THE STACKS Bun_n AND HECKE

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1. Why stacks?

In algebraic geometry one would like to have a classifying space BGL_n for vector bundles, such that

$$\mathrm{Hom}(S, \mathrm{BGL}_n) = \{\text{vector bundles of rank } n \text{ on } S\} / \sim .$$

Such an object can't be represented by a scheme, since a vector bundle is locally trivial, so any map $S \to B\,\mathrm{GL}_n$ would need to be locally constant, and for maps of schemes locally constant implies constant.

There are several possible ways to wriggle out of this situation.

(1) Add extra data (e.g. level structure) in order to eliminate automorphisms.
(2) Don't pass to isomorphism classes.

Stacks are the result of the second option.

2. Bun_n as a stack

Let k be a field.

Definition 2.1. A *stack* $\mathcal{M}$ is a sheaf of groupoids

$$\mathcal{M}\colon \mathrm{Sch}_k^{\mathrm{op}} \to \mathbf{Grp} \subset \mathbf{Cat}$$

i.e. an assignment

- for all S a groupoid $\mathcal{M}(S)$,
- for every $S \xrightarrow{f} S'$ a pullback functor $f^*\colon \mathcal{M}(S') \to \mathcal{M}(S)$,
- for all $S \xrightarrow{f} S' \xrightarrow{g} S''$ a transformation

  $$\varphi_{f,g}\colon f^* \circ g^* \implies (g \circ f)^*$$

  such that objects and morphisms glue (in the appropriate topology).

Example 2.2. The classifying stack

$$\mathrm{BGL}_n := [\mathrm{pt} / \mathrm{GL}_n]$$

takes S to the groupoid of vector bundles of rank n on S.

Example 2.3. Let X be a smooth, projective, connected curve over k. We define the stack Bun_n taking S to the groupoid of vector bundles of rank n on $X \times S$.

How do you make this geometric? We have a map $\mathrm{pt} \to \mathrm{BGL}_n$ corresponding to the trivial bundle. If $\mathcal{E}$ is a rank n vector bundle on S, then we get by definition a classifying map

$$f_{\mathcal{E}} \colon S \to \mathrm{BGL}_n.$$

Consider the fibered product

$$\begin{array}{ccc} S \times_{\mathrm{BGL}_n} \mathrm{pt} & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow \\ S & \xrightarrow{f_{\mathcal{E}}} & \mathrm{BGL}_n \end{array}$$

To understand what the fibered product is, let's compute its functor of points is.

$$\begin{array}{cccc} T & & & \\ & \searrow^{f} & & \searrow^{p} \\ & S \times_{\mathrm{BGL}_n} \mathrm{pt} & \longrightarrow & \mathrm{pt} \\ & \downarrow & & \downarrow \mathrm{Triv} \\ & S & \xrightarrow{f_{\mathcal{E}}} & \mathrm{BGL}_n \end{array}$$

Its T-valued points are

$$\{(f, \varphi \colon \mathrm{Triv} \circ p \xrightarrow{\sim} f_{\mathcal{E}} \circ f\} = \underline{\mathrm{Isom}}(\mathcal{O}_S^{\oplus n}, \mathcal{E})(T),$$

which is the frame bundle of $\mathcal{E}$. Let's think about what this means.

(1) We can recover $\mathcal{E} = \mathcal{O}_S^n \times_{\mathrm{GL}_n} \underline{\mathrm{Isom}}(\mathcal{O}_S^{\oplus n}, \mathcal{E})$, i.e. the map $\mathrm{pt} \to \mathrm{BGL}_n$ is the universal vector bundle.
(2) The map $\mathrm{pt} \to \mathrm{BGL}_n$ is a smooth surjection after every base change.

Inspired by these examples, we make a definition.

Definition 2.4. A stack $\mathcal{M}$ is called *algebraic* if

(1) For all maps $S \to \mathcal{M}$ and $S' \to \mathcal{M}$ from schemes S, S', the fibered product $\mathcal{S} \times_{\mathcal{M}} \mathcal{S}'$ is a scheme .
(2) There exists a scheme U together with a smooth surjection $U \to \mathcal{M}$ called an atlas.
(3) The map $U \times_{\mathcal{M}} U \to U \times U$ is qcqs.

An algebraic stack $\mathcal{M}$ is *smooth* (resp. locally of finite type, ...) if there is an atlas $U \twoheadrightarrow \mathcal{M}$ such that U is smooth (resp. locally of finite type, ...).

