THE ATIYAH-SINGER INDEX THEOREM

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Contents

1. OVERVIEW

The course is about the Atiyah-Singer Index Theorem. Its statement is given below, although it may not make sense at the moment; the beginning of the course will introduce the necessary ingredients.

Theorem 1.1 (Atiyah-Singer)**.** *The index of a Dirac operator D of a Clifford module $\mathscr{E}$ over a compact, oriented manifold M of even dimension is*

$$\operatorname{ind}(D) = \int_M \widehat{A}(M)\operatorname{ch}(\mathscr{E}/S).$$

For now, the interesting thing to take away about this theorem is that it relates two things that don't seem like they should be related. The left hand side

$$\operatorname{ind}(D) := \begin{cases} \text{“graded dimension of } D\text{”} \\ \dim(\ker D^+) - \dim \operatorname{coker}(D^+) \end{cases}$$

is an *analytic* object. Meanwhile, the right hand side can be written in terms of the cap product pairing on M as

$$\int_M \widehat{A}(M)\operatorname{ch}(\mathscr{E}/S) = \langle [\widehat{A}(M)\operatorname{ch}(\mathscr{E}/S)], [M]\rangle.$$

That makes it clear that it is a *topological quantity*. It is also “local” in the sense that integration is computed locally.

Remark 1.2. To elaborate a bit more on the formula, $\widehat{A}(M) = \det\left(\frac{R/2}{\sinh R/2}\right)^{1/2}$ where R is the Ricci curvature of M.

Even the form of the equation immediately yields interesting information. For example, it is obvious that the left hand side is an integer and highly non-obvious that the right hand side is an integer. So if, for example, you compute $\int_M \widehat{A}(M)\operatorname{ch}(\mathscr{E}/S)$ and you don't get an integer, you can deduce that the hypotheses were false (this would give, for instance, a proof that M is not orientable).

The Atiyah-Singer index theorem encompasses many other important results.

Chern-Gauss-Bonnet. The Chern-Gauss-Bonnet formula is

$$\chi(M) = \int_M \operatorname{Pf}(R).$$

This is a special case of the Atiyah-Singer index theorem. For now, let's content ourselves with figuring out what the corresponding objects $(D, \mathscr{E})$ in the index theorem should be. How can we realize the Euler characteristic as the index of some operator on a vector bundle? We should take the bundle $\mathscr{E} = \bigwedge^\bullet T^*M$ and $D = d + d^*$.

Since D is self-adjoint by construction, $\ker D = \ker D^2 = \ker \Delta$. By Hodge theory, this is $H^\bullet_{\mathrm{dR}}(M)$. Then $\operatorname{ind}(D)$ is the graded dimension of $H^\bullet_{dR}(M)$, which is the alternating sum of the Betti numbers, and that is the Euler characteristic. We'll come back and compute the right hand side later on in the course.

Hirzebruch Signature Theorem. Suppose M is a manifold of dimension $4k$. The Hirzebruch signature theorem says that

$$\sigma(M) = \int_M L(M)$$

where $\sigma(M)$ is the *signature* of M (the signature of the intersection form on $H_{2k}(M)$) and $L(M) = \det\left(\frac{R/2}{\tanh R/2}\right)^{1/2}$.

Again, this is a special case of the Index theorem applied to the bundle $\mathscr{E} = \bigwedge^\bullet T^*M$ where $D = d + d^*$, and $\ker D \cong H^\bullet_{\mathrm{dR}}(M)$. This time, we take the graded dimension induced by the Hodge $*$, normalized so that $*^2 = 1$. This graded dimension gives the signature.

Hirzebruch-Riemann-Roch. If M is a Kähler manifold and $W \to M$ is a holomorphic vector bundle, then the Hirzebruch-Riemann-Roch formula says that

$$\chi_{\mathrm{hol}}(W) = \int_M \mathrm{Td}(M)\,\mathrm{ch}(W).$$

This is the Atiyah-Singer index theorem applied with $\mathscr{E} = \bigwedge^{0,\bullet} T^*M \otimes W$ and $D = \overline{\partial} + \overline{\partial}^*$. Then $\ker D$ is essentially holomorphic sections, and $\mathrm{ind}(D)$ is the graded dimension, which is the holomorphic Euler characteristic.

Part 1. **Geometric Preliminaries**

2. CONNECTIONS

Definition 2.1. Let $E \to M$ be a vector bundle. A *covariant derivative* is a map

$$\nabla \colon \Gamma(M, E) \to \Gamma(M, T^*M \otimes E)s$$

such that for $f \in C^\infty(M)$,

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla s \quad \text{(Leibniz rule)}.$$

This is equivalent to the data of a *connection* on E, so we'll use the two terms interchangeably.

Denote $\Omega^\bullet(M, E) = \Gamma(M, \bigwedge^\bullet T^*M \otimes E)$. Then we can extend ∇ to an operator

$$\boxed{\nabla \colon \Omega^\bullet(M, E) \to \Omega^{\bullet+1}(M, E)}$$

by the Leibniz rule: for $\alpha \in \Omega^k(M)$ and $s \in \Omega^\bullet(M, E)$,

$$\nabla(\alpha \wedge s) = d\alpha \wedge s + (-1)^k \alpha \wedge \nabla s$$

Definition 2.2. For $v \in \Gamma(TM)$, recall that there is a contraction operator

$$\iota_v \colon \Omega^\bullet(M) \to \Omega^{\bullet-1}(M)$$

defined by

$$\iota_v \omega(v_1, \ldots, v_p) = \omega(v, v_1, \ldots, v_p).$$

For $v \in \Gamma(TM)$, we define $\nabla_v = \iota_v \circ \nabla$.

Definition 2.3. The *curvature* of ∇ is the form $F \in \Omega^2(M, \mathrm{End}(E))$ defined

$$F(v, w) = \nabla_v \nabla_w - \nabla_w \nabla_v - \nabla_{[v,w]} \quad \text{for } v, w \in \Gamma(TM).$$

One has to check that F is linear over $C^\infty(M)$, but this is easy.

Definition 2.4. We call (E, ∇) *flat* if $F = 0$.

Example 2.5. If $E = M \times V$ is the trivial bundle, then d is a flat connection.

Example 2.6. On $E = M \times V$, a connection takes the form $d + A = \nabla$ for $A \in \Omega^1(M, \mathrm{End}(E))$. Then $F = dA + A \wedge A$.

Since any vector bundle is locally trivial, we have *locally* $(E, \nabla) \cong (M \times V, d + A)$.

Definition 2.7. A *metric* on $E \to M$ is a map of vector bundles $E \otimes E \to \underline{\mathbb{R}}$ which is an inner product upon restriction to fibers.

On global sections, this induces a map $\Gamma(E) \otimes \Gamma(E) \to C^\infty(M)$.

Definition 2.8. Given $(E, \nabla, \langle -, - \rangle)$ we say that ∇ is *compatible* with $\langle -, - \rangle$ if

$$d\langle v, w \rangle = \langle \nabla v, w \rangle + \langle v, \nabla w \rangle \in \Omega^1(M).$$

Definition 2.9. For a connection ∇ on TM, the *torsion* T is

$$T(v, w) = \nabla_v w - \nabla_w v - [v, w].$$

If $T = 0$, then we say that ∇ is *torsion-free*.

It is a fact that given a metric on TM, there is a unique connection that is both torsion-free and compatible. This is called the *Levi-Civita connection*.

3. Super vector bundles

In this section we develop a formalism of "super vector bundles", which are essentially just $\mathbb{Z}/2$-graded vector bundles, which will be useful for encoding sign issues.

3.1. **Super vector spaces.**

Definition 3.1. A *super vector space* is a $\mathbb{Z}/2$-graded vector space:

$$E = E^{\text{even}} \oplus E^{\text{odd}}.$$

We might also denote $E^{\text{even}} = E^+ = E^0$ and $E^{\text{odd}} = E^- = E^1$.

Example 3.2. We denote by $\mathbb{R}^{n|n} = \underbrace{\mathbb{R}^n}_{\text{even}} \oplus \underbrace{\mathbb{R}^n}_{\text{odd}}$.

Definition 3.3. A *super-algebra* is an algebra A whose underlying vector space is a super vector space whose multiplication respects the grading, i.e. $E^i \otimes E^j \to E^{i+j}$.

Example 3.4. (Exterior algebra) $\bigwedge^\bullet V$ is a super-algebra, with

$$(\bigwedge^\bullet V)^{\text{even}} = \bigwedge^{\text{even}} V \quad \text{and} \quad (\bigwedge^\bullet V)^{\text{odd}} = \bigwedge^{\text{odd}} V.$$

Example 3.5. (*Clifford algebras*) For (V, Q) a vector space V with quadratic form Q, we can define the *Clifford algebra*

$$\text{Cl}(V, Q) = T(V)/\langle v \otimes v + 1 \cdot Q(v) \rangle.$$

Here $T(V)$ is the tensor algebra. Since we've identified a degree 2 element with a degree 0 element, the Clifford algebra is only $\mathbb{Z}/2$-graded. Note that $V \subset \text{Cl}(V, Q)$ has odd degree. This lets you figure out the grading of everything.

Definition 3.6. A super algebra is *super-commutative* if for $a, b \in A$,

$$a \cdot b = (-1)^{|a| \cdot |b|} b \cdot a,$$

where $|a| = \deg(a) \in \{0, 1\}$ and $|b| = \deg b \in \{0, 1\}$.

Example 3.7. $\bigwedge^\bullet V$ is super-commutative. However, $\text{Cl}(V, Q)$ is not super-commutative in general.

3.2. **Maps between super vector spaces.**

Let E, F be super vector spaces. Then

$$\text{Hom}(E, F)^+ = \text{Hom}(E^+, V^+) \oplus \text{Hom}(E^-, V^-)$$

and

$$\text{Hom}(E, F)^- = \text{Hom}(E^+, V^-) \oplus \text{Hom}(E^-, V^+).$$

Remark 3.8. There's a more abstract formulation of things which makes this a little more conceptual. There's a symmetric monoidal category of vector spaces whose objects are super vector spaces, and morphisms are *even* maps. The signs comes from the monoidal structure:

$$(E \otimes F)^{\text{even}} = (E^{\text{even}} \otimes F^{\text{even}}) \oplus (E^{\text{odd}} \otimes F^{\text{odd}})$$

and

$$(E \otimes F)^{\text{odd}} = (E^{\text{even}} \otimes F^{\text{odd}}) \oplus (E^{\text{odd}} \otimes F^{\text{even}}).$$

The signs come from an isomorphism $E\otimes F\cong F\otimes E$, sending $e\otimes f\mapsto(-1)^{|e|\cdot|f|}f\otimes e$ which is built into the monoidal structure.

It may be worth remarking that if we forget this sign, then we obtain simply the usual category of $\mathbb{Z}/2$-graded vector spaces.

For a superalgebra A, the *super-commutator* of $a,b\in A$ is

$$[a,b]=ab-(-1)^{|a|\cdot|b|}ba.$$

Notice that $[-,-]=0\iff A$ is super-commutative.

Definition 3.9. A *super trace* on A is a linear map $\varphi\colon A\to\mathbb{C}$ or $\mathbb{R}$ vanishing on super-commutators, i.e. $\varphi([a,b])=0$ for all $a,b\in A$.

Example 3.10. For $A=\mathrm{End}(E)$, and $a\in A$, we can define

$$\mathrm{str}(a)=\begin{cases}\mathrm{tr}(a|_{E^+})-\mathrm{tr}(a|_{E^-}) & a \text{ even}\\ 0 & a \text{ odd}\end{cases}.$$

We claim that this is a super trace. Indeed, if a,b have opposite parity, then $[a,b]$ is odd so $\mathrm{str}([a,b])=0$ tautologically. If a,b are even, then we may write $a=\begin{pmatrix}a_+ & 0\\ 0 & a_-\end{pmatrix}$ and $b=\begin{pmatrix}b_+ & 0\\ 0 & b_-\end{pmatrix}$ and $\mathrm{str}([a,b])=\mathrm{tr}([a_+,b_+])-\mathrm{tr}([a_-,b_-])=0-0=0$. Finally, if a,b are odd then we may write

$$a=\begin{pmatrix}0 & a_+\\ a_- & 0\end{pmatrix}\quad b=\begin{pmatrix}0 & b_+\\ b_- & 0\end{pmatrix}$$

Then

$$[a,b]=\begin{pmatrix}b_+a_-+a_+b_- & 0\\ 0 & b_-a_++a_-b_+\end{pmatrix}.$$

Again, we see that $\mathrm{str}([a,b])=0$.

3.3. **Hermitian super vector bundles.**

Definition 3.11. A *hermitian super vector bundle* is a super vector bundle $E=E^+\oplus E^-$, where E^+ and E^- are each equipped with a hermitian structure.

A consequence of the definition is that odd elements and even elements are automatically orthogonal to each other.

Definition 3.12. A family of odd operators $U\in\Gamma(M,\mathrm{End}(E))$ is *self-adjoint* if

$$U=\begin{pmatrix}0 & V\\ V^* & 0\end{pmatrix}.$$

Example 3.13. If $E=\bigwedge^\bullet T^*M$, then a family of odd operators is given by interior multiplication / contraction ι_v for $v\in\Gamma(TM)$. Fixing a Riemannian metric on TM, we also have $\wedge v^*$. These are odd, and $\iota_v+\wedge v^*$ is self-adjoint (really the point is that ι_v and $\wedge v^*$ are "adjoints," but we haven't defined this).

For a super vector bundle $E \to M$, we can form the space $\Omega^\bullet(M,E)$ with a $\mathbb{Z}\times\mathbb{Z}/2$-grading. We get a total $\mathbb{Z}/2$ grading from this by viewing the $\mathbb{Z}$ grading as a $\mathbb{Z}/2$ grading, and then summing. For example, $\omega_{\text{even}} \otimes s_{\text{odd}} \in \Omega^\bullet(M,E)^{\text{odd}}$ if $\omega_{\text{even}} \in \Omega^{\text{even}}(M)$ and $s_{\text{odd}} \in \Gamma(M, E^{\text{odd}})$.

Definition 3.14. A *super connection* on a super vector bundle is an odd map

$$\mathbb{A}\colon \Omega^\bullet(M,E) \to \Omega^\bullet(M,E)$$

such that $\mathbb{A}(\alpha\wedge s) = d\alpha\wedge s + (-1)^{|\alpha|}\alpha\wedge\mathbb{A}(s)$, where $\alpha \in \Omega^\bullet(M)$ and $s \in \Omega^\bullet(M,E)$.

Example 3.15. If M is a point, then E is a super vector space, and a super connection is an odd endomorphism of E. As we'll see soon, a Dirac operator is an odd endomorphism of a super Hilbert space.

Definition 3.16. The *curvature* of a super connection is $\mathbb{A}^2 = \mathbb{A}\circ\mathbb{A}$.

Proposition 3.17. *$\mathbb{A}^2$ is linear over $\Omega^\bullet(M)$, i.e. $\mathbb{A}^2 \in \Omega^\bullet(M, \mathrm{End}(E))$. Also, $[\mathbb{A},\mathbb{A}^2] = 0$ (corresponding to the usual Bianchi identity $\mathbb{A}(F) = 0$).*

Proof. Exercise. □

Note that $\mathbb{A}$ is determined by its restriction to $\Gamma(M,E) = \Omega^0(M,E)$. We can express $\mathbb{A}$ as $\sum_i \mathbb{A}_i$ where $\mathbb{A}_i\colon \Omega^0(M,E) \to \Omega^i(M,E)$. If we look at the effect of the Leibniz rule, then $\mathbb{A}_1 = \nabla$ is an ordinary connection and $\mathbb{A}_i \in \Omega^i(M,\mathrm{End}(E))^{\text{odd}}$.

From this description, we see that the space of super-connections is an affine space for $\Omega^1(M,\mathrm{End}(E))^{\text{odd}}$. This is analogous to the usual space of connections being an affine space for $\Omega^\bullet(M,\mathrm{End}(E))$.

4. CHARACTERISTIC CLASSES

4.1. **Chern-Weil Theory.** One nice feature of the curvature tensor is that it gives differential form representatives of characteristic classes of vector bundles; this is called *Chern-Weil* theory. We'll briefly review it.

Definition 4.1. Given a polynomial $f(z)$, let

$$f(\mathbb{A}^2)=\sum_{i=0}^{N}\frac{f^{(n)}(0)}{n!}(\mathbb{A}^2)^n.$$

Then $\operatorname{str}(f(\mathbb{A}^2))\in\Omega^\bullet(M)$.

Proposition 4.2. *If* $\mathbb{A}$ *be a superconnection, then* $\operatorname{str}(f(\mathbb{A}^2))$ *is a closed form of even degree.*

If $\mathbb{A}_t$ *is a one-parameter family of superconnections, then*

(1) We have

$$\frac{d}{dt}\operatorname{str}(f(\mathbb{A}_t^2))=d\operatorname{str}\left(\frac{d\mathbb{A}_t}{dt}f'(\mathbb{A}_t^2)\right).$$

(2) $[\operatorname{str}(f(\mathbb{A}_0^2))]=[\operatorname{str}(f(\mathbb{A}_1^2))]\in H^{\mathrm{even}}_{\mathrm{dR}}(M)$ *for* $\mathbb{A}_0,\mathbb{A}_1$ *super connections on* $E\to M$.

The usual Chern-Weil theory uses a connection between the trace and the de Rham d, and the analogous thing applies here.

Lemma 4.3. $\operatorname{str}([\mathbb{A},\alpha])=d\operatorname{str}(\alpha)$ *for* $\alpha\in\Omega^\bullet(M;\operatorname{End}(E))$.

Proof. Locally, $\mathbb{A}=d+\omega$. Then

$$\operatorname{str}([d+\omega,\alpha])=\operatorname{str}(d\alpha)+\operatorname{str}([\omega,\alpha])$$

(since $[d,\alpha]$ is multiplication by $d\alpha$). Since ω,α are endomorphism valued forms, and a supertrace vanishes on commutators by definition, we have $\operatorname{str}([\omega,\alpha])=0$. To finish, it is clear that $\operatorname{str}(d\alpha)=d\operatorname{str}(\alpha)$. □

Proof of Proposition 4.2. For the first part, the Lemma we have

$$\begin{aligned}d(\operatorname{str}(f(\mathbb{A}^2))&=\operatorname{str}([\mathbb{A},f(\mathbb{A}^2)])\\&=0\end{aligned}$$

with the latter inequality following from the Bianchi identity. This shows that $\operatorname{str}(f(\mathbb{A}^2))$ is closed.

The operator $\mathbb{A}^2$ is even, so $f(\mathbb{A}^2)$ is even. Since str preserves degree, $\operatorname{str}f(\mathbb{A}^2)\in\Omega^\bullet(X)$ is even. Caution: $f(\mathbb{A}^2)$ being does not mean that it doesn't have "odd parts" (since they can be valued in odd degree differential operators), but taking the supertrace kills those parts.

(1) We claim that $\frac{d}{dt}\operatorname{str}(f(\alpha_t))=\operatorname{str}(\frac{d\alpha_t}{dt}f'(\alpha_t))$. By linearity, it suffices to handle the case $f(z)=z^n$. In that case, we have

$$\frac{d}{dt}\operatorname{str}(\alpha_t^n)=\operatorname{str}\left(\sum_{i=0}^{n-1}\alpha_t^i\frac{d\alpha_t}{dt}\alpha_t^{n-i-1}\right).$$

Here we must be careful with the order of things, since the operators aren't commutative. But by the cyclicity of the supertrace, this is $n\,\mathrm{str}(\frac{d\alpha_t}{dt}\alpha_t^{n-1})$, as desired.

