Beilinson-Kato and Beilinson-Flach elements in p-adic families

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This was a slide talk, so I'm just jotting down some impressions.

1 Rubin's formula

Let $K/\mathbb{Q}$ be an imaginary quadratic extension and A an elliptic curve over $\mathbb{Q}$ with CM by $\mathcal{O}_K$. Katz defined a two-variable l-function $L_p(\nu)$ interpolating special values $L_A(\nu^{-1}, 0)$.

One considers ν which is in the realm of classical interpolation, but $\nu^* = \nu \circ c$ (where c is complex conjugation) is not. Rubin's theorem says that if $L(A, s)$ vanishes to order one at $s = 1$, then there exists $P \in A(\mathbb{Q})$ such that

$$L_p(\nu_A^*) = \Omega_p(A)^{-1} \log_{\omega_A}(P)^2 \mod K^*.$$

Here is the idea of the proof.

1. The starting point is a global class $\kappa_A \in S_p(A/\mathbb{Q})$ coming from elliptic units. On one hand, $\log \kappa_A \sim \Omega_p(A) L_p(\nu_A^*)$. On the other hand, the self-intersection of κ_A is $\sim L_p'(\nu_A) L_p(\nu_A^*)$, and the first value is in the realm of classical interpolation.

2. Use Perrin-Riou's p-adic analogue of Gross-Zagier:

$$\langle \kappa_P, \kappa_P \rangle = L_p'(\nu_A) \Omega_p(A)^{-1}$$

   where P comes from a Heegner point.

3. Since the Selmer group has rank 1, the generators are proportional. Combine the formulas.

The moral is that the proof is a comparison of two different Euler systems, coming from elliptic units (from which Katz's p-adic L-function arises) and Heegner points (used to construct points on elliptic curves).

B-Darmon-Prasanna use a p-adic Gross-Zagier formula and only use the Euler system of Heegner points.

2 Perrin-Riou's conjecture

Let A be a modular elliptic curve over $\mathbb{Q}$ and f the associated modular form. Let p be a prime of good reduction and α, β the roots of the Hecke polynomial at p:

$$x^2 - a_p(f)x + p$$

A construction of Mazur-Swinnerton-Dyer yields a cyclotomic p-adic L-function

$$L_{p,\alpha}(A, \chi_1) \in \widetilde{\Lambda}$$

where $\widetilde{\Lambda}$ is some extension of the usual Iwasawa algebra. If α and β are admissible (valuation < 1) then the construction gives a p-adic L-function.

Perrin-Riou also attempted to define a p-adic L-function in the critical case $\mathrm{ord}_p(\beta) = 1$ using an Euler system and p-adic interpolation of the Bloch-Kato logarithm. More precisely, she defined a "big logarithm"

$$\mathcal{L}\colon H^1(\mathbb{Q}_p, V_p(A) \otimes \widetilde{\Lambda}(\chi_1)) \to H^1_{\mathrm{dR}}(A/\mathbb{Q}_p) \otimes \widetilde{\Lambda}.$$

Given an auxiliary Dirichlet character χ_2, Kato constructed an Euler system $\kappa_{BK}(\chi_1, \chi_2) \in H^1(\mathbb{Q}, V_p(A) \otimes \widetilde{\Lambda}(\chi_1))$ arising from Beilinson-Kato elements in the second K-group of a modular curve, defined in terms of modular units associated to (χ_1, χ_2).

Perrin-Riou's "de Rham-valued" p-adic L-function is defined as the image of Kato's Euler system under the big logarithm map (so it's valued in de Rham cohomology).

Kato's reciprocity law saws that

$$\exp^*(\kappa_{BK}(\chi_1, \chi_2)) \sim \frac{L(A, \chi_1, 1)}{\Omega_{\chi_1}(A)} \cdot \frac{L(A, \chi_2, 1)}{\Omega_{\chi_2}(A)}$$

where $\exp^*$ is the dual exponential map.

By projecting onto α and β eigenspaces for the action of Frobenius, one can recover the Mazur-Swinnerton-Dyer conjecture.

