Geometric Class Field Theory

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In the first half we will explain the unramified picture from the geometric point of view, and in the second half we will sketch the generalization to the ramified situation.

1 The unramified case

Let p be a prime and $\mathbb{F}_q$ a finite field over $\mathbb{F}_p$. Let ℓ be a prime not equal to p.

The central actor in our story is a smooth projective geometrically connected curve $X/\mathbb{F}_q$. Let $K = K(X)$ be the field of rational functions on X. For $x \in |X|$ (the set of closed points of X), we denote $\mathcal{O}_x = \widehat{\mathcal{O}}_{X,x}$ and $K_x = \mathrm{Frac}(\mathcal{O}_x)$. Let

$$\mathbb{A}_K = \prod\nolimits' K_x.$$

The goal of unramified class field theory is to understand all abelian extensiosn of K which are everywhere unramified.

Theorem 1.1 (Unramified CFT)**.** *There is an isomorphism*

$$\left(\mathbb{G}_m(K)\backslash\mathbb{G}_m(\mathbb{A}_K)/\mathbb{G}_m(\prod \mathcal{O}_x)\right)^{\wedge} \cong (G_K^{\mathrm{unr}})^{\mathrm{ab}}$$

such that

$$(a_x) \mapsto \prod_x \mathrm{Frob}_x^{\mathrm{ord}_x(a_x)}$$

where the $\wedge$ means profinite completion.

1.1 First geometric reformulation

We want to understand this statement more geometrically. The right hand side can be interpreted as

$$G_K^{\mathrm{unr}} = \pi_1(X).$$

(We are suppressing the base points.) The left hand side can be interpreted as

$$\mathbb{G}_m(K)\backslash\mathbb{G}_m(\mathbb{A}_K)/\mathbb{G}_m(\prod \mathcal{O}_x) \cong \mathrm{Pic}(X).$$

So here is a geometric reformulation.

Theorem 1.2. *We have a natural bijection*

$$\{characters\ \pi_1(X) \to \overline{\mathbb{Z}}_\ell^\times\} \leftrightarrow \{characters\,\mathrm{Pic}(X) \to \overline{\mathbb{Z}}_\ell^\times\}.$$

If we denote it by $\rho \mapsto \chi_\rho$, then

$$\rho(\mathrm{Frob}_x) = \chi_\rho(O([x]))\ \textit{for all } x \in X.$$

1.2 Second reformulation: categorification

The goal is to upgrade this statement by categorifying both sides. The point is to obtain a formulation of local nature, so that you can apply things like descent. *Although the initial statement is specific to working over a finite field, the categorical reformulation will not be.*

Let's first categorify the left sidde of Theorem 1.2. That's easy: it is the same thing as rank 1 local systems on X, up to isomorphism:

$$\{\text{characters } \pi_1(X) \to \overline{\mathbb{Z}}_\ell^*\} = \pi_0(\mathrm{Loc}_1(X) := \{\text{rank 1 local systems}/X\}).$$

For the right hand side, recall the following fact.

Theorem 1.3. *If G is a connected commutative algebraic group over $\mathbb{F}_q$, then the set of characters $G(\mathbb{F}_q) \to \overline{\mathbb{Z}}_\ell^*\}$ is isomorphism classes (i.e. π_0) of the category of "character local systems"*

$$\mathrm{CharLoc}(G) := \{\textit{character local systems } /G\}.$$

This $\mathrm{CharLoc}(G)$ is the category of local systems $L \in \mathrm{Loc}_1(G)$ such that

$$m^* L \cong p_1^* L \otimes p_2^* L \text{ on } G \times G.$$

Alternatively, one can think of the isomorphism being given

$$\psi\colon m^* L \cong p_1^* L \otimes p_2^* L \text{ on } G \times G$$

but then one also has to guarantee a cocycle condition (it is non-trivial to show that there always exists a unique such datum, i.e. that the two definitions presented are equivalent).