Our calculation shows that

$$\frac{d}{dt}\mathrm{str}(f(\mathbb{A}_t^2)) = \mathrm{str}\left(\frac{d\mathbb{A}_t^2}{dt} f'(\mathbb{A}_t^2)\right)$$

and $\frac{d}{dt}\mathbb{A}_t^2 = A_t \frac{d\mathbb{A}_t}{dt} + \frac{d\mathbb{A}_t}{dt} A_t = [\mathbb{A}_t, \frac{d\mathbb{A}_t}{dt}]$. So by repeated application of the Bianchi identity and the lemma (in that order),

$$\begin{aligned}\frac{d}{dt}\mathrm{str}(f(\mathbb{A}_t^2)) &= \mathrm{str}\left(\left[\mathbb{A}_t, \frac{d\mathbb{A}_t}{dt} f'(\mathbb{A}_t^2)\right]\right) \\ &= d\,\mathrm{str}\left(\frac{d\mathbb{A}_t}{dt} f'(\mathbb{A}_t^2)\right).\end{aligned}$$

(2) Let $\omega = \mathbb{A}_1 - \mathbb{A}_0$. Take $\mathbb{A}_t = A_0 + t\omega$. Then

$$\mathrm{str}(f(\mathbb{A}_1^2)) - \mathrm{str}(f(\mathbb{A}_0^2)) = d\int_0^1 \mathrm{str}\left(\frac{d\mathbb{A}_t}{dt} f'(\mathbb{A}_t)^2\right) dt.$$

□

Remark 4.4. It is useful to have this explicit expression for the form interpolating between $\mathrm{str}(f(\mathbb{A}_1^2))$ and $\mathrm{str}(f(\mathbb{A}_0^2))$. This is sometimes called the Chern-Weil form.

Example 4.5. If $\mathbb{A} = \mathbb{A}_1 = \nabla$, then $f(\mathbb{A}^2)$ is the usual *Chern-Weil* form of f.

4.2. **Characteristic classes.**

4.3. **Chern classes.** For $f(z) = \exp(-z)$, $\mathrm{str}(f(\mathbb{A}))$ is the *Chern character* of $(E, \mathbb{A})$. Explicitly,

$$\mathrm{ch}(E, \mathbb{A}) = \mathrm{str}(e^{-\mathbb{A}^2}).$$

The cohomology class is actually independent of the choice of connection, so we may just write $\mathrm{ch}(E) = [\mathrm{str}(e^{-\mathbb{A}^2})]$.

If $E = E^+ \oplus E^-$, with superconnection $\mathbb{A} = (E^\pm, \nabla^\pm)$ then $\mathrm{ch}(E, \mathbb{A}) = \mathrm{ch}(E^+, \nabla^+) - \mathrm{ch}(E^-, \nabla^-)$.

Properties of the Chern character.

- $\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$.
- $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \smile \mathrm{ch}(F)$.

In fact, these even hold at the level of forms.

4.4. **Genera.**

Definition 4.6. We define the $\widehat{A}$ *genus* of M to be

$$\widehat{A}(\nabla) = \det\left(\frac{R/2}{\sinh(R/2)}\right)^{1/2}$$

where $R = \nabla^2$ (i.e. the curvature).

Remark 4.7. You might wonder, where in the above definition does the supertrace appear? It's hiding in the expression $\det(e^A) = e^{\operatorname{Tr} A}$.

Definition 4.8. With $R = \nabla^2$ as before, the *Todd class* is

$$\operatorname{Td}(\nabla) = \det\left(\frac{R}{e^R - 1}\right).$$

Properties.

- $\widehat{A}(\nabla_0 \oplus \nabla_1) = \widehat{A}(\nabla_0) \wedge \widehat{A}(\nabla_1)$.
- $\operatorname{Td}(\nabla_0 \oplus \nabla_1) = \operatorname{Td}(\nabla_0) \wedge \operatorname{Td}(\nabla_1)$.

Usually one considers $\widehat{A}$ for real vector bundles, and Td for complex vector bundles. For real vector bundles, $\widehat{A}(\nabla) \in \Omega^{4k}(M)$.

4.5. **Back to the big picture.**

Example 4.9. Recall the Hirzebruch-Riemann-Roch formula: for $E \to M$ a holomorphic vector bundle over a Kähler manifold,

$$\operatorname{ind}(\overline{\partial} + \overline{\partial}^*) = \int_M \operatorname{Td}(M) \operatorname{ch}(E).$$

The right hand side now makes sense.

Example 4.10. Recall the Hirzebruch signature formula:

$$\sigma(M) = \operatorname{ind}(d + d^*) = \int_M L(M).$$

Here $L(M) = \det\left(\frac{R/2}{\tanh(R/2)}\right)^{1/2}$ where ∇ is some connection on TM and R is the curvature of ∇.

Remark 4.11. When not specified, the bundle in question is the tangent bundle.

5. Clifford algebras

5.1. **The Clifford algebra.** Let V be a real vector space, Q a quadratic form on V (possibly degenerate). There are three perspectives on what a Clifford algebra is.

(1) *(Generators and relations)* Let $\mathrm{Cl}(V,Q)$ be the algebra generated by V with relations $v\cdot w + w\cdot v = -2\cdot Q(v,w)$.
(2) *(Universal property)* $\mathrm{Cl}(V,Q)$ is initial with respect to linear maps $c\colon V\to A$ to an algebra A such that $c(v)\cdot c(w)+c(w)\cdot c(v) = -2Q(v,w)$.

$$\begin{tikzcd} V \arrow[r, "c"] \arrow[dr] & A \\ & \mathrm{Cl}(V,Q) \arrow[u, dotted, "\exists!"] \end{tikzcd}$$

(3) *(Quotient of tensor algebra)* $\mathrm{Cl}(V,Q)$ is the following quotient of the tensor algebra:
$$T(V)/\{v\otimes w + w\otimes v = -2Q(v,w)\}.$$

The natural $\mathbb{Z}$-grading on $T(V)$ induces a $\mathbb{Z}/2$-grading on $\mathrm{Cl}(V,Q)=\mathrm{Cl}^+(V)\oplus\mathrm{Cl}^-(V)$, and $V\subset \mathrm{Cl}^-(V)$.

There is an anti-automorphism of $\mathrm{Cl}(V)$ determined by $v\mapsto -v$. (So it reverses the order of a “word.”) We denote this by $a\mapsto a^*$.

The natural action of $O(V)$ on V extends to an action on $\mathrm{Cl}(V)$.

5.2. **Clifford modules.** The exterior algebra $\bigwedge^\bullet V$ carries an action of $\mathrm{Cl}(V,Q)$ making it into a *Clifford module*. The action can be specified on V, provided that it satisfies the necessary relations. Let ι_v denote contraction with $Q(v,-)\in V^*$. Then we define the action of $v\in V$ on $\bigwedge^\bullet V$ via $\mathrm{cl}(v) := \wedge v - \iota_v$. This is *self-adjoint* (see Example 3.13).

We get from this a *symbol map* $\mathrm{Cl}(V)\to\bigwedge^\bullet V$ sending $a\mapsto \mathrm{cl}(a)\cdot 1$ (the action). This is *not* an algebra map, but it does give an isomorphism of $\mathbb{Z}/2$-graded $O(V)$-modules $\mathrm{Cl}(V)\cong\bigwedge^\bullet V$.

There's a filtration on $\mathrm{Cl}(V)$ with $\mathrm{Cl}(V)_0=\mathbb{R}$, $\mathrm{Cl}(V)_1 = \mathbb{R}\oplus V$, etc. and the associated graded is $\bigwedge^\bullet V$.

5.3. **Bundles of Clifford modules.** For M a Riemannian manifold, we have inner product spaces (T_xM, g) for each $x\in M$, so we get a bundle of algebras $\mathrm{Cl}(TM)$ whose fiber at $x\in M$ is $\mathrm{Cl}(T_xM,g)$.

Definition 5.1. A *bundle of Clifford modules* on M is a super vector bundle $E\to M$ with a smooth fiberwise action by $\mathrm{Cl}(TM)$, together with a hermitian metric and compatible super-connection on E, i.e.

(1) For all $s_1,s_2\in\Gamma(E)$ and $v\in\Gamma(M,TM)\subset\Gamma(M,\mathrm{Cl}(TM))$ we have
$$\langle \mathrm{cl}(v)\cdot s_1, s_2\rangle + \langle s_1, \mathrm{cl}(v)\cdot s_2\rangle = 0$$
and
(2) The connection $\mathbb{A}$ on E is compatible with the Levi-Civita connection ∇ on TM: for all $v\in\Gamma(M,TM)$
$$[\mathbb{A}, \mathrm{cl}(v)] = \mathrm{cl}(\nabla v)$$

as operators on $\Omega^\bullet(M, E)$.

The Clifford bundle has a symbol map $\sigma\colon \mathrm{Cl}(TM) \to \bigwedge^\bullet TM$ which is an isomorphism of $\mathbb{Z}/2$-graded vector bundles.

Definition 5.2. For E a Clifford module and W a $\mathbb{Z}/2$-graded vector bundle with a super connection $\mathbb{A}$, call $W \otimes E$ a *twisted* Clifford module with action $\mathrm{Id} \otimes \mathrm{cl}_E$.

6. Dirac Operators

6.1. Classical Dirac operators.

Definition 6.1. The *Dirac operator* associated to a Clifford module E (with ordinary connection ∇^E) is the composition

$$D:\Gamma(M,E)\xrightarrow{\nabla^E}\Gamma(M,T^*M\otimes E)\xrightarrow{\text{metric}}\Gamma(M,TM\otimes E)\xrightarrow{\text{cl}}\Gamma(M,E).$$

In local coordinates we have

$$D=\sum_i \mathrm{cl}(dx^i)\nabla_i,$$

so $[D,f]=\mathrm{cl}(df)$.

Example 6.2. The exterior bundle $E=\bigwedge^\bullet TM$ has a Clifford module structure from the symbol map σ. Explicitly, if $v\in\Gamma(TM)$ then $\mathrm{cl}(v)=v\wedge-\iota_v$.

Then the Dirac operator is

$$D(w)=\sum_i \mathrm{cl}(e^i)\nabla_i w=\underbrace{\sum_i e^i\wedge\nabla_i\,\omega}_{\approx d}-\underbrace{\sum_i \iota_{e_i}\nabla_i\,\omega}_{\approx d^*}$$

which is $(d+d^*)\omega$ if the connection is torsion-free.

6.2. Clifford superconnections and Dirac operators.

We can also define a Dirac operator for Clifford superconnections. Let $E\to M$ be a Clifford bundle. Recall that we defined a symbol map $\sigma\colon \mathrm{Cl}(TM)\cong\bigwedge^\bullet TM$. This induces an isomorphism (using the metric as well)

$$\sigma:\Omega^\bullet(M,E)\cong\Gamma(M,\mathrm{Cl}(TM)\otimes E).$$

For a Clifford superconnection $\mathbb{A}$ on E, we define the Dirac operator $D_{\mathbb{A}}$ by the composition

$$\Gamma(M,E)\xrightarrow{\mathbb{A}}\Omega^\bullet(M,E)\xrightarrow{\sigma^{-1}}\Gamma(M,\mathrm{Cl}(TM)\otimes E)\xrightarrow{\text{cl}}\Gamma(M,E).$$

What does this look like in local coordinates? Locally, we may write

$$\mathbb{A}=dx^i\otimes\nabla_i+\sum_{I\subset\{1,\dots,n\}}dx^I\otimes\mathbb{A}_I.$$

Then

$$D_{\mathbb{A}}=\sum_{i=1}^n \mathrm{cl}(dx^i)\otimes\nabla_i+\sum_I \mathrm{cl}(dx^I)\otimes\mathbb{A}_I.$$

Proposition 6.3. *For $\mathbb{A}$ a Clifford super-connection on E, we have*

$$\mathbb{A}^2=R^E+F^{E/S}$$

where $R^E\in\Omega^2(M,\mathrm{Cl}(TM))\hookrightarrow\Omega^2(M,\mathrm{End}(E))$ is defined by

$$R^E(e_i,e_j)=\frac{1}{2}\sum_{k<\ell}\langle R(e_i,e_j)e_k,e_\ell\rangle\,\mathrm{cl}(e^k)\,\mathrm{cl}(e^\ell).$$

and moreover $F^{E/S}\in\Omega^\bullet(M,\mathrm{End}_{\mathrm{Cl}(TM)}(E))$, i.e. is $\mathrm{Cl}(TM)$-linear.

Remark 6.4. Roughly speaking, R^E is a Clifford version of the curvature (of the Levi-Civita connection), and $F^{E/S}$ is what's left.

Proof. $[\mathbb{A}^2, \mathrm{cl}(a)] = [\mathbb{A}, [\mathbb{A}, \mathrm{cl}(a)]]$. By the property of being a Clifford superconnection, this is

$$\begin{aligned} [\mathbb{A}, \mathrm{cl}(\nabla^{LC} a)] &= \mathrm{cl}((\nabla^{LC})^2 a) \\ &= \mathrm{cl}(Ra) \\ (\text{computation}) &= [R^E, \mathrm{cl}(a)] \end{aligned}$$

which shows that $[\mathbb{A}^2 - R^E, \mathrm{cl}(a)] = [F^{E/S}, \mathrm{cl}(a)] = 0$. □

Definition 6.5. We define the *twisting curvature* of E to be $F^{E/S} \in \Omega^\bullet(M, \mathrm{End}_{\mathrm{Cl}(TM)}(E))$.

6.3. **Index of a Dirac operator.** For a Clifford module $(E, \mathbb{A})$ suppose that $D_{\mathbb{A}}$ is self-adjoint. Then let $D^\pm = D_{\mathbb{A}}|_{\Gamma(M,E^\pm)}$ so

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

with $(D^+)^* = D^-$.

Definition 6.6. We define the *index* of D to be

$$\mathrm{ind}(D) := \dim\ker D^+ - \dim\operatorname{coker} D^+$$

and the *superdimension* of D to be

$$\mathrm{sdim}(D) := \dim\ker D^+ - \dim\ker D^-.$$

Lemma 6.7. *If D is self-adjoint, then* $\mathrm{ind}(D) = \mathrm{sdim}(D)$.

Proof. We have

$$\begin{aligned} \mathrm{sdim}(\ker D) &= \dim(\ker D^+) - \dim(\ker D^-) \\ &= \dim(\ker D^+)) - \dim(\ker(D^+)^*) \\ &= \dim(\ker D^+) - \dim(\operatorname{coker} D^+) \\ &= \mathrm{ind}(D) \end{aligned}$$

□

Theorem 6.8 (Atiyah-Singer)**.** *The index of the Dirac operator of a Clifford module over a compact, oriented, even-dimensional manifold M is*

$$\mathrm{ind}(D) = (2\pi i)^{-n/2} \int_M \widehat{A}(M) \cdot \mathrm{ch}(F^{E/S}).$$

7. Spin

7.1. The spin groups.

Definition 7.1. Let $\mathrm{Pin}(V)$ be the multiplicative subgroup of $\mathrm{Cl}(V)$ generated by $V-\{0\}\subset \mathrm{Cl}(V)$. Let $\mathrm{Spin}(V)=\mathrm{Pin}(V)\cap \mathrm{Cl}(V)^0$.

Define a homomorphism $\mathrm{Pin}(V)\to O(V)$ whose value on $v\in V$ such that $||v||=1$ is

$$x\mapsto v\cdot x\cdot v^{-1}=-v\cdot x\cdot v=x-2\langle x,v\rangle v.$$

This element is in $\mathrm{GL}(V)$, and from its form you can see that it is actually a *reflection* through the plane perpendicular to v. Since $\mathrm{Pin}(V)$ is generated by $v\in V$, we can produce any composition of reflections, so $\mathrm{Pin}(V)\twoheadrightarrow O(V)$.

Furthermore, $\mathrm{Spin}(V)\subset \mathrm{Cl}(V)^0$, so its image is in $\mathrm{SO}(V)$, i.e.

$$\begin{tikzcd} \mathrm{Spin}(V) \arrow[r, hook] \arrow[d, two heads] & \mathrm{Pin}(V) \arrow[d, two heads] \\ \mathrm{SO}(V) \arrow[r, hook] & \mathrm{O}(V) \end{tikzcd}$$

What's the kernel?

Proposition 7.2. *There is an exact sequence*

$$\{\pm 1\}\hookrightarrow \mathrm{Spin}(V)\twoheadrightarrow \mathrm{SO}(V).$$

Proof Sketch. The kernel consists of all elements of $\mathrm{Spin}(V)$ super commuting with all $x\in V$. That implies (though we haven't shown it yet) that the kernel consists of scalars, because $\mathrm{Cl}(V,Q)$ is a (super)simple algebra over itself. Then we just have to check that the only scalars in $\mathrm{Spin}(V)$ are $\{\pm 1\}$.

The map $a\mapsto a^*\colon \mathrm{Cl}(V)\to \mathrm{Cl}(V)$ is an anti-homomorphism. (For $v\in V$, this takes $v\mapsto v^{-1}=: v^*$.) So any scalar $a\in\mathrm{Spin}(V)$ has the property that $a^*=a^{-1}$, but also $a^*=a$ ♠♠♠ TONY: [why??], hence scalars have order 2. ☐

Let $\mathrm{Spin}(k)=\mathrm{Spin}(\mathbb{R}^k)$ for $\mathbb{R}^k$ with the standard metric.

Proposition 7.3. *For $k\geq 2$, $\mathrm{Spin}(k)$ is connected and for $k\geq 3$, $\mathrm{Spin}(k)$ is simply-connected and is the universal cover of $\mathrm{SO}(k)$.*

Proof. From Proposition 7.2 we get a long exact sequence of homotopy groups

$$\begin{tikzcd} \pi_1(\mathbb{Z}/2) \arrow[r] & \pi_1(\mathrm{Spin}(k)) \arrow[r] & \pi_1(\mathrm{SO}(k)) \arrow[dll, "\delta"] \\ \pi_0(\mathbb{Z}/2) \arrow[r] & \pi_0(\mathrm{Spin}(k)) \arrow[r] & \pi_0(\mathrm{SO}(k)) \end{tikzcd}$$

and we know that $\pi_1(\mathrm{SO}(k))\cong\mathbb{Z}/2$ for $k\geq 3$. For $k\geq 2$, we need to connect ± 1 in $\mathrm{Spin}(k)$ by a path. One such $t\mapsto \cos t+e_1e_2\sin t$ where e_1,e_2 are linearly independent (so we are obviously using $k\geq 2$ here.) This implies that $\pi_0(\mathbb{Z}/2)\to\pi_0(\mathrm{Spin}(k))$ is 0, verifying connectedness.

For simply-connectedness, we have to check that $\pi_1(\mathbb{Z}/2)\to\pi_1(\mathrm{Spin}\,k)$ is also 0. But that is obvious.

□

Corollary 7.4. *We have* $\mathrm{Lie}(\mathrm{Spin}(V)) \cong \mathrm{Lie}(\mathrm{SO}(V)) \cong \bigwedge^2 V$.

For $A = (a_{ij}) \in \mathrm{Lie}(\mathrm{SO}(V))$, $A \mapsto \frac{1}{2}\sum_{i<j} a_{ij} e_i e_j \in \mathrm{Lie}(\mathrm{Spin}(V))$ gives an explicit inverse.

7.2. **Spin structures.** We will give three different (but equivalent) definitions of a spin structure.

Definition 7.5. A *spin structure* on M is a $\mathrm{Spin}(n = \dim M)$-principal bundle $\mathrm{Spin}\, M \to M$ that is a double cover of the frame bundle $\mathrm{SO}(M)$ such that on fibers the covering map is the double cover $\mathrm{Spin}(n) \to \mathrm{SO}(n)$. Equivalently,

$$\mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} \mathbb{R}^n \cong TM.$$

If M has a spin structure, then it is called a *spin manifold*.

There are some topological obstructions to the existence of a spin structure. From the short exact sequence

$$1 \to \mathbb{Z}/2 \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 1$$

and the associated long exact sequence in cohomology

$$\begin{tikzcd} H^1(m;\mathbb{Z}/2) \arrow[r] & H^11(M;\mathrm{Spin}(n)) \arrow[r] & H^1(M;\mathrm{SO}(n)) \arrow[dll, "\delta"] \\ H^2((\mathbb{Z}/2) \arrow[r] & \dots \end{tikzcd}$$

we get a class $\alpha \in H^1(M;\mathrm{SO}(n))$, classifying $\mathrm{SO}(M)$, and its image under $H^1(M;\mathrm{SO}(n)) \to H^2(M;\mathbb{Z}/2)$ is the Stiefel-Whitney class $w_2(M)$. It describes the obstruction of lifting $\mathrm{SO}(n)$ to a $\mathrm{Spin}(n)$ bundle. Also, since M is oriented $w_1(M) = 0$. Therefore, we can also define:

Definition 7.6. M is *spin* if $w_1(M) = w_2(M) = 0$. Its spin structures are then a torsor for $H^1(M;\mathbb{Z}/2)$.

Finally, we have one last definition.

Definition 7.7. A *spin structure on* M is a bundle of $\mathbb{Z}/2$-graded irreducible $\mathrm{Cl}(TM)-\mathrm{Cl}(\mathbb{R}^n)$ bimodules.