Conjecture 2.1 (Perrin-Riou). *There exists a global point $P_{\chi_1} \in A(\mathbb{Q}_{\chi_1})^{\chi_1}$ such that*

$$\langle \log \kappa_{BK}(\chi_1, \chi_2), \omega_A \rangle \sim_{\chi_2} \log P_{\chi_1} \cdot \log P_{\overline{\chi_1}}.$$

Theorem 2.2 (B-Darmon). *Under some technical hypotheses, and if χ_1, χ_2 are quadratic characters, the conjecture is true.*

The non-quadratic setting would involve a conjectural construction of global points over cyclotomic fields; of course we don't know how to do this.

3 Strategy of proof

Let K be the quadratic field associated to $\chi_1\chi_2$. Let ψ be the Hecke character attached to (χ_1, χ_2).

There is a *Beilinson-Flach class* $\kappa_{BF}(\psi) \in H^1(\mathbb{Q}, V_p(A) \otimes V_p(\theta_\psi))$ defined by B-Darmon-Rotger as a p-adic limit of Beilinson-Flach elements in K_1 of a product of modular curves. (The B-F elements are something like a collection of curves and rational functions on them, in a surface.)

One takes for P_ψ a Heegner point.

The strategy, again, is to compare two Euler systems.

1. Use a Λ-adic "Siegel-Weil formula" to compare the Beilinson-Kato and Beilinson-Flach classes (exact equation omitted).

2. Compare the Beilinson-Flach class (closer to Heegner points) and the Heegner point P_ψ. Use the BDP "p-adic Gross-Zagier" formula plus a factorization of p-adic L-function due to Darmon-Lauder-Rotger.

3. Assuming $L(A, \chi_1, 1) = 0$ and $L(A, \chi_2, 1)$ one can show that $\kappa_{BK}(\chi_2, \chi_1) = 0$. Then the result follows from the previous steps.

In the rest of the talk I will elaborate on Step 1.

4 Comparison of BK and BF classes

4.1 Hida-Garrett-Rankin p-adic L-functions

Let $\underline{g}$ and $\underline{h}$ be two Hida families (through Eisenstein series), so their coefficients in $\Lambda = \mathbb{Z}_p[[\Gamma]]$.

Let $\nu_{k,\xi} \colon \Lambda \to \mathbb{C}_p$ be a character. We'll use it to specialize the Hida families, thus obtaining p-adic modular forms.

If g is an overconvergent modular form, one can apply some operators

- p-depletion, basically deleting Fourier coefficients with index divisible by p,

- Serre's derivative operator $d = q\frac{d}{dq}$ (which arises conceptually from the Gauss-Manin connection, in terms of the geometric definition).

By some gymnastics with these operators, produce a p-adic modular form that can then be projected to de Rham cohomology $H^1_{\mathrm{dR}}(X_0(N)/\mathbb{Q}_p)$, following Coleman (the de Rham cohomology is related to rigid cohomology, whose rigid analytic sections are overconvergent modular forms).

The punchline is that specializing the Hide-Garrett-Ranking p-adic L-function lands in $\mathrm{Fil}^1 H^1_{\mathrm{dR}}$, up to applying some explicit polynomial in Frobenius.

Remark 4.1. This is a "de Rham counterpart" to Pollack-Stevens' and Bellaiche's attack on overconvergent modular forms in the critical case via overconvergent modular symbols.

There are two formulas of importance. One relates a p-adic L-function (arguments involved two Hida families) to Beilinson-Kato classes (due to Kato-Perrin-Riou). The second relates another p-adic L-function (arguments involved Hecke characters) to Beilinson-Flach classes (due to BDP).

Remark 4.2. The BF elements were generalized and exploited by Kings, Lei, Loeffler, and Zerbes in their work on the Iwasawa Main Conjecture.

What goes into the proof?

1. You use a local reciprocity law of Perrin-Riou to relate some derivatives of p-adic L-functions to the logarithms of BK classes (two equations with the order of characters reversed).

2. There is also a Λ-adic Siegel-Weil formula.

3. $\kappa_{BK}(\chi_2, \chi_1) = 0$.