Example 2.5. (Picard stack) We define $\mathrm{Pic}_X = \mathrm{Bun}_{X,1}$. Let Jac_X be the Jacobian of X. This is the coarse moduli space of Pic_X, so we have a map

$$\mathrm{Pic}_X \to \mathrm{Jac}_X$$

which preserves the labelling of connected components by degree. Suppose you have $x \in X(k) \neq \emptyset$. Then we actually have an isomorphism

$$\mathrm{Pic}_X \xrightarrow{\sim} \mathrm{Jac}_X \times B\mathbf{G}_m$$

where the map $\mathrm{Pic}_X \to B\mathbf{G}_m$ corresponds to the restriction of the universal line bundle on $X \times \mathrm{Pic}_X$ to $\{x\} \times \mathrm{Pic}_X$.

This shows that Pic_X is a smooth algebraic stack locally of finite type of dimension $g(X)-1$.

Theorem 2.6. Bun_n *is a smooth algebaic stack locally of finite type over* k*, of dimension* $n^2(g(X)-1)$*, and* $\pi_0(\mathrm{Bun}_n)=\mathbf{Z}$.

Proof. Choose an ample line bundle $\mathcal{O}_X(1)$ on X. Define U to be the union over N of $(\mathcal{E},\{s_i\})$ such that

- $\mathcal{E}(N)$ is globally generated,
- $H^1(X,\mathcal{E}(N))=0$, and
- the $\{s_i\}$ are a basis of $H^0(X,\mathcal{E}(N))$.

This U is represented by a smooth scheme, by the theory of Quot schemes, and $U\to\mathrm{Bun}_n$ is an atlas. (The obstruction to deforming a basis lies in $H^1(X,\mathcal{E}(N))$, which we have asked to vanish.) □

Example 2.7. Let $X=\mathbf{P}^1_k$. Then $[\mathrm{pt}/\mathrm{GL}_n]$, corresponding to the trivial bundle, is an open immersion in Bun_n because $H^1(\mathbf{P}^1_k,\mathfrak{g})=0$. For example, $\mathrm{Bun}_2^0(k)=\{\mathcal{O}^{\oplus 2},\mathcal{O}(1)\oplus\mathcal{O}(-1),\ldots\}$ so the automorphism groups get bigger as the points get more special.

3. Adelic uniformization of Bun_n

3.1. **Weil's uniformization.** Let $k=\mathbf{F}_q$. Let F be a the function field of X, and $|X|$ the set of closed points. For $x\in|X|$ denote by $\mathcal{O}_x$ the completed local ring at x. This is non-canonically isomorphic to $k_x[[\varpi_x]]$. We also set $F_x=\mathrm{Frac}(\mathcal{O}_x)$, which is non-canonically isomorphic to $k_x((\varpi_x))$. Recall the ring of adeles

$$\mathbf{A}=\prod_{x\in|X|}{}'(F_x,\mathcal{O}_x)=\{(a_x)\in\prod F_x\mid a_x\in\mathcal{O}_x\text{ for almost all }x\in|X|\}.$$

Theorem 3.1 (Weil)**.** *There is a canonical isomorphism of groupoids*

$$\mathrm{GL}_n(F)\backslash\left(\mathrm{GL}_n(\mathbf{A})/\prod_{x\in|X|}\mathrm{GL}_n(\mathcal{O}_x)\right)\xrightarrow{\sim}\mathrm{Bun}_n(k).$$

Here if S is a set with a group action of G, then S/G can be considered as a groupoid, whose objects are orbits and automorphisms are stabilizers.

Example 3.2. For $n=1$, this gives

$$F^\times\backslash\mathbf{A}^\times/\prod\mathcal{O}_X^\times=F^\times\backslash\left(\prod_x F_x^\times/\mathcal{O}_x^\times\right)=F^\times\backslash\mathrm{Div}(X)=\mathrm{Pic}_X(k).$$

Proof. Consider the set

$$\Sigma:=\left\{(\mathcal{E},\{\alpha_x\},\tau)\colon\begin{array}{c}\mathrm{rank}\,\mathcal{E}=n\\ \alpha_x\colon\mathcal{E}|_{\mathrm{Spec}\,\mathcal{O}_x}\cong\mathcal{O}_x^{\oplus n}\\ \tau\colon\mathcal{E}|_{\mathrm{Spec}\,F}\cong F^{\oplus n}\end{array}\right\}.$$