Why are these definitions equivalent? The equivalence of the first two has already been sketched. To compare the first and third, note that if $\mathrm{Spin}(M)$ is a spin structure in the first sense, then we can form the bundle $\mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} \mathrm{Cl}(\mathbb{R}^n)$ which has a Clifford action on the right and a $\mathrm{Cl}(TM)$-action on the left.

7.3. **Spinor representation.** Let $V \cong (\mathbb{R}^n, \langle -,- \rangle)$.

Definition 7.8. A *chirality operator* is $\Gamma \in \mathbb{C}\ell(V) := \mathrm{Cl}(V) \otimes \mathbb{C}$ defined by

$$\Gamma := i^p e_1 \dots e_n, \quad p = \begin{cases} n/2 & n \text{ even} \\ (n+1)/2 & n \text{ odd} \end{cases}$$

and $\{e_i\}$ an orthonormal basis of V.

Note that $\Gamma^2 = 1$, and

$$\Gamma v = \begin{cases} +v\Gamma & n \text{ odd,} \\ -v\Gamma & n \text{ even.} \end{cases}$$

For n even, we get a $\mathbb{Z}/2$-grading on complex Clifford modules of $\mathbb{C}\ell(V)$:

$$E^{\pm} = \{w \in E \mid \Gamma w = \pm w\}.$$

Definition 7.9. A *polarization* of $V \otimes \mathbb{C}$ is a subspace $P \subset V \otimes \mathbb{C}$ such that

(1) $Q_{\mathbb{C}}(w, w) = 0$ for all $w \in P$ (where $Q_{\mathbb{C}}$ is the $\mathbb{C}$-bilinear complexification of Q)
(2) $V \otimes \mathbb{C} \cong P \oplus \overline{P}$.

We'll use a polarization to build a spinor representation.

Proposition 7.10. *If* $\dim V$ *is even, then there exists a unique* $\mathrm{Cl}(V)$*-module* $\not{S} = \not{S}^+ \oplus \not{S}^-$ *such that*

$$\mathrm{End}(\not{S}) \cong \mathbb{C}\ell(V).$$

Definition 7.11. This $\not{S}$ is called the *spinor representation* of $\mathbb{C}\ell(V)$.

Proof. Choose $P \subset V \otimes \mathbb{C}$ a polarization. Let $\not{S} := \bigwedge^{\bullet} P$. Let $\not{S}^+ = \bigwedge^{\text{even}} P$ and $\not{S}^- = \bigwedge^{\text{odd}} P$. Then we define the action by

$$\mathrm{cl}(w) \cdot s = \begin{cases} \sqrt{2} w \wedge s & w \in P \\ -\sqrt{2} \cdot \iota_{\overline{w}} s & w \in \overline{P} \end{cases}$$

for $w \in V \otimes \mathbb{C} \subset \mathrm{Cl}(V) \otimes \mathbb{C}$. Since such w generate $\mathbb{C}\ell(V)$, that determines the representation.

Then $\mathrm{Cl}(V) \otimes \mathbb{C} \cong \mathrm{End}(\not{S})$, and the latter is a simple algebra (this is just a general fact about matrix algebras). Therefore, it has a unique irreducible module, which is $\not{S}$. □

Proposition 7.12. *There exists a unique inner product on* $\not{S}$ *(up to scalars) making* $\not{S}$ *a self-adjoint* $\mathbb{C}\ell(V)$*-module.*

Proof. Exercise. □

Proposition 7.13. *For* $\dim V$ *even, every* $\mathbb{Z}/2$*-graded* $\mathbb{C}\ell(V)$ *module* E *is isomorphic to* $W \otimes \not{S}$ *for* $W := \mathrm{Hom}(\not{S}, E)$.

Proof. We've already established that $\mathbb{C}\ell(V) \cong \mathrm{End}(\not{S})$, and the result follows from the linear algebra fact that any module for a matrix algebra takes this form. (We're sweeping the $\mathbb{Z}/2$-graded issue under the rug. That's why there are subtleties in odd dimensions; the linear algebra fact is true in general.) □

Definition 7.14. This W is called the *twisting space* for the $\mathbb{C}\ell(V)$-module E.

Since $\mathrm{Spin}(V) \subset \mathbb{C}\ell(V)$, we can also view $\not{S}$ as a $\mathrm{Spin}(V)$-representation.

7.4. **Spinor bundles.**

Definition 7.15. The *spinor bundle* on a manifold M is the associated bundle

$$\not{S}(M) := \not{S}_M := \mathrm{Spin}(M) \otimes_{\mathrm{Spin}(n)} \not{S}.$$

From the $\mathbb{C}\ell(\mathbb{R}^n)$-action on $\not{S}$, we get a $\mathrm{Cl}(TM)$ action on $\not{S}(M)$, so $\not{S}(M)$ is a $\mathrm{Cl}(TM)$-bundle.

Proposition 7.16. *If M is an even-dimensional spin manifold, then every Clifford module E is a* twisted spinor bundle, *i.e. $E \cong W \otimes \not{S}$ for W a $\mathbb{Z}/2$-graded vector bundle.*

Proof. Set $W = \mathrm{Hom}_{\mathrm{Cl}(TM)}(\not{S}, E)$. Then the map

$$\mathrm{Hom}_{\mathrm{Cl}(T,M)}(\not{S}, E) \otimes \not{S} \to E$$

taking $w \otimes s \mapsto w(s)$ is an isomorphism. □

Connection on $\not{S}$. The connection on $\not{S}$ comes from the Levi-Civita connection on M. Conceptually, the Levi-Civita connection on M induces a connection on $\mathrm{SO}(M)$, and $\mathrm{Spin}(M)$ is a double cover (in particular, a local diffeomorphism) of $\mathrm{SO}(M)$, hence inherits this connection. In local coordinates, if

$$\Delta_i^{LC} e_j = \omega_{ij}^k e_k$$

then

$$\nabla_i^{\not{S}} = \partial_i + \frac{1}{2}\sum_{j,k} \omega_{ij}^k c^i c^j$$

where $c^i = \mathrm{cl}(e_i)$.

Definition 7.17. The *Dirac operator* $\not{D}$ is the Dirac operator on $\mathrm{Spin}(M)$ associated to $\nabla^{\not{S}}$.

Exercise 7.18. Check that $\nabla^{\not{S}}$ is a Clifford superconnection.

7.5. **Twisted spinor bundles.** Let $(W, \mathbb{A}^W)$ be a super vector bundle with super-connection on a spin manifold M.

Proposition 7.19. $\mathbb{A} := \mathbb{A}^W \otimes 1 + 1 \otimes \nabla^{\not{S}}$ *is a Clifford super-connection on $W \otimes \not{S}$.*

Proof. This is just a computation:

$$\begin{aligned}
[\mathbb{A}, \mathrm{cl}(a)] &= [\mathbb{A}^W \otimes 1 + 1 \otimes \nabla^{\not{S}}, 1 \otimes \mathrm{cl}(a)] \\
&= 1 \otimes [\nabla^{\not{S}}, \mathrm{cl}(a)] = 1 \otimes \mathrm{cl}(\nabla^{\not{S}} a) \\
&= \mathrm{cl}(\nabla a).
\end{aligned}$$

□

Facts.

(1) Any Clifford super-connection on $\not{S} \otimes W$ is of this form. Combined with the fact that all Clifford modules on a spin manifold are of this form, this completely characterizes Dirac operators coming from Clifford modules on spin manifolds.
(2) For $E = \not{S} \otimes W$, $\mathbb{A}^2 = R^{TM} + F^{E/S}$ and in this case $F^{E/S} = R^W$. In particular, $F^{E/S}$ of $E = \not{S}$ vanishes ("the spinor bundle is untwisted").

8. Differential Operators

8.1. **Differential operators on vector bundles.**

Definition 8.1. We define the *space of differential operators* $D(M,E)$ to be the subalgebra of $\mathrm{End}(\Gamma(E))$ generated by $\Gamma(M,\mathrm{End}(E))$ and ∇_X for any covariant derivative ∇ and $X \in \Gamma(M,TM)$.

Here is an equivalent formulation.

Definition 8.2. For E a vector bundle and $\mathscr{E}$ its sheaf of sections, $D(M,E) = \mathrm{Hom}_{\mathbf{Shv}}(\mathscr{E},\mathscr{E})$ considered in the category of sheaves of vector spaces (rather than sheaves of $C^\infty(M)$-modules).

The equivalence is not obvious; it involves local analysis patched together with partitions of unity.

There's a useful filtration on $D(M,E)$ according to the number of covariant derivatives.

$$D_i(M,E) = \langle \Gamma(M,\mathrm{End}(E))\{\nabla^1_{X_1}\dots\nabla^1_{X_j} \mid j \le i\}\rangle.$$

We say that D is of *order* i if D is an ith order differential operator in local coordinates, i.e. i the smallest integer such that $D \in D_i(M,E) \subset D(M,E)$. Notice that by the Leibniz rule, D is of order i if and only if for all $f \in C^\infty$,

$$(\mathrm{ad}(f))^i D = [f,[f,\dots,[f,D]]\dots]$$

is of order zero, i.e. $C^\infty(M)$-linear.

Proposition 8.3. *The associated graded of* $D(M,E)$ *is* $\mathrm{Sym}(TM)\otimes\mathrm{End}(E)$.

Definition 8.4. If D has order k, then we define the *symbol map*

$$\sigma := \sigma_k \colon \mathrm{gr}_k(D(M,E)) \to \mathrm{Sym}^k(TM)\otimes\mathrm{End}(E)$$

by

$$\sigma_k(D)(x,\xi) = \lim_{t\to\infty} t^{-k}(e^{-itf}De^{itf})(x) \in \mathrm{End}(E)$$

for $x \in M$, $\xi \in T_xM$ and $df(x)$ dual to ξ.

Proof. By the Leibniz rule, for $D = a\nabla_{X_1}\dots\nabla_{X_k}$ we have

$$e^{-itf}De^{itf} = (it)^k\langle X_1,\xi\rangle\dots\langle X_k,\xi\rangle + O(t^{k-1}).$$

This gives the claimed isomorphism. □

Identify $\Gamma(M,\mathrm{Sym}^\bullet(TM)\otimes\mathrm{End}(E)) \subset \Gamma(T^*M,\pi^*\mathrm{End}(E))$ as the sections that are polynomial on fibers of $\pi\colon T^*M \to M$. We say that an operator D is *elliptic* if $\sigma(D) \in \Gamma(T^*M,\pi^*\mathrm{End}(E))$ is invertible away from the zero section of T^*M.

8.2. **Generalized Laplacians.**

Definition 8.5. A *generalized Laplacian* is a second-order differential operator H such that $\sigma(H)(x,\xi)=|\xi|^2$ for $x\in M,\xi\in\Gamma(TM)$. (Implicitly, we use a metric to make sense of the norm, and identify $\mathrm{End}(E)\cong E^*\otimes E^*$.)

Remark 8.6. Clearly any such H is elliptic.

Here are equivalent formulations of generalized Laplacians.

(1) $[[H,f],f]=-2|df|^2$ for any $f\in C^\infty(M)$.
(2) Locally H looks like

$$H=-\sum_{i,j}\underbrace{g^{ij}}_{\text{metric tensor}}\partial_i\partial_j+(\text{first-order}).$$

The punchline is that the Dirac operator is a "square root" of the Laplacian.

Laplacian of a connection. Let Δ^E be the composition

$$\Gamma(M,E)\xrightarrow{\nabla^E}\Gamma(M,T^*M\otimes E)\xrightarrow{\nabla^{LC}\otimes\nabla^E}\Gamma(M,T^*M\otimes T^*M\otimes E)\xrightarrow{\text{metric}}\Gamma(M,E).$$

In brief,

(1) $\Delta^E(s)=-\mathrm{Tr}(\nabla^{T^*M\otimes E}\nabla^E s)$. (The negative sign is chosen to make Δ^E positive-definite.)
(2) In a local frame,

$$\Delta^E=-\sum_{i,j}\left(\nabla^E_{e_i}\nabla^E_{e_j}-\nabla^E_{\nabla^{LC}_{e_i}e_j}\right).$$

Proposition 8.7. *For H a generalized Laplacian on E, there exists ∇^E such that $H-\Delta^E$ is 0th order, i.e. $H-\Delta^E\in\Gamma(\mathrm{End}(E))$.*

Proof. We need to show that $[H-\Delta^E,f]=0$. See BGV, page 65(ish). □

So any generalized Laplacian H can be written as

$$H=\Delta^E+F,\quad F\in\Gamma(\mathrm{End}(E)).$$

In summary, generalized Laplacians are given:

(1) a metric g on M which specifies the second-order part,
(2) a connection ∇^E on E which specifies the first-order piece.
(3) $F\in\Gamma(\mathrm{End}(E))$ which specifies the zeroth-order piece.

9. Densities and Divergences

9.1. **Density bundles.**

Definition 9.1. The *s-density bundle* of M is a line bundle associated to the frame bundle via the homomorphism

$$|\det|^{-s}\colon \mathrm{GL}(n,\mathbb{R})\to\mathbb{R}_{>0}.$$

Denote the s-density bundle by $|\Lambda|^s$. We can integrate 1-densities (just call them "densities") without orientations.

Remark 9.2. For this reason, compactly supported 1/2-densities have a canonical inner product, given by integration.

You can view a choice of orientation as an isomorphism between the top exterior power of the cotangent bundle and a density bundle.

Definition 9.3. Let $D\colon\Gamma(M,E_1)\to\Gamma(M,E_2)$ be a differential operator. Its *formal adjoint*

$$D^*\colon\Gamma(M,E_2^*\otimes|\Lambda|)^*\to\Gamma(M,E_1^*\otimes|\Lambda|)$$

is characterized by

$$\int_M\langle Ds_1,s_2\rangle=\int_M\langle s_1,D^*s_2\rangle\quad\text{for}\quad s_1\in\Gamma_c(M,E_1),s_2\in\Gamma(M,E_2^*\otimes|\Lambda|).$$

where $\Gamma_c(M,E_1)\subset\Gamma(M,E_1)$ is the subset of compactly supported functions.

If $E_1=E_2=E$ is hermitian and D acts on $\Gamma(M,E\otimes|\Lambda|^{1/2})$, then D is *symmetric* if $D=D^*$. Here we are using the canonical isomorphism $(|\Lambda|^s)^*\cong|\Lambda|^{1-s}$.

Note that $\sigma(D^*)=\sigma(D)^*\otimes 1_{|\Lambda|}$ where $1\in|\Lambda|^{1/2}\otimes(|\Lambda|^{1/2})^*$. In particular, D elliptic implies that D^* is elliptic.

Example 9.4. Suppose M is Riemannian and orientable. The exterior derivative $d\colon\Omega^\bullet(M)\to\Omega^{\bullet+1}(M)$ induces $d^\bullet\colon\Gamma(\bigwedge^\bullet TM\otimes|\Lambda|)\to\Gamma(\bigwedge^{\bullet+1}TM\otimes|\Lambda|)$. The metric induces an isomorphism $TM\cong T^*M$, which induces $\bigwedge^\bullet T^*M\cong\bigwedge^\bullet(TM\otimes|\Lambda|)$. So we get

$$d^*\colon\Omega^\bullet(M)\to\Omega^{\bullet-1}(M),$$

and using the volume form this is adjoint to d in the following sense:

$$\int_M\langle\omega,d\eta\rangle|d\,\mathrm{vol}|=\int_M\langle d^*\omega,\eta\rangle|d\,\mathrm{vol}|.$$

9.2. **Divergences.**

Definition 9.5. For $\alpha\in\Omega^1(M)$, we define the *divergence* of α to be $d^*\alpha\in C^\infty(M)$.

Observe that $\int d^*\alpha|dx|=0$ if α is compactly supported (this follows from the definition of the adjoint: $\langle d^*\alpha,1\rangle=\langle\alpha,0\rangle$).

Lemma 9.6. *For $\alpha\in\Omega^1(M)$,*

$$d^*\alpha=-\mathrm{tr}(\nabla^{LC}\alpha)$$

In particular, if $\alpha \in \Omega^1_c(M)$ we have

$$\int_M \mathrm{tr}(\nabla^{LC}\alpha)|dx| = 0.$$

Proof. In the argument we denote $\nabla = \nabla^{LC}$. For X a vector field on M, $\nabla X \in \Gamma(M, T^*M \otimes T^*M) \cong \Gamma(M, \mathrm{End}(TM))$. Since ∇ is torsion-free, we have

$$\mathscr{L}_X Y = [X,Y] = \nabla_X Y - (\nabla X)Y$$

so $\mathscr{L}_X |dx| = -\mathrm{tr}(\nabla X)|dx|$ ♠♠♠ TONY: [don't understand what's happening here]. Therefore,

$$\int_M \mathscr{L}_X f\,|dx| = \int_M X(f)|dx| = -\int_M \mathrm{tr}(\nabla X) f\,|dx|.$$

Say $X \mapsto \alpha$ under $TM \cong T^*M$. Then

$$\int_M \langle \alpha, df\rangle |dx| = \int_M X(f)|dx| = -\int_M \mathrm{tr}(\nabla\alpha) f |dx|$$

and the left hand side is $\int \langle d^*\alpha, f\rangle ||dx|$. Hence $d^*\alpha = -\mathrm{tr}(\nabla^{LC}\alpha)$. □

Proposition 9.7. *If $\iota \colon \Gamma(M, T^*M \otimes \bigwedge^\bullet T^*M) \to \Gamma(M, \bigwedge^\bullet T^*M)$ denotes the contraction operator, then we have*

$$d^* = -\iota \circ \nabla^{TM}.$$

Proof. Using the torsion-freeness of $d = \wedge \circ \nabla^{TM}$ ♠♠♠ TONY: [what is this?]

$$\langle \wedge \circ \nabla^{TM} \alpha, \beta\rangle + \langle \alpha, \iota \circ \nabla^{TM}\beta\rangle = \mathrm{Tr}(\nabla\gamma)$$

where $\gamma(X) = \langle \alpha, \iota_X \beta\rangle$. Now integration kills the right hand side, so the formal adjoint is as claimed. □

Definition 9.8. For $\mathbb{A} = \nabla + \omega$, define $\mathbb{A}^* \colon \nabla^* + \omega^*$ where $\omega^* = (-1)^{i(i+1)/2}\omega_i^\dagger$ (where † means the adjoint on the endomorphism part) for $\omega_i \in \Omega^i(M, \mathrm{End}(E))$.

Proposition 9.9. *For $\mathbb{A}$ a Clifford superconnection, $(D_{\mathbb{A}})^* = D_{\mathbb{A}^*}$.*

Proof. Consider

$$\langle D_{\mathbb{A}} s_1, s_2\rangle + \langle s_1, D_{\mathbb{A}^*} s_2\rangle.$$

By the signs in ω_i^*, the contribution from the differential form part of D drops out immediately for purely formal reasons, so this is equal to

$$\langle D_{\mathbb{A}} s_1, s_2\rangle + \langle s_1, D_{\mathbb{A}^*} s_2\rangle = \langle D_\nabla s_1, s_2\rangle + \langle s_1, D_{\nabla^*} s_2\rangle.$$

Let X be a vector field such that for $\alpha \in \Omega^1(M)$,

$$\alpha(X) = \langle s_1, \mathrm{cl}(\alpha) s_2\rangle.$$

Then ♠♠♠ TONY: [why???]

$$\langle D_\nabla s_1, s_2\rangle + \langle s_1, D_{\nabla^*} s_2\rangle = \mathrm{tr}(\nabla X).$$

The latter integrates to 0 on M, which is what we wanted (so $(D_\nabla)^* = D_{\nabla^*}$ too).

□

Now we study the formal adjoint of ∇^E.

Proposition 9.10. *For $s_1 \in \Gamma_c(E \otimes |\Lambda|^{1/2})$ and $s_2 \in \Gamma_c(E^* \otimes |\Lambda|^{1/2})$, we have*

$$\int_M \langle \nabla^{E\otimes|\Lambda|^{1/2}} s_1, s_2 \rangle = \int_M \langle s_1, \nabla^{E^*\otimes|\Lambda|^{1/2}} s_2 \rangle = \int_M \operatorname{tr}(\langle \nabla^{E\otimes|\Lambda|^{1/2}} s_1, \nabla^{E\otimes|\Lambda|^{1/2}} s_2 \rangle).$$

Proof. Exercise. $\square$

Remark 9.11. Here we are using the existence of a connection on $|\Lambda|^s$ coming from Levi-Civita such that $|dx|^s$ is parallel for all s (applied to $s = \frac{1}{2}$). In the equation above, the pairing is between forms valued in $\operatorname{End}(E) \otimes |\Lambda|^1$, and then the trace is a density, which you can integrate.