Remark 1.4. One can also think of a character local system as a *homomorphism* from G to $B\,\mathrm{GL}_1$. (A general rank 1 local system would be any morphism $G \to B\,\mathrm{GL}_1$.)

Proof. If $(M, \psi) \in \mathrm{CharLoc}(G)$, then we get a function $G(\mathbb{F}_q) \to \overline{\mathbb{Z}}_\ell^*$ by

$$g \mapsto \mathrm{Tr}(\mathrm{Frob}_y \mid M_g).$$

This is simply the function-sheaf correspondence.

The converse is trickier; it uses the *Lang isogeny*

$$L_G\colon G \to G$$

defined by $g \mapsto \mathrm{Frob}(g)g^{-1}$. This is an abelian étale cover of G with Galois group $G(\mathbb{F}_q)$. This construction gives an $N \in \mathrm{Loc}_1(G)$ for any $\chi\colon G(\mathbb{F}_q) \to \overline{\mathbb{Z}}_\ell^*$.

Exercise 1.5. Check that N is in fact a character local system, and that these constructions are inverse.

□

1.3 The Abel-Jacobi map

In the geometric formulation there is an obvious choice of bijection. To describe this we need to recall the Abel-Jacobi map

$$AJ\colon X \to \underline{\mathrm{Pic}}(X)$$

Here $\underline{\mathrm{Pic}}(X)$ denotes the Picard variety (as opposed to the group). For $x \in X$, we have

$$AJ(x) := \mathcal{O}([x]).$$

Theorem 1.6. *AJ^* induces an equivalence of categories*

$$\mathrm{CharLoc}(\underline{\mathrm{Pic}}(X)) \cong \mathrm{Loc}_1(X).$$

We denote the inverse by $L \mapsto \mathrm{Aut}_L$.

Remark 1.7. Although we stated the theorem for connected abelian G, we seem to be applying it to $\underline{\mathrm{Pic}}(X)$ which is not connected. Fortunately, it is also true for $G = \mathbb{Z}$ (but not necessarily for finite groups!).

Remark 1.8. Some observations:

1. This makes sense over any field, even $\mathbb{C}$. (We used the fact that we were over a finite field to arrive at this geometric formulation, but the final statement makes no reference to that.)

2. We have the following compatibility: for $x \in X(\mathbb{F}_q)$,

$$\mathrm{Tr}(\mathrm{Frob}_x \mid L_x) = \mathrm{Tr}(\mathrm{Frob}_{\mathcal{O}([x])} \mid \mathrm{Aut}_{L,\mathcal{O}([x])}).$$

   This is the desired compatibility condition from earlier.

3. There is a Hecke eigensheaf condition: if

$$h\colon X \times \underline{\mathrm{Pic}}(X) \to \underline{\mathrm{Pic}}(X)$$

   is the map

$$(x, \mathcal{L}) \mapsto \mathcal{L} \otimes \mathcal{O}([x]),$$

   then

$$h^* \mathrm{Aut}_L \cong L \boxtimes \mathrm{Aut}_L\,.$$

Proof. (Deligne) There is a clear map in one direction: given a character local system on $\underline{\mathrm{Pic}}(X)$, we can pull it back to one on X via the Abel-Jacobi map.

In the other direction, we will descend to the space of line bundles of sufficiently large degree, and then use the character sheaf property to extend to all of $\underline{\mathrm{Pic}}(X)$. Fix $d > 2g - 2$, we have

$$X^d \to [X^d/S_d] \to \mathrm{Sym}^d(X) \to \underline{\mathrm{Pic}}^d(X).$$

Some observations:

- The map $\mathrm{Sym}^d(X) \to \underline{\mathrm{Pic}}(X)$ is a projective space bundle for $d \gg 0$.
- The map $[X^d/S_d] \to \mathrm{Sym}^d(X)$ is a coarse moduli space. The only difference between the two spaces is that the stack has nontrivial stabilizers.
- The map $X^d \to [X^d/S_d]$ is a S_d-torsor.

