* $k \subset F' \subset F$ *such that*

$$(\delta_F)_F \to (\Theta_F(\delta_F))_F$$

   is a homomorphism.

This induces

$$\Theta_K \colon \varprojlim_{F\in\Omega} \Xi_F \to \Sigma(T)$$

4.3 Kolyvagin derivative

Let P be the set of prime powers q such that

- q splits completely in $k(\mu_M, (\mathcal{O}_k^*)^{1/M})$ and
- $A/(\mathrm{Frob}_q - 1)A \cong \mathbb{Z}/M$.

For $q \in P$, let $k(q)$ be the maximal p-power extension of k in the ray class field mod q. If $p \nmid |\mathrm{Cl}(\mathcal{O}_K)|$ then $\mathrm{Gal}(k(q)/k) = \langle \sigma_q \rangle$.

Let $\mathcal{N}$ be the set of squarefree products of P. For $n \in \mathcal{N}$, let

$$k(n) = \prod_{q|n} k(q).$$

We define for $n \in \mathcal{N}$

$$D_n = \prod_{q|n} \left(\sum_{i=1}^{\#G_q - 1} i\sigma_q^i \right) \in \mathbb{Z}_p[\mathrm{Gal}(k(n)/k)].$$

Let $\mathcal{O}_n = \mathcal{O}_{k, S \cup S_{\mathrm{ram}}/k) \cup \{q|n\}}$

Theorem 4.4. *For $c \in \Sigma(T)$,*

1. *For $n \in \mathcal{N}$, $D_n(c_{k(n)}) \pmod{M}$ gives $\kappa_n'(c) \in \bigwedge_{R_F}^r Inj_{\overline{R_F}}(H^1(\mathcal{O}_n, A_F))$. (Here* Inj *denotes an injective envelope.)*
2. *There exists $\Psi_F \in \mathrm{End}_{\overline{R_F}}\left(\prod_{n \in N} Inj_{\overline{R_F}}(H^1(\mathcal{O}_n, A_F))\right)$ with*
   - *(a) For $r = 1, F = k$*

     $$\Psi_k|_{\prod_{n \in \mathcal{N}} H^1(\mathcal{O}_n, A)}$$

     is the homomorphism used by Mazur-Rubin used to define the Kolyvagin derivative.
   - *(b) For $c \in \mathcal{E}_F^*(T)$,*

     $$\Psi((\kappa_n'(c)) \in \mathcal{K}(A_F).$$

Thus:

Definition 4.5. We define $KC(A_F) = \prod_{n \in \mathcal{N}} \bigwedge_{R_F}^r \mathrm{Inj}_{\overline{R_F}}(H^1(\mathcal{O}_n, A_F))$ and $D_L = \Psi \circ (\kappa_n'(-))_n$.

Let me just say that this approach gives an interpretation of the finite/singular comparison map of Mazur-Rubin, which is a key point, in terms of Bockstein homomorphisms.