Corollary 9.12. *If ∇^E is compatible with a hermitian structure on E (which we use to identify $E \cong E^*$ and the corresponding connections), then $\nabla^{E\otimes|\Lambda|^{1/2}}$ is symmetric.*

Example 9.13. For $E = \underline{\mathbb{C}}$ and $\nabla = d$,

$$\int_M \langle \Delta f, f \rangle = \int \langle df, df \rangle \implies \Delta = d^* d.$$

This confirms what we expect from the classical cases.

Part 2. Towards the Index Theorem

10. Heat kernels

10.1. **The big picture.** Here is a vague overview of the proof of the Atiyah-Singer index theorem via heat kernels. We'll study the operator e^{-tH} for $H = D^2$. In the limit as $t \to \infty$, $e^{-tH} = e^{-tD^2} =: P_t$ will be a projection onto $\ker H = \ker D^2$, which coincides with $\ker D$ if D is self-adjoint. In particular, its supertrace is the superdimension of $\ker D$, so by Lemma 6.7 we have

$$\mathrm{str}(e^{-tD^2}) = \mathrm{ind}(D).$$

The idea of the proof is to carry out the following two steps:

(1) show that $\mathrm{str}(e^{-tD^2})$ is independent of t,
(2) analyze $\mathrm{str}(e^{-tD^2})$ in the limit of small t, using an asymptotic expansion. We'll show that str is an integral of a differential form, which gives the right hand side of the index theorem.

The main tool used in proving the Index Theorem is analyzing the heat kernel on generalized Laplacians.

10.2. **Heat kernel on $\mathbb{R}$.** On $\mathbb{R}^n$ the Laplacian is $\Delta = -\frac{d}{dx^2}$. We define its associated *heat kernel* to be

$$\boxed{q_t(x,y) = (4\pi t)^{-1/2} e^{-(x-y)^2/4t}.}$$

This satisfies the heat equation in the following sense:

$$(\partial_t + \Delta_x) q_t(x,y) = 0.$$

You can use this kernel to cook up general solutions to the heat equation with boundary conditions. We'll approach this in a way that generalizes well to manifolds.

Definition 10.1. Let $C^\ell(\mathbb{R})$ be the space of C^ℓ functions on $\mathbb{R}$ with the norm

$$||\phi||_\ell := \sup_{k\le \ell} \sup_{x\in\mathbb{R}} \left| \frac{d^k}{dx^k}\phi \right|.$$

Define the operator

$$Q_t(\phi)(x) = \int_{\mathbb{R}} q_t(x,y)\phi(y)\,dy.$$

The point is that $Q_t(\phi)(x)$ should give solutions to the heat equation with initial condition ϕ. To this end, we have the following quantitative estimate:

Proposition 10.2. *Let $\ell \in 2\mathbb{Z}$ and $\phi \in C^{\ell+1}(\mathbb{R})$ with $||\phi||_{\ell+1} < \infty$. Then*

$$||Q_t\phi - \sum_{k=0}^{\ell/2} \frac{(-1)^k}{k!}\Delta^k \phi|| \le O(t^{\ell/2+1}).$$

Roughly speaking, $Q_t\phi \approx e^{-t\Delta}\phi$ for small t.

Remark 10.3. This Q shows up in quantum physics.

Proof. Just a computation, using Taylor's theorem. □

10.3. **Kernels.** For C^ℓ-sections $\Gamma^\ell(M,E)$ with a C^ℓ norm, denote the *topological dual* by $\Gamma^{-\ell}(M,E)$. In particular, $\Gamma^{-\infty}(M,E)$ is dual to $\Gamma^\infty(M,E):=\Gamma(M,E)$.

Remark 10.4. We construct a norm on M by partitions of unity. The norm is not canonical, but the topology is if M is compact.

Definition 10.5. For $E_1,E_2\to M$, the *exterior tensor product* of E_1 and E_2 is

$$E_1\boxtimes E_2 := p_1^* E_1\otimes p_2^* E_2,$$

a bundle on $M\times M$.

Definition 10.6. A section $p\in\Gamma(M\times M,(F\otimes|\Lambda|^{1/2})\boxtimes(E^*\otimes|\Lambda|^{1/2}))$ is a *kernel*. Given such a kernel p, define an operator

$$P:\Gamma^{-\infty}(M,E\otimes|\Lambda|^{1/2})\to\Gamma(M,F\otimes|\Lambda|^{1/2})$$

by

$$P(s)(x)=\int_M p(x,y)s(y)\,dy$$

Linear maps $P:\Gamma^{-\infty}(M,E\otimes|\Lambda|^{1/2})\to\Gamma(M,F\otimes|\Lambda|^{1/2})$ are called *smoothing operators.* If P_1 and P_2 have kernels p_1 and p_2, then $P_1\circ P_2$ has kernel

$$p_{12}(x,y)=\int_M p_1(x,z)p_2(z,y)\,dz.$$

Theorem 10.7 (Schwartz)**.** *The assignment $p(x,y)\mapsto P$ from kernels to smoothing operators is an isomorphism between smooth kernels and bounded linear operators.*

This is a remarkable fact!

Since this field is highly intertwined with physics, we'll use the *Dirac notation*:

$$p(x,y)=\langle x\mid P\mid y\rangle$$

so

$$(P\phi)(x)=\int_{Y\in M}\langle x\mid P\mid y\rangle\phi(y)\,dy.$$

There are two approaches to analyzing generalized Laplacians:

(1) Given a symmetric H, find a self-adjoint extension $\overline{H}$ on some L^2 completion of sections, apply the spectral theorem and use the theory of elliptic operators to analyze the kernel of H.
(2) Find a smooth kernel for H and analyze it.

10.4. **Heat kernels in general.**

Definition 10.8. Let H be a generalized Laplacian on $E\otimes|\Lambda|^{1/2}$. A *heat kernel* for H is a continuous section $p_t(x,y)$ of $(E\otimes|\Lambda|^{1/2})\boxtimes(E^*\otimes|\Lambda|^{1/2})$ over $\mathbb{R}_{>0}\times M\times M$ such that

(1) $\frac{\partial p_t(x,y)}{\partial t}$ is continuous (i.e. $p_t(x,y)$ is C^1 in t),
(2) $\frac{\partial^2 p_t(x,y)}{\partial x^i\partial x^j}$ is continuous (i.e. $p_t(x,y)$ is C^2 in $x\in M$),
(3) $(\partial_t+H_x)p_t(x,y)=0$,

(4) For $s \in \Gamma(E \otimes |\Lambda|^{1/2})$ and

$$(P_t s)(x) := \int_{y \in M} p_t(x,y)s(y),$$

we have $\boxed{\lim_{t \to 0} P_t s = s}$ for the limit in the sup norm:

$$||s||_0 = \sup_{x \in M} ||s(x)||_E.$$

Goal. Figure out when heat kernels exist and when they are unique.

We will tackle uniqueness first.

Lemma 10.9 (Uniqueness of homogeneous solution)**.** *Assume H^* has heat kernel p_t^*. Let $s(x,t)\colon \mathbb{R}_{>0} \to C^0(E \otimes |\Lambda|^{1/2})$ be a 1-parameter family of sections such that*

(1) s_t is C^1 in t,
(2) $s_t(x)$ is C^2 in x,
(3) $(\partial_t + H^*)s_t = 0$,
(4) $\lim_{t\to 0} s_t = 0$.

Then $s_t \equiv 0$.

Proof. For a test section $u \in \Gamma_c(E \otimes |\Lambda|^{1/2})$, define

$$f_t(\theta) = \int_{(x,y)\in M\times M} \langle s_\theta(x), p^*_{t-\theta}(x,y)u(y)\rangle.$$

Differentiating with respect to θ, and using (3) in the assumptions, one gets that f_t is constant in θ. Now taking the limit as $\theta \to t$, we find that

$$\int_{x\in M} \langle s_t(x), u(x)\rangle \equiv \text{constant}.$$

Taking $t \to 0$, we find that the constant is 0. Since this holds for all u, $s_t(x) \equiv 0$. □

Exercise 10.10. Start with the usual energy estimate for uniqueness of solutions to the heat equation and try to derive this argument.

Proposition 10.11. *Suppose there exists a heat kernel p_t^* for H^*. Then there exists at most one heat kernel for H, which is described by $(p_t^*(x,y))^* = p_t(x,y)$.*

Proof. Let

$$f(\theta) = \int_M \langle (P_\theta s)(x), (P^*_{t-\theta} u)(x)\rangle$$

for $0 < \theta < t$ and $s, u \in \Gamma(M, E \otimes |\Lambda|^{1/2})$. We claim that this is independent of θ, and $\langle P_t s, u\rangle = \langle s, P_t^* u\rangle$. This implies uniqueness and the formula. □

10.5. **Construction of the heat kernel.** We now go into the construction of the heat kernel. As a first guess, one might try to construct a heat kernel by patching together local heat kernels. This doesn't quite work; it only produces an approximation. One can prove, with some effort, that these approximations converge to the right thing in some appropriate setting.

Warm-up: the finite-dimensional case. Let V be a *finite*-dimensional vector space. (Later, we'll want to take $V = \Gamma(M, E \otimes |\Lambda|^{1/2})$, which is not finite-dimensional.) Let $H \in \mathrm{End}(V)$ (later $H = \Delta^{E\otimes|\Lambda|^{1/2}} + F$).

Suppose we are given "approximation" solutions $K_t : \mathbb{R}_{>0} \to \mathrm{End}(V)$ with "remainder" R_t, i.e.

$$R_t = \frac{dK_t}{dt} + HK_t = O(t^\alpha) \quad \alpha > 0$$

with $K_0 = 1$. We want to give a recipe for e^{-tH} by approximation in terms of these K_t.

In the finite-dimensional case, formally we have $e^{-tH} = \sum_k \frac{(-tH)^k}{k!}$. The K_t will be something like the finite truncations of this power series.

Definition 10.12. The *k-simplex* $\Delta^k \subset \mathbb{R}^k$ is

$$\{(t_0, \dots, t_k) \mid 0 \le t_1 \le \dots \le t_k \le 1\}.$$

Useful coordinates are $\sigma_1 = t_1, \sigma_i = t_i - t_{i-1}$. These make it clear that $\mathrm{vol}(\Delta^k) = \frac{1}{k!}$. Write $t \cdot \Delta^k$ for the rescaled k-simplex with

$$0 \le t_1 \le t_2 \le \dots \le t.$$

Then $\mathrm{vol}(t \cdot \Delta^k) = \frac{t^k}{k!}$.

Proposition 10.13. *Let $Q_t^k : \mathbb{R}_{>0} \to \mathrm{End}(V)$ be defined by*

$$Q_t^k = \int_{t\Delta^k} K_{t-t_k} R_{t_k - t_{k-1}} \dots R_{t_2 - t_1} R_{t_1}\, dt_1 \dots dt_k.$$

In particular, $Q_t^0 = K_t$. Then

$$\sum_{k\ge 0} (-1)^k Q_t^k = e^{-tH} =: P_t$$

where $P_t = K_t + O(t^{\alpha+1})$.

Remark 10.14. Q_t^k is something like a path integral.

Proof. Observe the elementary identity

$$\frac{d}{dt} \int_0^t a(t-s) b(s)\, ds = a(0) b(t) + \int_0^t \frac{da}{dt}(t-s) b(s)\, ds.$$

Applying this with $a(s) = K_s$ and $b(s) = R_s^{(k)}$ with

$$R_s^{(k)} = \int_{s\Delta^{k-1}} R_{s-t_{k-1}} \dots R_{t_2 - t_1} R_{t_1}\, dt_1 \dots dt_{k-1}$$

and noting that $K_0 = 1$, we get

$$(\partial_t + H)Q_t^k = R_t^{(k+1)} + R_t^{(k)}.$$

Then the point is that $\sum_{k\geq 0}(-1)^k Q_t^k$ telescopes after applying $\partial_t + H$, hence is formally a solution to the heat equation.

We need to show that the sum also converges, and to prove the estimate on $P_t - K_t$. For small t, we have uniform bounds on $t\Delta^k$: $|K_{t-t_k}| \leq C_0$ and $|R_{t_{i+1}-t_i}| \leq C t^\alpha$ by assumption. Therefore, by the volume estimate

$$|Q_t^k| \leq C_0 C^k t^{k\cdot\alpha} \frac{t^k}{k!}$$

so the sum converges with the claimed estimate. □

Example 10.15. A special case is the *Volterra series.* Let $H = H_0 + H_1$, so we know that the solution is "$e^{-t(H_0+H_1)}$. Suppose we take $K_t = e^{-tH_0}$. Then $R_t = H_1 e^{-tH_0}$, and the recipe gives

$$e^{-t(H_0+H_1)} = e^{-tH_0} + \sum_{k=1}^{\infty}(-1)^k I_k$$

where

$$I_k = \int_{t\Delta^k} e^{-(t-t_k)}H_0 H_1 e^{-(t_k - t_{K-1})H_0} \dots H_1 e^{-t_1}.$$

Example 10.16. For a smooth, one-parameter family of operators H_z and

$$H_1 = \epsilon \frac{dH_z}{dz}$$

we get

$$\frac{d}{dz}(e^{-tH_z}) = \int_0^t e^{-(t-s)H_z} \frac{dH_z}{dz} e^{-sH_z}\, dz.$$

This kind of formula is important once we start considering deforming our Laplacians.

The plan is to generalize what we did above to infinite-dimensional spaces. We proceed as follows.

(1) Construct an approximate solution $K_t(x,y)$ to the heat equation and study the remainder $r_t(x,y) = (\partial_t + H_x)K_t(x,y)$,
(2) Prove the convergence of

$$\sum_{k=0}^{\infty}(-1)^k \int_{t\Delta^k}\int_{M^k} K_{t-t_k}(x, z_k) r_{t_k - t_{k-1}}(z_k, z_{k-1}) \dots r_{t_1}(z,x).$$

Theorem 10.17. *For every $N > 0$, there exists $K_t^N(x,y)$ such that for all ℓ, the following three conditions are met:*

(1) for all T, there exists a uniform bound on the operators $K_t^N \in \mathscr{L}(\Gamma^\ell(M, E\otimes|\Lambda|^{1/2}))$ for $0 < t < T$, and
(2) for every $s \in \Gamma^\ell(M, E\otimes|\Lambda|^{1/2})$ we have $\lim_{t\to 0} K_t s = s$ with respect to the $\|-\|_\ell$-norm,

(3) $r_t(x,y) = (\partial_t + H_x)K_t^N(x,y)$ satisfies $\|r_k\|_\ell \le C(\ell)t^{(N-n/2)-\ell/2}$.

For now, let's just accept this and move on to complete the construction assuming it.

Fix N and write K_t for K_t^N. Consider the operator

$$Q_t^i = \int_{t\Delta^i} K_{t-t_i} R_{t_i - t_{i-1}} \dots R_{t_1}$$

with kernel

$$q_t^i(x,y) = \int_{t\Delta^i}\int_{M^k} K_{t-t_i}(x,z_i) r_{t_i-t_{i-1}}(z_i, z_{i-1}) \dots r_{t_1}(z_1,y).$$

For sufficiently large N, we want to show that this integral converges and q_t^i is differentiable up to a degree depending on N.

Estimates. We use the following estimates:

(1) (Remainder estimate) Let

$$r_t^{i+1} = \int_{t\Delta^i}\int_{M^i} r_{t-t_i}(x,z_i) r_{t_i-t_{i-1}}(z_i,z_{i-1}) \dots r_{t_1}(z,y).$$

For $N > \frac{n+\ell}{2}$, r_t^{i+1} is C^ℓ with respect to x,y and

$$\|r_t^{i+1}\|_\ell \le C^{i+1} t^{(i+1)(N-n/2)-\ell/2} \operatorname{vol}(M)^i \frac{t^i}{i!}.$$

(2) Let $N > \frac{n+\ell}{2}$ and $\ell \ge 1$. Then:
- $q_t^i(x,y)$ is C^ℓ with respect to x,y and there exists a constant $\widetilde{C}$ such that
$$\|q_t^i\|_\ell < \widetilde{C} C^i \operatorname{vol}(M)^{i-1} t^{(N-n/2)i-\ell/2} t^i/(i-1)!$$
- $q_t^i(x,y)$ is C^1 in t and
$$(\partial_t + H_x) q_t^i(x,y) = r_t^{i+1}(x,y) + r_t^i(x,y).$$

Theorem 10.18. *Assume that $K_t^N(x,y)$ satisfies the conditions of Theorem 10.17 with $N > n/2+1$. Then:*

(1) For all ℓ such that $N > (n+\ell+1)/2$,

$$P_t(x,y) = \sum_{k=0}^\infty (-1)^i q_t^i(x,y),$$

converges in the $\|\cdot\|_{\ell+1}$-norm over $M\times M$, and defines a C^1 map

$$\mathbb{R}_{>0} \to \Gamma^\ell(M\times M, (E\otimes |\Lambda|^{1/2}) \boxtimes (E^* \otimes |\Lambda|^{1/2}))$$

with $(\partial_t + H_x)P_t(x,y) = 0$.

(2) *K_t^N approximates P_t:*

$$|\partial_t^i (P_t - K_t^N)||_\ell = O(t^{(N-n/2)-i-\ell/2+1})$$

as $t \to 0$.

(3) *P_t is a heat kernel for H.*

Proof Sketch. The estimate on $||q_t^i(x,y)||_{\ell+1}$ proves the convergence of the series in (1). That this satisfies the heat equation follows from telescoping plus the estimate.

(2) follows from the estimates.

(3) only requires that P_t has the right initial condition, since K_t^N has the correct initial condition and the estimate in (2) gives the same for P_t.

□

Now we embark on the proof of Theorem 10.17. To get K_t^N, we'll construct a "formal" solution to the heat equation. Roughly, we locally set

$$K_t(x,y) = q_t(x,y) \sum_{i=0}^{\infty} t^i \Phi_i(x,y,H)$$

where q_t is the Euclidean heat kernel

$$q_t(x,y) = (4\pi t)^{-n/2} e^{-||x||^2/4t} |dx|^{1/2}$$

and $\Phi_i \in \Gamma(E \boxtimes E^*)$ are sections defined only on a neighborhood of the diagonal in $M \times M$.

Riemann normal coordinates. The exponential map $T_{x_0}M \to M$ restricts to a diffeomorphism onto its image

$$B_\epsilon(0) \xrightarrow{\exp} M$$

for some $\epsilon > 0$. This gives local coordinates on M, with the special property that if

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} \sum_{k,\ell} R_{ijk\ell}(x_0) x^k x^\ell + \text{(higher-order terms)}$$

are the coordinates of the metric, then the first derivatives of g_{ij} vanish at the origin.

Let $j(x)$ be the Jacobian of $\exp_x$, i.e.

$$\exp_x^* d\,\mathrm{vol}_M = j(x) d\,\mathrm{vol}_{\mathbb{R}^n} = \det(g_{ij}(x))^{1/2} d\,\mathrm{vol}_{\mathbb{R}^n}.$$

Let R be the radial vector field $R = \sum_i x^i \partial_{x_i}$.

Proposition 10.19. *In Riemann normal coordinates,*

(1) *For $f \in C^\infty(M)$,*

$$\nabla(f|dx|^{1/2}) = (df - \frac{1}{2} f\, d\log(j))|dx|^{1/2}.$$

(2) *For $f \in C^\infty(M)$,*

$$\Delta(f|dx|^{1/2}) = (j^{1/2} \circ \Delta \circ j^{-1/2})(f)|dx|^{1/2}$$

where on the right hand side f is regarded as a function on $\mathbb{R}^n$ via Riemann normal coordinates.

(3) $\Delta(||x||^2) = -2(n + R\log j)$.
(4) *For* $q_t(x) = (4\pi t)^{-n/2} e^{-||||^2/4t} |dx|^{1/2}$,

$$[\partial_t + \Delta - j^{1/2}(\Delta \cdot j^{-1/2})]q_t = 0.$$

Let's rephrase. For $y \in M$ and $v \in \mathbb{R}^n \cong T_y M$ set $x = \exp_y(v) \in M)$. Then we have

$$\begin{aligned} q_t(x,y) &= (4\pi t)^{-n/2} e^{-||v||^2/4t} \, |dv|^{1/2} \\ &= (4\pi t)^{-n/2} e^{-d(x,y)/4t} \, j(v)^{-1/2} |dx|^{1/2}. \end{aligned}$$

Let $j^{1/2}$ denote the function on M (in a neighborhood of fixed $y \in M$) given by $j^{1/2}(x) := j^{1/2}(v)$ for $x = \exp_y(v)$.