Step 1. Given L, we form $L^{\boxtimes d} \in \mathrm{Loc}_1(X^d)$. This is evidently invariant under S_d, so descends at least to $\widetilde{L}^{(d)} \in \mathrm{Loc}_1([X^d/S_d])$. To descend to the coarse space $\mathrm{Sym}^d X$, you need to check that stabilizers act trivially. That works here *because* we are in the rank 1 situation.

Example 1.9. For $d = 2$, we have $(x, x) \in \Delta \subset X \times X$. This has a stabilizer S_2. A local system on $L^{\boxtimes 2}_{(x,x)}$ has stalk $L_x \otimes L_x$, which is also the stalk $\widetilde{L}^{(2)}_{(x,x)}$. The S_2 action here is the switch action. But *because we're in the rank one case*, the switch map is the identity.

Step 2. This implies that $L^{\boxtimes d}$ descends to $L^{(d)}$ on $\mathrm{Sym}^d(X)$ for $d \gg 0$. The last step is easy, thanks to Deligne's observation that if we have a projective space bundle then the source and target have the same fundamental group:

$$\pi_1(\mathrm{Sym}^d(X)) \cong \pi_1(\underline{\mathrm{Pic}}^d(X))$$

so $L^{(d)}$ descends to Pic^d_X.

Step 3. To extend to all of $\underline{\mathrm{Pic}}(X)$, we use the following fact: for $d, e > 2g - 2$ the map

$$+\colon\ \mathrm{Pic}^d_X \times \mathrm{Pic}^e_X \to \mathrm{Pic}^{d+e}_X$$

obtained by tensoring the corresponding line bundles has the following "character sheaf" property. If we call the descended object $\mathrm{Aut}_{L,d} \in \mathrm{Loc}_1(\mathrm{Pic}^d_X)$ then

$$+^* \mathrm{Aut}_{L,d+e} \cong \mathrm{Aut}_{L,d} \boxtimes \mathrm{Aut}_{L,e}\,.$$

Because of this we can extend to all of $\underline{\mathrm{Pic}}_X$.

$\square$

2 The ramified case

2.1 Generalized Picard varieties

We want to do something similar on open curves. Fix $S = \{x_1, \ldots, x_n\} \subset |X|$ (we may and do assume that $n \geq 1$.) Let $U = X - S$. We want to understand rank 1 local systems on U, i.e. extensions of K which are unramified outside U.

This involves "generalized Picard varieties". To introduce these, we need some setup. Let $D := [x_1] + \ldots + [x_n]$ and $D_m := mD$. As m varies we get a tower

$$D_1 \subset D_2 \subset \ldots \subset D_\infty := \text{formal completion of } X \text{ along } D.$$

Definition 2.1. We define $\underline{\mathrm{Pic}}_{D_n}(X)$ to be the moduli space for pairs $\{L \in \underline{\mathrm{Pic}}(X), \psi\colon L|_{D_n} \cong \mathcal{O}_{D_n}\}$.

We get a tower

$$\underline{\mathrm{Pic}}_{D_\infty}(X) = \lim\left(\ldots \to \mathrm{Pic}_{D_2}(X) \to \mathrm{Pic}_{D_1}(X) \to \mathrm{Pic}(X)\right).$$

The map $\mathrm{Pic}_{D_1}(X) \to \mathrm{Pic}(X)$ is a $\mathbb{G}_m^n$-bundle. Then $\mathrm{Pic}_{D_2}(X) \to \mathrm{Pic}_{D_1}(X)$ is a $\mathbb{G}_a^n$ bundle, and similarly for the rest of the maps. In particular, since the transition maps are affine morphisms the limit makes sense. This $\underline{\mathrm{Pic}}_{D_\infty}(X)$ is a *pro-algebraic group* over the ground field.