We seek to define a $\mathrm{GL}_n(F) \times \prod \mathrm{GL}_n(\mathcal{O}_x)$-equivariant map

$$\Sigma \to \mathrm{GL}_n(\mathbf{A}). \tag{3.1}$$

Once we have this, we get a map of quotients

$$\begin{array}{ccc} \Sigma & \longrightarrow & \mathrm{GL}_n(\mathbf{A}) \\ \downarrow & & \downarrow \\ \mathrm{Bun}_n(k) & \dashrightarrow & \mathrm{GL}_n(F)\backslash \left(\mathrm{GL}_n(\mathbf{A})/\prod \mathrm{GL}_n(\mathcal{O}_x)\right). \end{array}$$

We'll just show you how to define the map (3.1). Given $(\mathcal{E}, \{\alpha_x\}, \tau) \in \Sigma$, we get $gx \in \mathrm{GL}_n(F_x)$ given by

$$F_x^n \xrightarrow{\alpha_x^{-1}} \mathcal{E}|_{\mathrm{Spec}\ F_x} \xrightarrow{\tau} F_x^{\oplus n}.$$

□

3.2. **Level structure.** Given $D = \sum d_x \cdot x$ an effective divisor, we can look at the double quotient

$$\mathrm{GL}_n(F)\backslash (\mathrm{GL}_n(\mathbf{A})/K_D) \cong \left\{(\mathcal{E}, \alpha) \mid \alpha \colon \mathcal{E}|_D \cong \mathcal{O}_D^{\oplus n}\right\}$$

where $K_D = \ker\left(\prod_{x\in|X|} \mathrm{GL}_n(\mathcal{O}_x) \to \prod_{x\in|X|} \mathrm{GL}_n(\mathcal{O}_x/\varpi_x^{d_x})\right)$.

3.3. **Split groups.** If G is any (not necessarily reductive) algebraic group split over k, then

$$G(F)\backslash \left(G(\mathbf{A})/\prod_{x\in|X|} G(\mathcal{O}_X)\right) \cong \mathrm{Bun}_G(k).$$

If G is not split, then we instead get an injection, with the right side having terms related to inner twists of G.

4. Hecke stacks

Let $r \geq 0$ and $\mu = (\mu_1, \dots, \mu_r)$ a sequence of dominant coweights of GL_n such that μ_i is either $\mu_+ = (1, 0, \dots, 0)$ or $\mu_- = (0, \dots, 0, -1)$.

Definition 4.1. The *Hecke stack* Hk_n^μ is the stack defined by $\mathrm{Hk}_n^\mu(S)$ is the groupoid classifying the following data:

- a sequence $(\mathcal{E}_0, \dots, \mathcal{E}_r)$ of rank n vector bundles on $X \times S$.
- a sequence $(x_1, \dots, x_r)$ of morphisms $x_i \colon S \to X$, with graphs $\Gamma_{x_i} \subset X \times S$,
- maps $(f_1, \dots, f_r)$ with

  $$f_i \colon \mathcal{E}_{i-1}|_{X\times S\backslash\Gamma_{x_i}} \xrightarrow{\sim} \mathcal{E}_i|_{X\times S\backslash\Gamma_{x_i}}$$

  such that if $\mu_i = \mu_+$, then f_i extends to $\mathcal{E}_{i-1} \hookrightarrow \mathcal{E}_i$ whose cokernel is an invertible sheaf on Γ_{x_i}, and if $\mu_i = \mu_-$ then f_i^{-1} extends to $\mathcal{E}_i \hookrightarrow \mathcal{E}_{i-1}$ whose cokernel is an invertible sheaf on Γ_{x_i}.

For $i = 0, \dots, r$ we have a map

$$p_i \colon \mathrm{Hk}_n^{\mu} \to \mathrm{Bun}_n$$

sending $(\underline{\mathcal{E}}, \underline{x}, \underline{f}) \mapsto \mathcal{E}_i$ and

$$p_X \colon \mathrm{Hk}_n^{\mu} \to X^r$$

sending $(\underline{\mathcal{E}}, \underline{x}, \underline{f}) \mapsto \underline{x}$.

Lemma 4.2. *The morphism*

$$(p_0, p_X) \colon \mathrm{Hk}_n^{\mu} \to \mathrm{Bun}_n \times X^r$$

is representable by a proper smooth morphism of relative dimension $r(n-1)$, whose fibers are iterated $\mathbf{P}^{n-1}$-bundles.

Proof. Once we have fixed a reference bundle, the fibers are iterated modifications, which amounts to a choice of a hyperplane in an n-dimensional vector space. □