Fix $y \in M$ and define $q_t \in \Gamma(|\Lambda|^{1/2})$ by $x \mapsto q_t(x,y)$, which is a section of some open neighborhood of y.

Proposition 10.20. *Define an operator B on $\Gamma(E)$ in a neighborhood of y by*

$$B := (j^{-1/2}|dx|^{-1/2}) \circ H \circ (j^{1/2}|dx|^{1/2}).$$

Then for $s_t \in \Gamma(E)$ depending smoothly on t,

$$(\partial_t + H)(s_t q_t) = ((\partial_t + t^{-1}\nabla^E_R + B)s_t)q_t$$

where R is the radial vector field in Riemann normal coordinates.

Remark 10.21. The B is basically the "transport" of H to $\mathbb{R}^n$, and then the equation expresses the "correction" relating the heat kernel on $\mathbb{R}^n$ with that on M.

Definition 10.22. Let $\sum_{i=0}^{\infty} t^i \Phi_i(x,y)$ be a formal power series in t whose coefficients are smooth sections of $E \boxtimes E^*$ defined in a neighborhood of the diagonal in $M \times M$. We say that $q_t(x,y)\sum_{i=0}^{\infty} t^i \Phi_i(x,y)|dy|^{1/2}$ is a *formal solution of the heat equation near y* if the local section $(x \mapsto \sum t^i \Phi_i(x,y)) \in \Gamma(\mathrm{End} \otimes E_y^*)$ in a neighborhood of y satisfies

$$(\partial_t + t^{-1}\nabla^E_R + B)\sum_i t^i \Phi_i(-,y) = 0.$$

Theorem 10.23. *There exists a unique formal solution*

$$k_t(x,y) = q_t(x,y)\sum_{i=0}^{\infty} t^i \Phi_i(x,y,H)|dy|^{1/2}$$

such that $\Phi_0(y,y,H) = \mathrm{Id} \in \mathrm{End}(E_y) \cong E_y^* \otimes E_y$. *Moreover* $\Phi_0(x,y,H)$ *satisfies* $\Phi_0(x,y,H) =$ *Par*$(x,y): E_y \to E_x$, *parallel transport using ∇^E along the unique geodesic from y to x.*

Remark 10.24. There are "explicit formulas" for the Φ_i.

Proof Sketch. From $(\partial_t + t^{-1}\nabla^E_R + B)\sum_i t^i \Phi_i(-,y) = 0$, we get:
(1) $\nabla_R \Phi_0 = 0$,
(2) $(\nabla_R + i)\Phi_i = -B\Phi_{i-1}$ for $i > 0$,

Together with $\Phi_0(y,y) = \mathrm{Id}$, the first condition implies that Φ_0 is parallel transport.

For higher i, use existence and uniqueness for solutions to ODE. □

Let $\psi_\epsilon : \mathbb{R}_{>0} \to [0,1]$ be a smooth function satisfying

$$\begin{cases} \psi_\epsilon(s) = 1 & s < \frac{\epsilon^2}{4}, \\ \psi_\epsilon(s) = 0 & s > \epsilon^2. \end{cases}$$

This is a cutoff function we'll use to patch solutions. For ϵ smaller than the injectivity radius of exp, define

$$k_t^N(x,y) = \psi_\epsilon(d(x,y)^2) q_t(x,y) \sum_{i=0}^{N} t^i \Phi_i(x,y,H) |dy|^{1/2}.$$

Theorem 10.25. *Let ℓ be an even, positive integer. Then*

(1) For any $T > 0$, the kernels k_t^N for $0 \le t \le T$ define a uniformly bounded family of operators K_t^N on $\Gamma^\ell(M, E \otimes |\Lambda|^{1/2})$, and

$$\lim_{t \to 0} \|K_t^N s - s\|_\ell = 0.$$

(2) For all j, there exist differential operators D_k of order $\le 2k$ such that $D_0 = \mathrm{Id}$ and for all $s \in \Gamma^{\ell+1}(M, E \otimes |\Lambda|^{1/2})$,

$$\|K_t^N s - \sum_{k=0}^{\ell/2-j} D_k s\|_{2j} = O(t^{(\ell+1)/2-j}).$$

(3) The kernel $r_t^N(x,y) = (\partial_t + H_x) k_t^N(x,y)$ satisfies

$$\|\partial_t^k r_t^N\| < C(\ell,k) t^{(N-n/2)-k-\ell/2}.$$

Proof Sketch. (1) follows from chasing definitions.
(2) follows from Taylor's remainder theorem.
(3) follows from using features of formal solutionsto the heat equation. □

Using the "path integral" trick, we can use $k_t^N(x,y)$ to build the heat kernel for H, which is a priori C^ℓ. But by uniqueness, these must agree for all ℓ, so $k_t^N(x,y)$ is actually C^∞. Call this smooth kernel $p_t(x,y)$.

Summary. Let $p_t(x,y)$ be the heat kernel of $H = \Delta^{E \otimes |\Lambda|^{1/2}} + F$. There exist smooth sections $\Phi_i \in \Gamma(M \times M, E \boxtimes E^*)$ such that for all $N > \frac{\dim M}{2}$, the kernel $k_t^N(x,y,H)$ defined by

$$(4\pi t)^{-n/2} e^{-d(x,y)^2/4t} + (d(x,y)^2) \sum_{i=0}^{N} t^i \Phi_i(x,y,H) j(x,y)^{-1/2} |dx|^{1/2} \otimes |dy|^{1/2}$$

is asymptotic to $p_t(x,y,H)$:

$$\|\partial_t^k (p_t(x,y,H) - k_t^N(x,y,H)\| = O(t^{N-n/2-\ell/2-k}).$$

The leading term $\Phi_0(x,y,H)$ is parallel transport in E via ∇^E along the unique geodesic from y to x.

For $s \in \Gamma(M, E \otimes |\Lambda|^{1/2})$,

$$(P_t s)(x) = \int p_t(x, y, H) s(y)$$

with asymptotic expansion

$$||P_t s - \sum_{j=0}^{k} \frac{(-tH)^j}{j!} s|| = O(t^{k+1})$$

so $P_t \sim e^{-tH}$.

11. Some Harmonic Analysis

11.1. **Operators on Hilbert space.** As outlined earlier, the general plan of proof of the Index Theorem is as follows: we'll want to calculate $\mathrm{str}(e^{-tH})$ in two ways, where $H = \Delta^{E\otimes|\Lambda|^{1/2}} + F$ is a generalized Laplacian, and then prove that the two are equal.

In the limit of large t, the super-trace measures the index. In the small t limit, we will use the asymptotic expansion.

For the rest of this section, we assume that M is a *compact* manifold.

Definition 11.1. Let $\mathscr{H}$ be a Hilbert space. For B a linear operator on $\mathscr{H}$, we define the *operator norm* of B to be

$$||B|| = \inf\{C \geq 0 \colon ||Bv|| \leq C||v|| \text{ for all } v \in \mathscr{H}\}$$

Let $\mathscr{B}(\mathscr{H})$ denote the space of bounded linear operators on a Hilbert space $\mathscr{H}$.

Definition 11.2. A bounded operator is *Hilbert-Schmidt* if

$$||A||^2_{HS} := \sum ||Ae_i||^2 = \mathrm{Tr}(A^*A) = \sum_{i,j} |\langle Ae_i, e_j\rangle|$$

is finite (intuitively, this is the "matrix norm"). The set of Hilbert-Schmidt operators forms a Hilbert space, with inner product

$$\langle A, B\rangle = \mathrm{Tr}(A^*B) = \sum_{i,j} \langle Ae_i, Be_j\rangle.$$

Let $\Gamma_{L^2}(M, E\otimes|\Lambda|^{1/2})$ be the Hilbert space of square integrable sections of $E\otimes|\Lambda|^{1/2}$ for a choice of hermitian inner product on E, i.e. the completion of $\Gamma_c(M, E\otimes|\Lambda|^{1/2})$ with respect to the induced inner product.

Proposition 11.3. *The topological vector space underlying the Hilbert space $\Gamma_{L^2}(M, E\otimes|\Lambda|^{1/2})$ is independent of a choice of hermitian structure on E.*

Proof. Let h_1, h_2 be two such hermitian pairings on E. Then there exists $A \in \Gamma(M, \mathrm{End}(E))$ invertible with

$$h_1(s,s) = h_2(As, As).$$

Then A determines a bounded operator on $\Gamma_{L^2}(M, E\otimes|\Lambda|^{1/2})$ with inverse given by the bounded (because M is compact!) operator A^{-1}. □

Example 11.4. For an operator K determined by a kernel $k(x,y)$ in $\Gamma_{L^2}(M\times M, (E\otimes|\Lambda|^{1/2})\boxtimes(E^*\otimes|\Lambda|^{1/2}))$, we have

$$||K||_{HS} = \int_{M\times M} \mathrm{Tr}(k(x,y)^* k(x,y)).$$

As this is finite, K is Hilbert-Schmidt.

Definition 11.5. An operator K is *trace class* if it can be written as $K = AB$ for Hilbert-Schmidt operators A, B. For such a K, we can define the *trace* $\mathrm{Tr}(K) = \mathrm{Tr}(AB)$. Then $\sum_i \langle Ke_i, e_i\rangle$ is absolutely summable so $\mathrm{Tr}\, K = \sum_i \langle Ke_i, e_i\rangle$ is well-defined.

Example 11.6. The inclusions of operators on Hilbert spaces

$$\{\text{trace class}\} \subset \{\text{Hilbert Schmidt}\} \subset \{\text{compact}\} \subset \{\text{bounded}\}$$

is analogous to the inclusion for sequence spaces

$$\ell^1 \subset \ell^2 \subset c_0 \subset \ell^\infty$$

and the norm on trace class is $||A|| = \mathrm{Tr}(AA^*)^{1/2}$.

The restriction of $k(x,y)$ to the diagonal $M \xrightarrow{\Delta} M \times M$ gives a section of $\mathrm{End}(E) \otimes |\Lambda|$. Then taking the trace gives a section $\mathrm{Tr}\, k(x,x) \in \Gamma(M, |\Lambda|)$.

Proposition 11.7. *Let p_t be the heat kernel of a generalized Laplacian on M and P_t the corresponding operator. Then P_t is trace class with trace*

$$\mathrm{Tr}(P_t) = \int_M \mathrm{Tr}(p_t(x,x)).$$

Proof. To show that P_t is trace class, we use the semigroup property

$$P_t \circ P_s = P_{t+s}.$$

Then we can write $P_t = P_{t/2} \circ P_{t/2}$, and each $P_{t/2}$ is Hilbert-Schmidt (since it has a smooth kernel as constructed in the previous section), P_t is trace class.

For the second part,

$$\begin{aligned}
\mathrm{Tr}\, P_t &= \langle P_{t/2}, P_{t/2}^* \rangle_{HS} \\
&= \int_{(x,y)\in M^2} \mathrm{Tr}(p_{t/2}(x,y) p_{t/2}(y,x)) \\
&= \int_M \mathrm{Tr}(p_t(x,x))
\end{aligned}$$

□

11.2. **Unbounded operators.** One of the “problems” with the Laplace operator is that it is unbounded. We’ll sketch how to work around this.

Definition 11.8. An *unbounded operator* H on $\mathscr{H}$ is a subspace $V \subset \mathscr{H}$ and a map $V \to \mathscr{H}$.

Example 11.9. The model to keep in mind is where V is the subspace of differentiable functions of $L_2(M)$, and the map is differentiation.

A *formal adjoint* of H is another unbounded operator H^* (implicitly, with the same domain) such that

$$\langle H^* v, w \rangle = \langle v, H w \rangle.$$

Our case of interest is the unbounded operator $H = \Delta^{E \otimes |\Lambda|^{1/2}} + F$ on $\Gamma_{L^2}(M, E \otimes |\Lambda|^{1/2})$ with domain $\Gamma(M, E \otimes |\Lambda|^{1/2})$ (smooth sections). Its formal adjoint has the same domain. When H is symmetric, there exist a unique self-adjoint extension of H to $\Gamma_{L^2}(M, E \otimes |\Lambda|^{1/2})$, which we’ll call $\overline{H}$, or just H when context makes it clear.

From now on, assume that H is symmetric. Then $P_t \circ P_s = P_{t+s} = P_{s+t} = P_s \circ P_t$, so we can simultaneously diagonalize P_t for all t (noting that P_t is Hilbert-Schmidt, hence compact, hence diagonalizable). We get eigensections ϕ_i with $P_t \phi_i = e^{-t\lambda_i}\phi_i$ for some λ_i.

Facts.

- ϕ_i are smooth sections,
- λ_i are bounded below, because the P_t are bounded
- $P_t\phi_i$ satisfies the heat equation, so $H\phi_i = \lambda_i\phi_i$
- $\sum_i e^{-t\lambda_i}$ is finite for all $t > 0$

Proposition 11.10. *If H is a symmetric generalized Laplacian, then $\overline{H}$ has discrete spectrum, bounded below, and each eigenspace is finite-dimensional with eigenvectors given by smooth sections of $E \otimes |\Lambda|^{1/2}$.*

It follows from the spectral theorem that

$$\boxed{p_t(x,y) = \sum_j e^{-t\lambda_j}\phi_i(x) \otimes \overline{\phi_j(y)}.}$$

11.3. **Green operators.**

Definition 11.11. The *Green operator* of a positive Laplacian H is the L^2-inverse of H, denoted G.

Remark 11.12. G is bounded.

Proposition 11.13. *We have the integral formula*

$$G^k = \frac{1}{(t-1)!}\int_0^\infty e^{-tH}t^{k-1}\,dt.$$

Proof. Let ϕ be an eigenvector of H with eigenavlue λ. Then

$$G^k\phi = \frac{1}{(k-1)!}\int_0^\infty e^{-t\lambda}t^{k-1}\,dt\,\phi = \lambda^{-k}\phi.$$

It remains to show that

$$g^k(x,y) = \frac{1}{(k-1)!}\int_0^\infty p_t(x,x)t^{k-1}\,dt$$

converges and defines a kernel that is the kernel of G^k. For large t, this is an eigenvalue estimate, and for small t one uses the asymptotic expansion for p_t. □

From the proof, we get an estimate on the smoothness of $g^k(x,x)$: it's C^ℓ for $\ell \le 2k - \dim M - 1$.

For non-positive H, define G to be the inverse of H on $(\ker H)^\perp$. Then if H has eigenvalues $\lambda_{-m} \le \ldots \le \lambda_{-1} \le 0$, we have

$$G^k = \frac{1}{(k-1)!}\int_0^\infty P_{(0,\infty)}e^{-tH}t^{k-1}\,dt + \sum_{i=1}^m (\lambda_{-i})^{-k}P_{-\lambda_i}.$$

Here the $P_?$ are the projections to various eigenspaces. ♠♠♠ TONY: [I don't see why there would be only finitely many negative eigenvalues]

Corollary 11.14 (Elliptic regularity)**.** *Let H be a generalized Laplacian.*

(1) If $s \in \Gamma_{L^2}(M, E \otimes |\Lambda|^{1/2})$ has $H^k s$ is in L^2 for all k, then s is smooth.
(2) If A commutes with H and is bounded on Γ_{L^2} and $s \in \Gamma(M, E \otimes |\Lambda|)$ is smooth, then $A(s)$ is smooth.

♠♠♠ TONY: [something doesn't make sense; isn't H supposed $L^2 \to L^2$ and invertible?]

Proof. Without loss of generality H is positive (otherwise add a large constant to H). Then $\mathrm{Id} = G^k \circ H^k = H^k \circ G^k$, so $s = G^k \circ H^k s$. Since G^k smooths, we get that $s \in C^\ell$ for $\ell \leq 2k - \dim M - 1$. So as $k \to \infty$ we get (1).

(2) is similar: $As = (G^k \circ H^k)As = G^k(H^k As) = G^k A(H^k s)$, and then apply the same argument as in (1). □

11.4. **"Can you hear the shape of a drum?"** Kac raised this queston in '66, basically asking if you can detect the "geometry" of a manifold from the analytic data of its Laplacian.

We know that

$$\mathrm{Tr}(e^{-t\Delta}) = \int_M \mathrm{Tr}(p_t(x,x)) \sim (4\pi t)^{-n/2} \sum_i t^i \int_M \mathrm{Tr}(\Phi_i(x,x,H))\, dx.$$

We know that $\Phi_0(x,x,H) = \mathrm{Id}_E$.

Theorem 11.15 (Weyl)**.** *As $t \to 0$,*

$$\mathrm{Tr}(e^{-t\Delta}) \sim \sum_i e^{-t\lambda_i} (4\pi t)^{-n/2} \mathrm{rank}(E)\mathrm{vol}(M) + O(t^{-n/2+1}).$$

Remark 11.16. In two dimensions, for $H = \Delta$ a scalar Laplacian the integral

$$\int \mathrm{Tr}\,\Phi_1(x,x,H)$$

becomes

$$\frac{1}{6}\int k(x)\,dx$$

where $k(x)$ is the curvature. So by Gauss-Bonnet, Δ has the information of the topology of M.

Theorem 11.17 (Karamata, Weyl)**.** *If $N(\lambda)$ is the number of eigenvalues of H less than λ, then*

$$N(\lambda) \sim \frac{\mathrm{rank}(E)\mathrm{vol}(M)}{(4\pi)^{n/2}\Gamma(n/2+1)} \lambda^{n/2}.$$

12. Proof of the Index Theorem

12.1. The McKean-Singer Formula.

Proposition 12.1. *If D is a Dirac operator on E and $f \in C^\infty(M)$, then there is an equality of operators*

$$[D, f] = \mathrm{cl}(df).$$

Proof. Recall that D is the composition

$$\Gamma(E) \xrightarrow{\mathbb{A}} \Gamma(E) \otimes \bigwedge^{\bullet} T^*M \xrightarrow{\mathrm{cl}} \Gamma(E).$$

By the Leibniz rule for $\mathbb{A}$, we basically get the formula immediately: if we try to commute multiplication by f with $\mathbb{A}$, the part left is the "connection part" of D, giving df, and then we compose with cl. □

Corollary 12.2. *The operator D is first-order elliptic. The operator D^2 has symbol* $\mathrm{cl}(df)^2 = -||df||^2$*, so D^2 is a generalized Laplacian.*

Theorem 12.3 (McKean-Singer, 1967)**.** *Let $\langle x \mid e^{-tD^2} \mid y \rangle$ be the heat kernel for D^2. Then for $t > 0$,*

$$\mathrm{ind}(D) = \mathrm{str}(e^{-tD^2}) = \int_M \mathrm{str}(\langle x \mid e^{-tD^2} \mid x \rangle)\, dx.$$

Proof. The trick is to use that D^2 has a square root! We can write

$$\mathrm{str}(tD^2) = \sum_{\lambda \in \mathbb{R}} (n_\lambda^+ - n_\lambda^-) e^{-t\lambda_n^2}$$

where $n_\lambda^\pm = \dim \mathscr{H}_\lambda^\pm$, the dimension of the odd/even pieces of the λ-eigenspace of $\mathscr{H}$. We are crucially using that D^2 is even, hence "commutes" with the grading!

For $\lambda > 0$, using that D is odd (by the definition of Dirac operator) we have the isomorphism

$$\mathscr{H}_\Lambda^+ \xrightarrow{D} \mathscr{H}_\lambda^-$$

so $n_\lambda^+ = n_\lambda^-$ for $\lambda > 0$. (This is called "supersymmetric cancellation.") That means that contributions from non-zero λ all cancel out, and we get $\sum n_0^+ - n_0^- = \mathrm{ind}(D)$. □

Remark 12.4. Since e^{-tD^2} is trace class (hence compact), its 1-eigenspace (which is the kernel of D) is finite-dimensional.

Let M be compact and H^z a family of generalized Laplacians, and suppose we have

(1) g_z a smooth family of metrics, for $z \in \mathbb{R}$,
(2) $\nabla^z = \nabla + \omega^z$ a smooth family of connections,
(3) a smooth family $F^z \in \Gamma(M, \mathrm{End}(E))$. ($H = \Delta^E + F$, so F is the order 0 part.)