2.2 The generalized Abel-Jacobi map

We want to do an analog of the previous story in the unramified case. To do that we need an Abel-Jacobi map

$$AJ\colon U \to \underline{\mathrm{Pic}}_{D_\infty}(X)$$

which sends

$$y \mapsto (\mathcal{O}(y), ?)$$

We need to also say how to trivialize this at D_n. But there is a *canonica*l trivialization of $\mathcal{O}(y)$ at every D_n since $y \in U$ is disjoint from D_n; the map AJ can then be described

$$y \mapsto (\mathcal{O}(y), \text{canon.}).$$

Theorem 2.2. *The pullback AJ^* induces an isomorphism*

$$\mathrm{CharLoc}(\underline{\mathrm{Pic}}_{D_\infty}(X)) \cong \mathrm{Loc}_1(U).$$

This encodes class field theory because it tells us how to translate local systems into bundle-theoretic data, and you can translate that into an adelic description. In order to do that, we need a sheaf-function correspondence for pro-algebraic groups.

The goal of the rest of the talk is to explain why this theorem amounts to a local statement. First, however, we remark on connections with the more classical versions.

2.3 Some remarks

1. There exists a version with bounded ramification. It basically says that if we restrict to $\underline{\mathrm{Pic}}_{D_n}$, then we get Galois extensions such that in the upper numbering of the ramification groups, everything above n acts trivially.

2. We get the classical formulation of CFT via the function-sheaf dictionary.

3. In characteristic 0, we have $\pi_1(\mathbb{A}^n) = 0$ so

$$\mathrm{CharLoc}(\underline{\mathrm{Pic}}_{D_\infty}(X)) = \mathrm{CharLoc}(\underline{\mathrm{Pic}}_{D}(X)).$$

So in this case we can proceed as before using the following observation:

There exists $d \gg 0$ such that

$$\operatorname{Sym}^d U \to \underline{\operatorname{Pic}}^d_D(X)$$

is an affine space bundle.

You can also do something like in the case of tame ramification, but wild ramification truly presents new difficulties.

4. Serre's classifcal proof (as in *"Algebraic groups and class fields"*) uses the following two results.

   Theorem 2.3 (Rosenlicht)**.** *The Abel-Jacobi map $AJ\colon U \to \underline{\operatorname{Pic}}_{D_\infty}(X)$ is the universal map for $U \to G$ for G a commutative smooth algebraic group.*

   This tells us that if we know that local systems are always pulled back from commutative groups then they are even pulled back from $\underline{\operatorname{Pic}}_{D_\infty}(X)$; this turns out to apply here.

   Theorem 2.4. *If A is a finite abelian group, then any A-torsor $V \to U$ is pulled back via*

$$\begin{tikzcd} V \arrow[r] \arrow[d] & G' \arrow[d, "\pi"] \\ U \arrow[r] & G \end{tikzcd}$$

   where $\pi\colon G' \to G$ is an isogeny of commutative smooth algebraic groups with kernel A.

   Example 2.5. If $A = \mathbb{Z}/p$, you can see this by Artin-Schreier theory.

2.4 The descent step

Instead of discussing this classical stuff we want to focus on explaining what happens if you try to imitate the proof in the unramified case.

Proof. Assume $D = [x]$. (This isn't necessary but simplifies the discussion.) Fix $L \in \operatorname{Loc}_1(U)$. We want to use the descent; we get $L^{(d)} \in \operatorname{Loc}_1(\operatorname{Sym}^d(U))$. The hard step is to descend $L^{(d)}$ to $\underline{\operatorname{Pic}}_{D_\infty}(X)$ along $\operatorname{Sym}^d X \to \underline{\operatorname{Pic}}^d_{D_\infty}(X)$.