Then we get:

Theorem 12.5. *For $t > 0$, we have a smooth family of heat kernels $p_t(x, y, z)$ for H^z, and*

$$\frac{\partial}{\partial z} p_t(x, y, z) = -\int_0^t \left(\int_{y \in M} p_{t-s}(x, y, z) \partial_z H^z p_s(x, y, z) \right) dx.$$

or in terms of operators:

$$\frac{\partial}{\partial z} e^{-tH^z} = -\int_0^t e^{-(t-s)H^z} \partial_z H^z e^{-sH^z}\, ds.$$

Recall that we made choices (metric, connection) in defining the Laplacian.

Theorem 12.6. *The quantity* $\operatorname{ind}(D)$ *is independent of the metric on* M *and the metric and connection on* E*.*

Proof. We just proved the McKean-Singer formula

$$\operatorname{ind}(D^z) = \int_M \operatorname{str} p_t(x,x)\, dx.$$

Since $p_t^z(x,y)$ depends smoothly on z, this equation implies that the index must also vary smoothly with z, but $\operatorname{ind}(D^z) \in \mathbb{Z}$ so it must be constant. □

Corollary 12.7. *The quantity* $\operatorname{ind}(\not{D})$ *for a Dirac operator* $\not{D}$ *on a spin manifold is actually an invariant of the spin manifold. The quantity* $\operatorname{ind}(d+d^*)$ *on an oriented manifold is actually an invariant of that manifold.*

Theorem 12.8 (Lichnerowicz)**.** *We have*

$$D_{\mathbb{A}}^2 = \nabla^{\mathbb{A}} + \operatorname{cl}(F^{E/S}) + \frac{r_M}{4}.$$

where r_M *is the scalar curvature of* M*.*

Proof. A computation. Omitted. □

Corollary 12.9. *For a compact spin manifold with non-negative scalar curvature and positive scalar curvature at some point,* $\operatorname{ind}(\not{D}) = 0$.

So if you know that the index is non-zero, you know that there can't be a metric with positive curvature.

Proof. In the spin case we have

$$\not{D}^2 = \Delta^{\not{S}} + \frac{r_M}{4} \geq 0$$

so the kernel of $\not{D}^2$ is zero-dimensional, so $\operatorname{sdim}(\ker \not{D}^2) = 0 = \operatorname{ind}(\not{D})$. □

12.2. **Asymptotic expansion.** Now that we have established the McKean-Singer formula

$$\operatorname{ind}(D) = \operatorname{str}(e^{-tD^2}) = \int_M \operatorname{str}(K_t(x,x))\, dx$$

it remains to study the expansion of the right hand side for $t \to 0$, using the asymptotic expansion.

We will use the proof of Getzler. The original proofs (e.g. of Atiyah-Singer) used that this is not only a topological invariant but an *cobordism invariant,* combined with explicit knowledge of the cobordism ring.

The main insight is that there is a filtration that divides this up into nice parts. We'll use the filtration on $\mathrm{End}(E)$ induced by the isomorphism

$$\mathrm{End}(E) \cong \mathrm{Cl}(TM) \otimes \mathrm{End}_{\mathrm{Cl}(TM)}(E),$$

i.e. obtained from the old filtration on $\mathrm{Cl}(TM)$ with $\mathrm{End}_{\mathrm{Cl}(TM)}(E)$ in the zeroth level of the filtration. Then we have a symbol map induced from the one on the Clifford algebra:

$$\sigma\colon \mathrm{End}(E) \cong \bigwedge^{\bullet} T^*M \otimes \mathrm{End}_{\mathrm{Cl}(TM)}(E).$$

Theorem 12.10 (Getzler)**.** *Consider the asymptotic expansion of $k_t(x,x)$:*

$$k_t(x,x) \sim (4\pi t)^{-n/2} \sum_{i=0}^{\infty} t^i k_i(x)$$

for $k_i(x) \in \Gamma(M, \mathrm{Cl}(TM) \otimes \mathrm{End}_{\mathrm{Cl}(TM)}(E))$.

(1) Then $k_i(x) \in \Gamma(M, \mathrm{Cl}_{2i}(TM) \otimes \mathrm{End}_{\mathrm{Cl}(TM)}(E))$.

(2) Let $\sigma(K) = \sum_{i=0}^{n/2} \sigma_{2i}(K_i) \in \Omega^{\bullet}(M, \mathrm{End}_{\mathrm{Cl}(TM)}(E))$. Then

$$\sigma(K) = \det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right) \cdot \exp(-F^{E/S}).$$

Proof Sketch. We compute using a clever rescaling. Roughly, this rescaling is $(t,x) \mapsto (ut, u^{1/2}x)$ which takes $\omega \in \Omega^k(M) \mapsto u^{-k/2} \cdot \omega$. This is the big insight of Getzler.

Let $U \subset X$ be an open set over which E is trivial. Then

$$\Gamma(U, \mathrm{End}(E)) \cong C^{\infty}(U, \mathrm{End}(E_{x_0})) \cong C^{\infty}(U, \bigwedge^{\bullet} \mathbb{R}^n \otimes \mathrm{End}_{\mathrm{Cl}(n)}(E_{x_0})).$$

For $a \in \Gamma(U, \mathrm{End}(E))$ and $t \in \mathbb{R}_+$, define the operator

$$(\delta_u a)(t,v) = \sum_{i=0}^{n} u^{-i/2} a(ut, u^{1/2}v)_i.$$

The restatement of the theorem for $k(t,v) := k_t(x_0, \exp_{x_0}(v))$ is:

$$\left(\lim_{u\to 0} u^{n/2} \delta_u K(t,v)\right)\Big|_{(t,v)=(1,0)} = (4\pi)^{-n/2} \widehat{A}(M) \exp(-F^{E/S}).$$

This rescaling is the *crucial* insight of Getzler, which makes the analysis tractable.

We'll then show that the rescaled heat kernel satisfies a "rescaled" the heat equation:

$$(\partial_t + H) k_t(x,y) = 0.$$

Recall Lichnerewicz's formula

$$H = \Delta^E + \frac{r_M}{4} + F^{E/S}.$$

Finally, we'll compute that the rescaled heat equation, in the limit $u \to 0$, approaches a "heat equation associated to a harmonic oscillator," which is something very specific for which we can write down the kernel explicitly. So using this expression for H and some facts about "harmonic oscillator" heat kernels, we'll be able to compute the $u \to 0$ limit. $\square$

12.3. **Harmonic oscillator.** The classical (one-dimensional) harmonic oscillator corresponds to the differential operator $H = -\frac{d^2}{dx^2} + x^2$ on $\mathbb{R}$.

Lemma 12.11 (Mehler's formula)**.** *The kernel associated to H is*

$$p_t(x,y) = (2\pi \sinh(2t))^{-1/2} \exp(-\frac{1}{2}(\coth(2t)(x^2+y^2) - 2\operatorname{cosech}(2t)xy)).$$

Proof. Exercise (just a computation). □

From a change of variables, we get that

$$p_t(x,r,f) = (4\pi t)^{-1/2}\left(\frac{tr/2}{\sinh(tr/2)}\right)^{1/2} \exp(-\frac{tr}{2}\coth(tr/2)\frac{x^2}{4t} - tf)$$

is the kernel for $(\partial_t - \frac{d^2}{dx^2} + \frac{r^2x^2}{16} + f)p = 0$. This looks like the local coordinate expression for the heat kernel, using the Lichnerewicz formula. The point is that as $u \to 0$, the term $-\frac{tr}{2}\coth(tr/2)\frac{x^2}{4t} - tf$ will die.

Definition 12.12. For R an $n \times n$ antisymmetric matrix and F an $N \times N$ matrix with coefficients in $\Omega^\bullet(\mathbb{R}^n)$, the *generalized harmonic operator* is

$$H = -\sum_i (\partial_i + \frac{1}{4}\sum_j R_{ij}x_j)^2 + F$$

acting on $\Omega^\bullet(\mathbb{R}^n) \otimes \operatorname{End}(\mathbb{C}^N)$.

We'll use formal techniques and estimates to construct the heat kernel of H.

Theorem 12.13. *For any "initial condition" $a_0 \in \Omega^\bullet(U) \otimes \operatorname{End}(\mathbb{C}^N)$, there exists a unique formal solution $p_t(x,R,F,a_0)$ of the heat equation of the form*

$$p_t(x) = q_t(x)\sum_{k=0}^{\infty} t^k \Phi_k(x)$$

such that

$$\Phi_0(0) = a_0$$

and explicitly,

$$p_t(x,R,F,a_0) = (4\pi t)^{-n/2} \underbrace{j^{-1/2}(tR)}_{\left(\frac{\sinh(tR/2)}{tR/2}\right)^{-1/2}} \exp(-\frac{1}{45}\langle x \mid \frac{tR}{2}\coth(\frac{tR}{2}) \mid x\rangle)\exp(-tF)a_0.$$

Staring with our original D^2 and rescaling by u to get a family D_u^2, we'll show as $u \to 0$ that $D_u^2 = H + O(u^{1/2})$.

Clifford supertraces. Suppose that $E = W \otimes \$$ (this is at always true *locally* for E a Clifford module). Then for $a \in \Gamma(M, \operatorname{End}(W)) \cong \Gamma(M, \operatorname{End}_{\mathrm{Cl}(M)}(E))$,

$$\operatorname{str}_W(a) = 2^{-n/2}\operatorname{str}_E(\gamma \cdot a)$$

where $\gamma \in \Gamma(M, \mathrm{Cl}(TM))$ is the *chirality operator*, equal to $i^{n/2}e_1 \dots e_n$ (if n is even, which we are assuming it is). While the left hand side only makes sense when E has a spin structure, the right hand side *always* makes sense. Therefore, when we don't have such a decomposition of E, we will use the right hand side to define the objects.

From this we define the *relative* Clifford trace:

$$\mathrm{str}_{E/S}(a) := 2^{-n/2}\,\mathrm{str}_E(\gamma\cdot a).$$

Similarly for a Clifford superconnection, we define the *relative Chern character*

$$\mathrm{ch}(E/S) = \mathrm{str}_{E/S}(\exp(-F^{E/S})).$$

Then we have

$$\mathrm{str}_E(K_{n/2}x) = (2i)^{n/2}\,\mathrm{str}_{E/S}(\sigma_n(K_{n/2}(x))).$$

Outline of the rest of the proof.

(1) First we show that we can rescale the heat kernel so that as $u \to 0$, we get the right hand side of the index theorem.
(2) Then we show that the rescaled heat kernel satisfies a rescaled heat equation.
(3) Computing the rescaled heat operator in the limit $u \to 0$, we get the harmonic operator.
(4) Conclude that as $u \to 0$, the heat kernel approaches a form given by Mehler's formula.
(5) Plug in and deduce the theorem, using McKean-Singer.

The rescaling for $\alpha \in C^\infty(\mathbb{R}_+ \times U, \bigwedge V^* \otimes \mathrm{End}(W))$ is

$$(\delta_u\alpha)(t,x) = \sum_{i=0}^{n} u^{-i/2}\alpha(u\cdot t, u^{1/2}\cdot x)_{[i]}.$$

Then we define the rescaled heat kernel:

$$r(u,t,x) = u^{n/2}(\delta_u k)(t,x).$$

Observations:

(1) the Euclidean heat kernel

$$(4\pi t)^{-n/2}e^{-|x|^2/4t}$$

is invariant under rescaling,
(2) $\lim_{u\to 0} r(u,t=1,x=0) = \lim_{u\to 0}\sum_{i=0}^{n} u^{(n-i)/2}k_u(x_0,x_0)_{[i]}$ and we want to check that this is $(\frac{R/2}{\sinh(R/2)})^{1/2}e^{-F^{E/S}}$.

The rescaled heat eqwuation is

$$(\partial_t + u\cdot\delta_u D^2\delta_u^{-1})r(u,t,x) = 0.$$

Lemma 12.14.

$$\lim_{u\to 0} u\delta_u D^2\delta_u^{-1} = -\sum_i(\partial_i - \frac{1}{4}R_{ij}x_j)^2 + F^{E/S}.$$

Proof. A computation. Using Lichnerewicz, the key step is that

$$\nabla^E_{\partial_i} = \partial_i + \frac{1}{4}\sum_{j,k,c,\ell} R_{k\ell ij}x^j c^k c^\ell + O(|x|^2)$$

where $c^k = \epsilon^k + \iota^k$, and the ϵ sticks around while the ι gets killed. □

So as $u \to 0$, the kernel of the rescaled equation looks like the term of a harmonic oscillator equation. Therefore, in the limit $u \to 0$ $r(u,t,x)$ approaches the kernel of the harmonic operator, with

$$-\sum_i (\partial_i - \frac{1}{4} R_{ij} x_j)^2 + F^{E/S} + O(u^{1/2})$$

From last time,

$$\lim_{u \to 0} r(u,t,x) = (4\pi t)^{-n/2} \det{}^{1/2}\left(\frac{tR/2}{\sinh(tR/2)}\right) \exp\left(-\frac{1}{4t}\langle x \mid \frac{tR}{2} \coth \frac{tR}{2} | x\rangle - tF^{E/S}\right).$$

Finally, setting $t = 1, x = 0$ we get

$$\lim_{u \to 0} r(u,t,x) = (4\pi)^{-n/2} \det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right) \exp(-F^{E/S}).$$

Theorem 12.15 (Atiyah-Singer)**.**

$$\operatorname{ind}(D) = \int_M \widehat{A}(M) \operatorname{ch}(\mathscr{E}/S).$$

Proof. By McKean-Singer,

$$\operatorname{ind}(D) = \operatorname{str}(e^{-tD^2}) = \int_M \operatorname{str}(k_t(x,x))\,dx.$$

By Getzler, the right hand side is

$$(4\pi)^{-n/2} \int_M \det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right) \exp(-F^{E/S}).$$

□

Remark 12.16. This factor $(4\pi)^{-n/2}$ is sometimes included in the normalization of the characteristic classes.

Part 3. **Applications**

13. Chern-Gauss-Bonnet and Hirzebruch Signature Theorem

We will now realize the Chern-Gauss-Bonnet Theorem and Hirzebruch Signature Theorem as consequences of the Atiyah-Singer index theorem. Since they are basically the same framework, we will do them both at once.

We take $E = \bigwedge^\bullet T^*M$ and ∇^E to be the Levi-Civita connection. As we saw before, $d + d^*$ is a Dirac operator. There are two possible gradings, and hence two supertraces, two consider.

13.1. **Linear algebra background.**

Let V be a quadratic space, $\mathrm{Cl}(V)$ the associated Clifford algebra, and $\bigwedge^\bullet V$ the associated Clifford module. If S is a spinor representation, then

$$S^* \otimes S \cong \mathrm{End}(S) \cong \mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sigma} \wedge^\bullet V_{\mathbb{C}}.$$

We always take S to have the ordinary $\mathbb{Z}/2$ grading. However, we have *two* options for the grading on S^*. We can take the $\mathbb{Z}/2$-grading, *or* the trivial grading. (Moral: "spin structure is a square root of the exterior algebra," although it is not quite true on the nose.)

So what are the two relative supertraces?

(1) If we take the $\mathbb{Z}/2$-grading on S^*, then we obtain the *Euler supertrace*

$$\mathrm{str}_\chi(a) = \mathrm{str}_{\wedge V}(\gamma \cdot a)$$

where γ is the chirality operator defined earlier.

(2) If we take the trivial grading on S^*, then we obtain

$$\mathrm{str}_\sigma(a) = 2^{-n/2}\,\mathrm{Tr}_{\wedge V}(a).$$

Choose an orthonormal basis e^i of V. Define

$$c^i = \epsilon(e^i) - \iota(e^i)$$

and

$$b^i = \epsilon(e^i) + \iota(e^i).$$

where ϵ and ι are the wedge and contraction operators. Then

$$[c^i, c^j] = -2\delta^{ij}$$

$$[b^i, b^j] = 2\delta^{ij}$$

$$[c^i, b^j] = 0.$$

Identify $\mathrm{Cl}^2(V) \cong \bigwedge^2 V$ via the symbol map σ. For

$$a = \sum_{i<j} a_{ij} e^i e^j \in \wedge^2 V \quad a_{ij} \text{ skew},$$

$$b(a) = \sum_{i<j} a_{ij} b^i b^j \in \mathrm{End}_{\mathrm{Cl}(V)}(\wedge^\bullet V)$$

we have

$$\mathrm{str}_\chi(e^{b(a)}) = (-2i)^{n/2} \det(\sinh(a/2))^{1/2} \mathrm{Pf}(a)$$

and

$$\mathrm{str}_\sigma(e^{b(a)}) = 2^{n/2}\det(\cosh(a/2))^{1/2}.$$

If $T\colon V \to V^*$ is skew, corresponding to $\omega_T \in \bigwedge^2 V^*$, then

$$\mathrm{Pf}(T) = \frac{1}{(\dim V/2)!}\omega_T^{\dim V/2} \in \bigwedge^{\mathrm{top}} V^*.$$

Globalizing. Now back to M:

$$\begin{aligned}(\nabla^E)^2 &= \sum_{i,j,k,\ell} R_{ijk\ell}\epsilon^i\iota^j e^k \wedge e^\ell \\ &= -\frac{1}{4}\sum R_{ijk\ell}(c^ic^j - b^ib^j)e^k\wedge e^\ell.\end{aligned}$$

Then $F^{E/S} = -\frac{1}{4}\sum_{i,j}\langle R(e_i), e_j\rangle b^ib^j$.

13.2. **Chern-Gauss-Bonnet.** We want to prove that

$$\chi(M) = \mathrm{ind}(d+d^*) = (2\pi)^{-n/2}\int_M \mathrm{Pf}(-R).$$

By Atiyah-Singer,

$$\begin{aligned}\mathrm{ind}(d+d^*) &= (4\pi)^{-n/2}\int_M \widehat{A}(M)\exp(-F^{E/S}) \\ &= (2\pi)^{-n/2}\int_M \frac{\widehat{A}(M)}{\widehat{A}(M)}\mathrm{Pf}(-R) \\ &= (2\pi)^{-n/2}\int_M \mathrm{Pf}(-R).\end{aligned}$$

13.3. **Hirzebruch Signature.**

$$\begin{aligned}\sigma(M) &= \mathrm{ind}(d+d^*) \\ &= (2\pi i)^{-n/2}\int_M \widehat{A}(M)\exp(-F^{E/S}) \\ &= (\pi i)^{-n/2}\int_M \det\left(\frac{R/2}{\sinh(R/2)}\cosh(R/2)\right)^{1/2} \\ &= (\pi i)^{-n/2}\int_M L(M)\end{aligned}$$

We have omitted some ork to identify the gradings, chirality operator and Hodge $*$.

13.4. **Hirzebruch Genera.** The Hirzebruch signature theorem actually says something stronger, namely that the signature is even a *cobordism* invariant. We take a detour to see how this fits with our framework.

Definition 13.1. Fix a ring R. An R-valued genus is an assignment ϕ to each oriented manifold such that

(1) $\phi(X \coprod Y) = \phi(X) + \phi(Y)$,
(2) $\phi(X \times Y) = \phi(X) \cdot \phi(Y)$,
(3) $\phi(\partial W) = 0$.

Write Ω^{SO} for the set of oriented manifolds modulo oriented cobordism.

Theorem 13.2. $(\Omega^{\mathrm{SO}}, \coprod, \times)$ *is a ring.*

So an R-valued genus is the same as a map $\Omega^{\mathrm{SO}} \to R$.

Example 13.3. L-genus, $\widehat{A}$-genus, Todd genus (stably complex cobordism).

In order to characterize these invariants, we should try to characterize the cobordism ring. Rationally, we get a very nice answer.

Theorem 13.4. $\Omega^{\mathrm{SO}} \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{CP}^2, \mathbb{CP}^4, \mathbb{CP}^6, \ldots]$.

Theorem 13.5 (Thom)**.** *Two manifolds represent the same class in oriented cobordism if and only if their Pontrjagin numbers agree.*

This allows us to get genera from power series in the following manner: if

$$Q(x) = 1 + a_2 x^2 + a_4 x^4 + \ldots \in R[[x^2]]$$

then we define polynomials $K_1, K_2, \ldots$ by

$$Q(x_1) \ldots Q(x_n) = 1 + a_2 K_1(p_1) + K_2(p_1, p_2) + \ldots$$

where the p_i are the elementary symmetric polynomials in the x_i. Then we define the genus ϕ_Q by

$$\phi_Q(M^n) = \langle K_n(TM), [M] \rangle \in R.$$

Thom's theorem says that this process produces all genera.