Consider the cartesian diagram

$$\begin{tikzcd} \operatorname{Sym}^d(X) \arrow[d] & T \arrow[l] \arrow[d] \\ \underline{\operatorname{Pic}}^d(X) & \underline{\operatorname{Pic}}^d_{D_\infty}(X) \arrow[l] \end{tikzcd}$$

The map $\underline{\mathrm{Pic}}^d(X) \leftarrow \underline{\mathrm{Pic}}^d_{D_\infty}(X)$ is a torsor for O_x^*, since the fiber over a point is the space of rigidifications of D_∞ (remember that we are assuming that $D = \{[x]\}$). The fiber product T is the moduli space for the datum of $D' \in \mathrm{Sym}^d(X)$ plus a trivialization for $O(D')$ on the formal completion along x.

Now consider the base change with respect to $\mathrm{Sym}^d(U) \to \mathrm{Sym}^d(X)$.

$$\begin{tikzcd}
\mathrm{Sym}^d(U) \arrow[d] & ??? \arrow[l] \arrow[d] \\
\mathrm{Sym}^d(X) \arrow[d] & T \arrow[l] \arrow[d] \\
\underline{\mathrm{Pic}}^d(X) & \underline{\mathrm{Pic}}^d_{D_\infty}(X) \arrow[l]
\end{tikzcd}$$

Since U is disjoint from $\{x\}$, we get a 0-section

$$\mathrm{Sym}^d(U) \to \mathrm{Sym}^d(U) \times O_x^*.$$

so the fibered product will be a trivial O_x^* torsor over $\mathrm{Sym}^d(U)$:

$$\begin{tikzcd}
\mathrm{Sym}^d(U) \arrow[d] & \mathrm{Sym}^d(U) \times O_X^* \arrow[l] \arrow[d] \\
\mathrm{Sym}^d(X) \arrow[d] & T \arrow[l] \arrow[d] \\
\underline{\mathrm{Pic}}^d(X) & \underline{\mathrm{Pic}}^d_{D_\infty}(X) \arrow[l]
\end{tikzcd}$$

Let's remind ourselves of our goal: we want to descend a local system from $\mathrm{Sym}^d(U)$ to $\underline{\mathrm{Pic}}^d_{D_\infty}(X)$. The map $T \to \underline{\mathrm{Pic}}^d_{D_\infty}(X)$ is a projective space bundle, so we can descend any local system along it; therefore it suffices to descend to T.

For this, the strategy is to find some $M \in \mathrm{CharLoc}(O_x^*)$ such that $L^{(d)} \boxtimes M \in \mathrm{Loc}_1(\mathrm{Sym}^d(U) \times O_x^*)$ extends to T. Since everything is smooth we only have to extend along codimension-one points of the complement. Since the map is base-changed from $\mathrm{Sym}^d(U) \to \mathrm{Sym}^d(X)$ against a 0-dimensional torsor, the situation basically looks the same as for the two maps. What are the codimension-one points of $\mathrm{Sym}^d(X) - \mathrm{Sym}^d(U)$? They correspond to the subset parametrizing divisors where *two points collide*, so in codimension 1 we are reduced to the $d = 1$ case. To do this we use a local analogue of this story, which we explain presently.

2.5 Local geometric class field theory

For X the formal disk and $x \in X$ the closed point, U the punctured disk, we get by analogous constructions an O_x^*-torsor $T \to X$ whose fiber over $y \in X$ is the space of trivializations of $O(y)$ at x. This splits canonically over U, so $T|_U \cong U \times O_x^*$.

Theorem 2.6 (Local class field theory)**.** *There is an equivalence*

$$\mathrm{Loc}_1(U)/\,\mathrm{Loc}_1(X) \cong \mathrm{CharLoc}(\mathcal{O}_x^*).$$

Moreover, $[L] \in \mathrm{Loc}_1(U)/\,\mathrm{Loc}_1(X)$ *corresponds to* $M \in \mathrm{CharLoc}(\mathcal{O}_x^*)$ *if and only if* $L \boxtimes M|_{\mathcal{O}_x^*}$ *extends to* T.

This informs us how to choose M locally.

$\square$