Corollary 13.6. *The Hirzebruch signature* $\mathrm{ind}(d + d^*) = \sigma(M)$ *is a cobordism invariant.*

Warning: the Euler characteristic $\chi(M) = \mathrm{Ind}(d + d^*)$ is *not* a cobordism invariant, e.g. $\chi(S^2) = \chi(\partial D^3) = 2 \neq 0$.

If M is a compact *spin* manifold and $\not{D}$ the Dirac operator, then we have

$$\mathrm{ind}(\not{D}) = \int_M \widehat{A}(M)$$

because $F^{E/S}$ vanishes for the spinor bundle itself.

Corollary 13.7. *(1)* $\mathrm{ind}(\not{D})$ *is independent of the spin structure and is a cobordism invariant.*
(2) If $\int_M \widehat{A}(M) \notin \mathbb{Z}$*, then* M *has no spin structure.*
(3) If M *is spin and* $R_M > 0$, $\widehat{A}(M) = 0$ *by the Lichnerewicz formula.*

For (3), recall Corollary 12.9.

Remark 13.8. (1) applies for spin and oriented cobordisms, if the manifolds are spin (which is necessary to make sense of ind($\not{D}$) in the first place).

We have a diagram

$$\begin{tikzcd} \Omega^{\mathrm{spin}} \arrow[r, "\mathrm{ind}(\not{D})"] \arrow[d] & \mathbb{Z} \arrow[d] \\ \Omega^{\mathrm{SO}} \arrow[r, "\widehat{A}"] & \mathbb{Q} \end{tikzcd}$$

This diagram is realized as a diagram of coefficients in the family index theorem for some cohomology theories. Namely, Ω^{Spin} is the coefficients for the cohomology theory MSpin (manifolds over X, with cobordisms over X) and Ω^{SO} is the coefficient ring for MSO. Then these fit into a diagram

$$\begin{tikzcd} \mathrm{MSpin}(X) \arrow[r, "\alpha"] \arrow[d] & \mathrm{KO}(X) \arrow[d, "\mathrm{ch}"] \\ \mathrm{MSO}(X) \arrow[r] & \mathrm{H}(X;\mathbb{Q}) \end{tikzcd}$$

The special case $X = \mathrm{pt}$ recovers the previous diagram. The bottom map is "fiberwise integration."

13.5. **Hirzebruch-Riemann-Roch.** Let M be a Kähler manifold, W a holomorphic vector bundle with holomorphic connection and hermitian metric. Let $E = \bigwedge^\bullet (T^{0,1}M)^* \otimes W)$ be the Clifford module with connection coming from the Levi-Civita connection and connection on W.

The Clifford action is determined by the rule that for $v \in \Omega^1_{\mathbb{C}}(M)$, $v = v^{1,0} + v^{0,1}$ on $s \in \Gamma(M, \bigwedge (T^{0,1})^* \otimes W)$ is

$$\mathrm{cl}(v)\cdot s = \sqrt{2}(\epsilon(v^{1,0}) - \iota(v^{0,1}))s.$$

You have to check that this is self-adjoint, but this is at least plausible: the adjoint interchanges ϵ and ι and $v^{1,0}$ and $v^{0,1}$.

The Dirac operator associated to this Clifford action is $D = \sqrt{2}(\overline{\partial} + \overline{\partial}^*)$.

Exercise 13.9. Check this.

Theorem 13.10 (Hirzebruch-Riemann-Roch)**.**

$$\chi_{hol}(W) = \mathrm{ind}(\sqrt{2}(\overline{\partial} + \overline{\partial}^*)) = (2\pi)^{-n/2} \int_M \mathrm{Td}(M)\,\mathrm{ch}(E).$$

Proof. We first show that $\chi_{\mathrm{hol}}(W) = \mathrm{ind}(\sqrt{2}(\overline{\partial} + \overline{\partial}^*))$. Since the operator is self-adjoint, its kernel is the same as that of its square, which is the space of holomorphic sections.

Now to understand the right hand side of Atiyah-Singer, we first compute the twisting curvature of E. We start with $E = \bigwedge^\bullet (T^{0,1}M)^*$ with Lev-Civicta connection. Then

$$(\nabla^E)^2 = \sum_{k,\ell} R(w_k, \overline{w_k})\epsilon(\overline{w}^\ell s)\iota(w^k)$$

for w_k a basis of $T^{0,1}M$ and w^k its dual basis. Next we have to calculate

$$\begin{aligned} R^E &= \frac{1}{4}\sum_{k,\ell}\langle R w_k, \overline{w_\ell}\rangle \operatorname{cl}(w^k)\operatorname{cl}(w^\ell) \\ &+ \frac{1}{4}\sum_{k,\ell}\langle R\overline{w_k}, w_\ell\rangle \operatorname{cl}(\overline{w}^k)\operatorname{cl}(w^\ell). \end{aligned}$$

By inspection,

$$(\nabla^E)^2 = R^E + \frac{1}{2}\sum_k \langle R w_k, \overline{w}_k\rangle.$$

So in this special case,

$$\boxed{F^{E/S} = \frac{1}{2}\sum_k \langle R w_k, \overline{w}_k\rangle.}$$

More generally, if $E = \bigwedge^\bullet (T^{0,1}M)^\bullet \otimes W$, then

$$F^{E/S} = \frac{1}{2}\sum_k \langle R w_k, \overline{w_k}\rangle + F^W$$

because curvature simply adds under tensor product. We can rewrite this as

$$F^{E/S} = \frac{1}{2}\operatorname{tr}(R^+) + F^W$$

where R^+ is the curvature of the Levi-Civita connection on $T^{0,1}M$. Since

$$TM \otimes_R \mathbb{C} \cong T^{1,0}M \oplus T^{0,1}M$$

we have

$$\widehat{A}(M) = \det\left(\frac{R^+}{e^{R^+/2} - e^{-R^+/2}}\right).$$

Putting this together,

$$\begin{aligned} \operatorname{ind}(\sqrt{2}(\overline{\partial} + \overline{\partial}^*)) &= (2\pi)^{-n/2}\int_M \det\left(\frac{R^+}{e^{R^+/2} - e^{-R^+/2}}\right)\operatorname{tr}\exp(-[\frac{1}{2}\operatorname{tr}(R^+) + F^W]) \\ &= (2\pi i)^{-n/2}\int_M \operatorname{Td}(M)\operatorname{tr}\exp(-F^W) \end{aligned}$$

where

$$\operatorname{Td}(M) = \det\left(\frac{R^+}{e^{R^+} - 1}\right) = \det\left(\frac{R^+}{e^{R^+/2} - e^{-R^+/2}}\right)\exp\left(-\frac{1}{2}\operatorname{tr}(R^+)\right).$$

□

Example 13.11. Let M be a Riemann surface and $W = \mathscr{L}$ a line bundle on M. We have $R^+ \in \Omega^2(M)$ and $F^{\mathscr{L}} \in \Omega^2(M)$. Then $\operatorname{Td}(M) = 1 - R^+/2$ and $\operatorname{ch}(\mathscr{L}) = 1 - F^{\mathscr{L}}$, so Hirzebruch-Riemann-Roch says

$$\operatorname{ind}(\overline{\partial}_{\mathscr{L}}) = \dim H^0(\mathscr{L}) - \dim H^1(\mathscr{L}) = \frac{-1}{4\pi i}\int (R^+ + 2F).$$

For $\mathscr{L} = \underline{\mathbb{C}}$ the trivial line bundle, we find that

$$\dim H^0 - \dim H^1 = 1 - g = -\frac{1}{4\pi i}\int_M R^+$$

where $g = \dim H^{1,0}(M) = \dim H^{0,1}(M) = \frac{1}{2}\dim H^1(M)$. If we define $\deg(\mathscr{L}) = -\frac{1}{2\pi i}\int_M F$, then we recover the familiar equation

$$\chi(\mathscr{L}) = 1 - g + \deg \mathscr{L}.$$

14. K-THEORY

14.1. Bott Periodicity.

In this section we assume that X is reasonably "nice" (a compact manifold or finite CW complex).

Definition 14.1. Two complex vector bundles E_1, E_2 on X are *stably isomorphic* if there exist m, n such that $E_1 \oplus \underline{\mathbb{C}}^m \cong E_2 \oplus \underline{\mathbb{C}}^n$.

Define the set $\widetilde{K}(X)$ to consist of complex vector bundles on X modulo stable isomorphism. Define $K(X)$ as formal differences of isomorphism classes of vector bundles, i.e. $[V]-[W]$. These form a ring under $\oplus$ and $\otimes$. This is the *K-group* of X.

If we choose a basepoint of X, then we get $K(X) \to K(\text{pt})$ whose kernel is $\widehat{K}(X)$. This is called the *reduced K-group* of X.

Definition 14.2. We define $\widehat{K}^{-n}(X) = K(S^n \wedge X) = \widehat{K}_{\text{cvs}}(\mathbb{R}^n \times X)$ where cvs means "compact vertical support".

Theorem 14.3 (Bott)**.** *There exists a "Bott class" $\beta \in K^{-2}(pt)$ with the property that the map $K(X) \to K^{-2}(X)$ given by $\alpha \mapsto \alpha \smile \beta$ is an isomorphism.*

Alternatively, $K(X) = [X, BU \times \mathbb{Z}]$, and $\Omega^2 BU \cong BU$.

Proof. $K_{cs}(\mathbb{R}^2) \cong K_{cs}(\mathbb{C}) \cong \widehat{K}(\mathbb{CP}^1)$. We want to define a map $\widetilde{K}(\mathbb{CP}^1) \to K(\text{pt})$ which is inverse to $\smile \beta$. First, we say that $\beta = [\underline{\mathbb{C}}] - [H]$, where H is the tautological bundle on $\mathbb{CP}^1$.

The inverse map in question is $\pi_!(W) := \text{ind}(\overline{\partial}_W) \in \mathbb{Z} \cong K(\text{pt})$. Then one shows that $\text{ind}(\overline{\partial}_\beta) = 1 \in \mathbb{Z}$ because H doesn't have holomorphic sections, so it's at least a one-sided inverse to the Bott map. Atiyah shows that this is enough to deduce that the Bott map is an isomorphism.

□

Periodicity allows us to define $K^n(X)$ for $n \in \mathbb{Z}$, which fit into a cohomology theory $X \mapsto K^\bullet(X)$, meaning that there are Mayer-Vietoris sequences.

There is also a story for real vector bundles, leading to KO-theory which is 8-fold periodic (alternatively, $\Omega^8 BO \cong BO$).

One way to realize the periodicities is through Clifford algebras. The complex Clifford algebras have a 2-fold periodicity with respect to Morita equivalence, and the real Clifford algebras have an 8-fold periodicity. This crucially depends on the $\mathbb{Z}/2$-grading.

14.2. A different description.

There is a different description of $K^{-n}(X)$ as bundles of $\mathbb{Z}/2$-graded $\mathbb{C}\ell_n$-modules mod bundles where the action can be extended to $\mathbb{C}\ell_{n+1}$. The idea here is that $K^0(M)$ are $\mathbb{Z}/2$-graded $\mathbb{C}\ell_0 \cong \mathbb{C}$-modules up to equivalence, which are $\mathbb{Z}/2$-graded complex vector bundles up to equivalence. The map from a $\mathbb{Z}/2$-graded complex vector bundles to virtual vector bundles is given by $[V] \mapsto [V^0] - [V^1]$.

If $V = V^0 \oplus V^1$ represents zero in the "old" description of K-theory, then we have $V^0 \cong V^1$ via maps $e^0 \colon V^0 \to V^1, e^1 \colon V^1 \to V^0$, so an odd map $e \colon V \to V$ which generates the action of $\mathbb{C}\ell_{-1} \cong \mathbb{C}[e]$.

We get $KO^n(X)$ using the real Clifford algebra where Cl_{-n} uses the negative-definite quadratic form on $\mathbb{R}^n$.

In this formulation, Bott periodicity corresponds to the Morita equivalences of Clifford algebras: $\mathbb{C}\ell_n$ is Morita equivalent to $\mathbb{C}\ell_{n+2}$, and Cl_n is Morita equialent to Cl_{n+8}.

Recall that we say that R and S are Morita equivalent rings if $R-\mathbf{Mod} \cong S-\mathbf{Mod}$. A *Morita equivalence* is a module ${}_S P_R$ such that

$$P \otimes_R -\colon R-\mathbf{Mod} \longleftrightarrow S-\mathbf{Mod}\colon \mathrm{Hom}({}_S P, -)$$

Note that the important thing is the $\mathbb{Z}/2$ grading on Clifford modules, as without this they are Morita equivalent to their centers ♠♠♠ TONY: [why?!], which is just the ground field. For Clifford algebras,

$$\mathbb{C}\ell_n \cong \mathbb{C}\ell^{\otimes n}, \mathbb{C}\ell_2 \cong \mathrm{End}(\mathbb{C}^{1|1})$$

Here $\mathbb{C}^{1|1} = \mathbb{C}^{\text{even}} \oplus \mathbb{C}^{\text{odd}}$. So

$${}_{\mathbb{C}\ell_2} P_{\mathbb{C}} := \mathbb{C}^{1|1}$$

and $\mathrm{Cl}_8 \cong \mathrm{End}(\mathbb{R}^{8|8})$, so we get

$${}_{\mathrm{Cl}_8} P_{\mathbb{R}} = \mathbb{R}^{8|8}.$$

Then by definition in this formulation of K-theory, we have

$$K^n(X) \cong K^{n+2}(X)$$

and

$$KO^n(X) \cong KO^{n+8}(X).$$

The coefficients for complex K-theory are

$K^0(\mathrm{pt})$	$\mathbb{Z}$
$K^1(\mathrm{pt})$	0

The coefficients for real K-theory are

$KO^0(\mathrm{pt})$	$\mathbb{Z}$
$KO^1(\mathrm{pt})$	$\mathbb{Z}/2$
$KO^2(\mathrm{pt})$	$\mathbb{Z}/2$
$KO^3(\mathrm{pt})$	0
$KO^4(\mathrm{pt})$	$\mathbb{Z}$
$KO^5(\mathrm{pt})$	0
$KO^6(\mathrm{pt})$	0
$KO^7(\mathrm{pt})$	0

14.3. **Chern character.** There is a map

$$K^0(X) \to HP^0(X) = \bigoplus_{i \in 2\mathbb{Z}} H^i(X; \mathbb{C}).$$

Then $[V] - [W] \mapsto [\mathrm{ch}(V)] - [\mathrm{ch}(W)]$.

Theorem 14.4 (Chern)**.** *The Chern character gives an isomorphism*

$$K^0(X) \otimes_{\mathbb{Z}} \mathbb{C} \cong HP^0(X).$$

Remark 14.5. This is also true over $\mathbb{Q}$ instead of $\mathbb{C}$.

Similarly, the Pontrjagin classes are encoded in a map

$$KO^0(X) \to HP^0(X) := \bigoplus_{i\in 4\mathbb{Z}} H^i(X,\mathbb{C}).$$

Remark 14.6. One way to imagine proving this: every rational spectrum is a wedge of Eilenberg-Maclane spaces, so it suffices to check their homotopy groups, so it suffices to check the case where X is a sphere.

14.4. **Index as a pushforward in K-theory.** Let M be a compact spin manifold of even dimension. For $[V] \in K^0(M)$, we get an "analytic pushfoward" $\pi_!^{\mathrm{an}}$ to $K^0(\mathrm{pt}) \ni [\mathrm{ind}(\not{D} \otimes V)]$ (as a virtual vector space).

On the other hand, we have a map $HP^0(M) \to HP^0(\mathrm{pt})$ given by integration of forms. Thus the following diagram encodes the index theorem:

$$\begin{tikzcd} K^0(M) \arrow[r, "\mathrm{ch}"] \arrow[d, "\pi_!^{\mathrm{an}}"'] & HP^0(M) \arrow[d, "\int_M \widehat{A}(M)\wedge -"] \\ K^0(\mathrm{pt}) \arrow[r, "\mathrm{ch}"] & HP^0(\mathrm{pt}) \end{tikzcd}$$

i.e.

$$\mathrm{ind}(\not{D} \otimes V) = \int_M \widehat{A}(M) \wedge \mathrm{ch}(V).$$

Thus suggestions some generalizations. One is to replace the point by a manifold (which is Grothendieck's relativization of Hirzebruch-Riemann-Roch to Grothendieck-Riemann-Roch). In this version, we have a map $M \to N$ viewed as a family of spin manifolds with even-dimensional fibers, and a commuting diagram

$$\begin{tikzcd} K^0(M) \arrow[r, "\mathrm{ch}"] \arrow[d, "\pi_!^{\mathrm{an}}"'] & HP^0(M) \arrow[d, "\int \widehat{A}(M/N)\wedge -"] \\ K^0(N) \arrow[r, "\mathrm{ch}"'] & HP^0(N) \end{tikzcd}$$

Note that there are two notions of pushforward in cohomology. One is by integration (say with real coefficients) which we might call "analytic", and the other is by relating to homology via Poincaré duality, which we might call "topological." These are the same maps on cohomology.

We've defined an analytic pushforward in K-theory via the index. There is also a pushforward $\pi_!^{\mathrm{top}}\colon K^0(M) \to K^0(\mathrm{pt})$ that uses the Thom isomorphism / Poincaré duality for K-theory, where the analog of an orientation is a spin structure. There's a K-theory version of the index theorem that says

$$\pi_!^{\mathrm{an}} = \pi_!^{\mathrm{top}}\colon K^0(M) \to K^0(\mathrm{pt}).$$

So last time we interpreted $KO^n(X)$ as Cl_{-n}-modules modulo the ones that extend to $\mathrm{Cl}_{-(n+1)}$-modules. If X is spin, then we can form the vector bundle $P_{\mathrm{spin}} \times_{\mathrm{Spin}(n)} \mathrm{Cl}_n$ (here P_{spin} is the principal spin bundle of X). The associated Dirac operator is Cl_n-linear, so $\ker D^2$ is a Cl_n-module.

This gives a map

$$\begin{array}{ccc} KO(X) & & [V] \\ \downarrow & & \downarrow \\ KO^{i-n}(\mathrm{pt}) & & \ker(D\otimes V) \end{array}$$

Then you see the index theorem by applying the Chern characer. From this, we get some mod 2 invariants of, e.g. $4k+2$ or $4k+1$-dimensional manifolds.

14.5. Spin$^{\mathbb{C}}$ **structures.** We'll see that there is a way to do this even without a spin structure. Think to orientations: you can always integrate sections of the orientation line. Analogously, you can always push forward Clifford modules, and if you have an actual spin structure then you can "untwist" back to a Cl_n-module.

Definition 14.7. We define the group $\mathrm{Spin}^{\mathbb{C}} \subset \mathrm{Cl}(V)\otimes\mathbb{C}$ to be generated by $\mathrm{Spin}(V)$ and $\mathrm{U}(1)\subset\mathbb{C}$. Alternatively,

$$\mathrm{Spin}^{\mathbb{C}} \cong \mathrm{Spin}(V)\times_{\mathbb{Z}/2}\mathrm{U}(1)$$

where $\mathbb{Z}/2$ is generated by $(-1,-1)$.

A "$U(1)$-structure" is a line bundle, so a spin$^{\mathbb{C}}$ structure combines a spin structure with a line bundle (when the line bundle is trivial, you recover the usual notion). There's an obvious map $\mathrm{Spin}^{\mathbb{C}} \to \mathrm{SO}(V)\times\mathrm{U}(1)$.

Definition 14.8. A spin$^{\mathbb{C}}$-*structure* on X is a line bundle $L\to X$ and a lift of its unitary structure to a fiber bundle with fibers Spin$^{\mathbb{C}}$:

$$\begin{tikzcd} \mathrm{Spin}^{\mathbb{C}} \arrow[r] \arrow[d] & \mathrm{SO}(n)\times\mathrm{U}(1) \arrow[d] \\ P_{\mathrm{Spin}^{\mathbb{C}}}(X,L) \arrow[r] \arrow[dr] & P_{\mathrm{SO}(n)}(X)\times P_{\mathscr{L}} \arrow[d] \\ & X \end{tikzcd}$$

Here $P_{\mathrm{SO}}(X)$ is a principal $\mathrm{SO}(n)$ bundle and $P_{\mathscr{L}}$ is the principal $U(1)$-bundle corresponding to L.

The obstruction to a spin$^{\mathbb{C}}$-structure on (X,L) is $w_2(X)+c_1(L)$ (mod 2).

Facts.

(1) If X is spin, then X is spin$^{\mathbb{C}}$ with L trivial.
(2) If X is an almost complex manifold, then X is spin$^{\mathbb{C}}$ with $L=\bigwedge^n_{\mathbb{C}} TX$.
(3) $\mathrm{Spin}^{\mathbb{C}}\to\mathbb{C}\ell(V)$ so we get a spinor representation and spinor bundle for spin$^{\mathbb{C}}$-manifolds.

If X is complex, then $\not{S}\cong\bigwedge^{0,\bullet}T^*X$ and the Dirac operator is $\overline{\partial}+\overline{\partial}^*$. (From a spin$^{\mathbb{C}}$-structure, we can form a spinor bundle, which has a Clifford module, which has a Dirac operator).

From this, we get for any spin$^{\mathbb{C}}$-manifold a map

$$\begin{tikzcd} K^0(X) \arrow[d, "\pi_!"'] & {[V]} \arrow[d] \\ K^0(\mathrm{pt}) & {[\ker(\not{D}^{\mathrm{spin}^{\mathbb{C}}}) \otimes V]} \end{tikzcd}$$

14.6. **Orientations and spin structures.** Denote the orientation bundle over X by $\mathscr{P}_{O(1)}(X)$ (a principal $\mathbb{Z}/2$-bundle). Then we have a map $H(X;\mathscr{P}_{O(1)}(X)) \to H(\mathrm{pt})$, which is an isomorphism.

Analogously, $\mathrm{Cl}(TX)$ admits a pushfoward $\pi_!$ to $K^0(\mathrm{pt})$. So Clifford modules are like orientations for K-theory.

A choice of orientation is a choice of isomorphism $H(X;\mathscr{P}_{O(1)}(X)) \cong H(X)$. Similarly, a choice of spin structure (as a $\mathrm{Cl}(TX)-\mathrm{Cl}(n)$ bimodule) is a choice of isomorphism

$$\mathrm{Cl}(TX)-\mathbf{mod} \cong \mathrm{Cl}(n)-\mathbf{mod} \cong K^n(X)$$

14.7. **Fredholm operators.** Let $T\colon V \to W$ be a linear operator between Banach spaces.

Definition 14.9. We say that T is *Fredholm* if $\ker T$ and $\operatorname{coker} T$ are finite-dimensional.

The main example is $\not{D}^+ \colon \Gamma(\not{S}^+) \to \Gamma(\not{S}^-)$.

It is a fact that for a separable Hilbert space $\mathscr{H}$ (say ℓ^2) the space of Fredholm operators $\mathrm{Fred}(H)$ represents K-theory, i.e.

$$K^0(X) \cong [X, \mathrm{Fred}(H)].$$

Note that this implies that $\mathrm{Fred}(H) = BU \times \mathbb{Z}$.

14.8. **Thom spectra.** There are maps $\mathrm{MSpin} \to KO$ and $\mathrm{MSpin}^{\mathbb{C}} \to K$ obtained by taking the (family version) index. There are also topological definitions of these maps, and the miracle of the index theorem is that they coincide.

15. Seiberg-Witten Theory

15.1. **Outline.** Let X^4 be a spin$^{\mathbb{C}}$ 4-manifold (it turns out that every 4-manifold admits a spin$^{\mathbb{C}}$-structure). Let $\mathscr{C}$ be the space of configurations (A, ϕ) where A is a Clifford connection and ϕ is a spinor. There is a "Seiberg-Witten map" $SW : \mathscr{C} \to \mathscr{B}$, where $\mathscr{B}$ will be described later. Here $\mathscr{C}$ is a Banach manifold, an infinite-dimensional analogue of manifolds.

This is nice in many ways. One is that there is a good notion of implicit/inverse function theorems. Roughly, by the Sard-Smale theorem and implicit function theorem for Banach spaces, $SW^{-1}(q)$ is a smooth submanifold of $\mathscr{C}$, with dimension computed from a Fredholm index. (The conditions that you need on a general map on Banach manifolds is that the differential is Fredholm). Think of these as moduli spaces for some decoration.

15.2. spin$^{\mathbb{C}}$**-structures on** 4**-manifolds.** Every oriented (closed) 4-manifold X has a spin$^{\mathbb{C}}$-structure. From this we get a spinor bundle $W = W^+ \oplus W^-$ (with $\dim W^{\pm} = 2$). Let

$$\rho : T^*X \to \mathrm{End}(W)$$

be the Clifford module structure, where $\rho(\alpha)^2 = -|\alpha|^2$. Locally for a frame e_1, e_2, e_3, e_4 of T^*X, using $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$ we have

$$W^{\pm}_{\rho} \cong \mathbb{H} = \{x_0 + I x_1 + J x_2 + K x_3\}$$

and

$$\rho(e_1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$\rho(e_2) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$
$$\rho(e_3) = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}$$
$$\rho(e_4) = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}.$$

Recall that we have an action by complex line bundles on Spin$^{\mathbb{C}}$ structures. In terms of $W^{\pm}$ and ρ, this action is

$$(W^+, W^-, \rho) \mapsto (W^+ \otimes L, W^- \otimes L, \widetilde{\rho})$$

where $\widetilde{\rho} : T^*X \to \mathrm{End}(W \otimes L) \cong \mathrm{End}(W)$ (using that $L \otimes L^{\vee}$ is trivial) is the same as ρ.

Relation to self-dual 2**-forms.** What makes the theory of higher-dimensional manifolds "easier" is that you can do the Whitney trick, h-cobordism, etc. The problem with 4-manifolds is that "$2+2=4$," i.e. disks intersect. We have a decomposition

$$\bigwedge^2 T^*M \cong \bigwedge^+ \oplus \bigwedge^-$$

into self-dual and anti-self dual 2-forms, i.e. the eigenspaces for the Hodge $*$. An explicit basis for $\bigwedge^+$ is

$$\{e_1 \wedge e_2 + e_3 \wedge e_4, e_1 \wedge e_3 - e_2 \wedge e_4, e_1 \wedge e_4 + e_2 \wedge e_3\}.$$

Now,

$$\rho(e_1 \wedge e_2 + e_3 \wedge e_4) = \begin{pmatrix} 2I & 0 \\ 0 & 0 \end{pmatrix}$$

and more generally, the self-dual 2-forms act on $\bigwedge^+$ while the anti-self-dual ones act on W^-: $\rho(\bigwedge^+) \subset \mathrm{SU}(W^+)$ and $\rho(\bigwedge^-) \subset \mathrm{SU}(W^-)$.

15.3. **Seiberg-Witten map.** Now, we have a Clifford connection $\mathbb{A}$ on a $\mathrm{Spin}^{\mathbb{C}}$ spinor bundle. The set of all such A form an affine space. To describe one, we only have to specify a connection on a line bundle, since the connection on Spin is determined by Levi-Civita. Therefore, these form an affine space for imaginary 1-forms. ♠♠♠ TONY: [why imaginary? some unitarity conditon?]

The Dirac operator is then

$$\Gamma(W^+) \xrightarrow{A} \Gamma(T^*X \otimes W^+) \xrightarrow{\rho} \Gamma(W^-).$$

Now we can define the Seiberg-Witten map. Let $\mathscr{A}_W$ be the space of Clifford connections and $\mathscr{C} := \mathscr{A}_W \times \Gamma(W^+)$. Define

$$SW : \mathscr{C} \to i \cdot \mathfrak{su}(W^+) \times \Gamma(W^-)$$

taking

$$SW(A, \phi) = (\rho(F^+_{A^\tau}) - (\phi \otimes \phi^*)_0, D_A \phi)$$

where $F^+_{A^\tau}$ is the self-dual part of the curvature of the determinant line bundle associated to the spinor bundle ♠♠♠ TONY: [needs some motivation...], A^τ is the connection on $\det W$ coming from the Clifford connection A, and $(\phi \otimes \phi^*)_0$ is the traceless part of the endomorphism $\phi \otimes \phi^*$: locally, if $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$, then

$$\phi \otimes \phi^* = \begin{pmatrix} |\phi_1|^2 & \phi_1 \overline{\phi}_2 \\ \phi_2 \overline{\phi}_1 & |\phi_2|^2 \end{pmatrix}$$

so

$$(\phi \otimes \phi^*)_0 = \phi \otimes \phi^* - \frac{|\phi|^2}{2} \mathrm{Id}.$$

♠♠♠ TONY: [what does this setup look like if you take the $\mathbb{Z}/2$-graded Dirac operator?]

Remark 15.1. What's the motivation for this? It's basically an easier version of Donaldson theory. There is also some physical motivation. The important part is the quadratic term $(\phi \otimes \phi^*)_0$, because that's non-linear.

Remark 15.2. We can deform SW by adding a specified self-dual 2-form σ, to get SW_σ.

Symmetries. $\mathscr{G} := \mathrm{Map}(X, S^1)$ acts on $\mathscr{C}$ and $i \cdot \mathfrak{su}(W^+) \times \Gamma(W^-)$. For $g \in \mathscr{G}$, the action on $(A, \phi) \in \mathscr{C}$ is described by

$$g \cdot A = A - g\, dg^{-1} \cdot \mathbf{1}_W$$

and $g \cdot \phi = g\phi$ (viewing $S^1 = U(1) \subset \mathbb{C}$), and the action on the target is $g \cdot w = gw$ for $w \in i \cdot \mathfrak{su}(W^+)$ and $g \cdot \psi = g\psi$ for $\psi \in \Gamma(W^-)$.

You can check by explicit computation that SW is equivariant with respect to this action. ♠♠♠ TONY: [does it though? 1. gdg^{-1} doesn't seem to be an imaginary 1-form, and 2. the map SW seems to be quadratic in the ϕ argument]

We want to understand $SW^{-1}(0)/\mathscr{G}$. Let's start with $\mathscr{C}/\mathscr{G}$. For X connected, we can understand stabilizers completely: we claim that the stabilizer of (A, ϕ) is trivial unless $\phi = 0$ in which case the stabilizer is $S^1 \subset \mathrm{Map}(X, S^1)$ (embedded via constant maps).

Proof. Fixing A requires that $dg = 0$, i.e. g is constant. Fixing non-zero ϕ, g must be trivial, unless $\phi = 0$, in which case g could be any constant map. □

Slice for $\mathscr{G}_0$-action. Choose a "base" connection A_0 so that

$$\mathscr{C} = \{(A_0 - ia, \phi) \mid a \in \Omega^1(X, \mathbb{R}), \phi \in \Gamma(W^+)\}.$$

The $\mathscr{G}$-action is then explicitly

$$(A_0 - ia, \phi) \mapsto (A_0 - ia + g\, dg^{-1}, g\phi).$$

We then define the *based Gauge transformations*

$$\mathscr{G}_0 = \{g \in \mathscr{G} \mid g(p_0) = 1\}$$

for $p_0 \in X$ a basepoint.

Proposition 15.3. *For* 1*-connected X, each base point of $\mathscr{C}/\mathscr{G}_0$ has a unique representative of the form $(A_0 - ia, \phi)$ for a satisfying $d^* a = 0$.*

Sketch. From the 1-connectedness, $g = e^{iu}$ for some $u \colon X \to \mathbb{R}$. Then the $\mathscr{G}_0$-action is $a \mapsto a + du$. We want to show that there exists a unique u such that $d^*(a + du) = 0$. This is basically the Poisson equation, and follows from Hodge theory. □

Let $(\mathscr{C}/\mathscr{G}_0)^*$ denote the complement of "reducible solutions", i.e. where $\phi = 0$. We claim that $(\mathscr{C}/\mathscr{G})^* \cong K(H^1(X, \mathbb{Z}), 1) \times \mathbb{CP}^\infty$. Why?

The configuration space $\mathscr{C}$ is affine hence contractible, and $\mathscr{C}^*$ is *also* affine ♠♠♠ TONY: [eh? what's affine mean?] and thus contractible. When you mod out by $\mathrm{Map}(X, S^1)$, whose identity component is S^1 ♠♠♠ TONY: [why is that true, by the way?], you get the $BS^1 \cong \mathbb{CP}^\infty$. The space of connected components, i.e. homotopy classes of maps to S^1, is described by $H^1(X; \mathbb{Z}) = \pi_0(\mathrm{Map}(X, S^1))$ since S^1 is a $K(1, \mathbb{Z})$. There's a fiber sequence $\mathscr{G}_0 \hookrightarrow \mathscr{G} \xrightarrow{\mathrm{ev}_{x_0}} S^1$, so the quotient map $\mathscr{C}^*/\mathscr{G}_0 \to \mathscr{C}^*/\mathscr{G}$ is an S^1-bundle, with total space

$$K(H^1(X, \mathbb{Z}), 1) \times \mathscr{L}_{\mathrm{Hopf}} \to K(H^1(X, \mathbb{Z}), 1) \times \mathbb{CP}^\infty.$$

15.4. **Transversality.** We want to study $SW^{-1}(0)/\mathscr{G} \subset \mathscr{C}/\mathscr{G}$. There's a good theory for submanifolds $F^{-1}(q)$ for $F\colon M \to N$ a Fredholm map between Banach manifolds.

Definition 15.4. A map $F\colon M \to N$ between Banach spaces is *Fredholm* if dF_p is Fredholm for each $p \in M$.

Remark 15.5. Our Banach manifolds will always be embedded in honest Banach spaces, so the notion of tangent spaces and differentials are easy.

Fact. If $q \in N$ is a regular value, then $F^{-1}(q)$ is a smooth manifold of dimension $\mathrm{Ind}(dF_p)$. This is a Banach version of the implicit function theorem.

Theorem 15.6 (Sard-Smale)**.** *Regular values are generic.*

We'll use this to show that (generically) the SW moduli space is a smooth manifold, and we'll compute its dimension using Atiyah-Singer.

15.5. **Perturbed Seiberg-Witten map.** Recall that we defined the perturbed Seiberg-Witten map

$$SW_\phi\colon \mathscr{C} \times \Omega^2_+(M) \to \Gamma(W^-) \times \Omega^2_+(M)$$

sending $(A, \psi, \phi) \mapsto (D_A\psi, F_A^+ - (\psi \otimes \psi^*)_0 - \phi)$. When the context is clear, we may omit the ϕ.

Then we claim that

$$\boxed{dSW_\phi(A,\psi,\phi)(a,\psi',\phi') = (D_A\psi' - (ia)\cdot\psi, (da)^+ - \partial(\psi \otimes (\psi')^*)_0 - \phi').}$$

Why? We're deforming the map infinitesimally. The tangent space to Clifford connections is purely imaginary one-form, which is parametrized by $a \in \Omega^1(X, \mathbb{R})$, a spinor $\psi' \in \Gamma(W^-)$, and $\phi' \in \Omega^2_+(M')$. So the differential measures the effect of the perturbation $A \mapsto A + a, \psi \mapsto \psi + \psi', \phi \mapsto \phi + \phi'$, which unravels to the above equation.

We claim that for A, ψ, ϕ with $SW_\phi(A,\psi,\phi) = 0$ such that $\psi \neq 0$, the differential $dSW_\phi(A,\psi,\phi)$ is surjective. We'll skip this; the proof uses nontrivial elliptic operator theory.

Define $N = F^{-1}(0,0)$. The tangent space at $(A,\psi,\phi) \in N$ is

$$T_{(A,\psi,\phi)}N = \{(a,\psi',\phi') \mid L(a,\psi') = (0,0,\phi')\}$$

where $L\colon \Gamma(W^+) \otimes \Omega^1(M) \to \Gamma(W^-) \oplus \widetilde{\Omega}^0(M) \oplus \Omega^2_+(M)$ is defined by

$$L(a,\psi') = (D_A\psi' - ia\cdot\psi, d^*a, (da)^+ - 2(\psi \otimes (\psi')^*)_0).$$

Recall that $L(a,\psi')$ furnished some gauge slicing, so this a is uniquely determined by the requirement that $d^*a = 0$.

Proposition 15.7. *The projection map $\pi\colon N \to \Omega^2_+(M)$ sending $(A,\psi,\phi) \mapsto \phi$ is Fredholm.*

Sketch. We have $\ker(d\pi) = \ker L$ and $\mathrm{Im}\, d\pi = \mathrm{Im}\, L \cap (0,0,\Omega^2_+(M))$. But L is Fredholm because it is a "0th order perturbation" of a Fredholm operator (namely the Dirac operator). ∎

15.6. **Expected dimension.** For $\phi \in \Omega^2_+(M)$ a regular value of π, $F^{-1}(\phi) \subset N$ is a smooth submanifold. Then $\dim F^{-1}(\phi)$ is the (real) index of L, which is the index of $L_0 = D_A \oplus d^* \oplus d^+$ ♠♠♠ TONY: [why??]. So we just need to compute that index.

So how do we compute the index? The kernel of L consists of a with $(da)^+ = 0$, throwing away the 0th order deformationof ψ, which implies $da = 0$ ♠♠♠ TONY: [unclear why]. So that means the kernel consists of harmonic forms.

The cokernel is dual to the kernel of the adjoint map which is $d^* \colon \Omega^2_+(M) \to \Omega^1(M)$ gives a contribution of b_2^+, since $d^* a = 0$ and a self-dual implies $da = 0$, i.e. a harmonic.

♠♠♠ TONY: [ehhh]

Now we focus on the index of D_A. By Atiyah-Singer, the complex index is

$$\begin{aligned}\operatorname{ind}_{\mathbb{C}}(D_A) &= (?) \int \widehat{A} e^{-c_1(\mathscr{L})/2} \\ &= (?) \int_M -\frac{p_1(M)}{24} + \frac{c_1(\mathscr{L})^2}{4} \\ &= -\frac{\sigma(M)}{8} + \frac{c_1(\mathscr{L})^2}{4}.\end{aligned}$$

We need to double things to get the real index, so we seem to be missing a factor of 2: it should be $\frac{\sigma(M) - c_1(\mathscr{L})^2}{4}$. The last line used the Hirzebruch Signature Theorem.

Corollary 15.8 (Rochlin)**.** *Thus the signature of a* 4*-manifold is divisible by* 8 *(in fact divisible by* 16*, because of some extra structure).*

Now, the complex index is $\operatorname{ind}(D_A) = \frac{c_1(\mathscr{L}) - \sigma(X)}{8}$, so the real index is twice this:

$$\operatorname{ind}(L_0) = 2 \operatorname{ind}_{\mathbb{C}}(D_A) + 1 - b_1 + b_2^+ = \frac{c_1(L)^2 - \sigma(X)}{4} + 1 - b_1 + b_2^+.$$

By some fiddling around, you see that this is $2\chi(X) + 3\sigma(X) - \frac{c_1(\mathscr{L})^2}{4}$. This is the "expected" dimension of the SW moduli space.

Using that $\chi(M) = b_0 - b_1 + b_2 - b_3 + b_4 = 2 - 2b_1 + b_2^+ + b_2^-$.

Theorem 15.9. *Given a Riemannian metric on an oriented* 4*-manifold* X^4*, there are finitely many* $\operatorname{spin}^{\mathbb{C}}$ *structures up to isomorphism for which the SW moduli space is non-empty (for a generic perturbation). (Equivalently, there are finitely many* $\operatorname{spin}^{\mathbb{C}}$*-structures for which the expected dimension is non-negative.)*

Main tool: (analogue of Lichnerewicz formula)

$$D_A \circ D_A = \nabla_A^* \nabla_A + \frac{s}{4} + \frac{F_A}{4}. \tag{1}$$

where s is the scalar curvature of X.

Lemma 15.10. *If* (A, ψ) *is a solution to the SW equations, then*

$$||\nabla_A \psi||^2_{L^2} + \frac{s}{4}\langle \psi, \psi \rangle_{L^2} + \frac{||\psi||^4_{L^4}}{4} = 0.$$

♠♠♠ TONY: [haven't actually defined what it means to be a "solution to the SW equations"]

Proof. By the definition of being a solution to the SW equation, $D_A\psi = 0$. Hence by (1),

$$\nabla_A^*\nabla_A\psi + \frac{s}{4}\psi + \frac{F_A}{4}\psi = 0.$$

The fact that ψ is a $+$-spinor implies that $F_A\psi = F_A^+\psi$. Then

$$\begin{aligned} 0 &= \nabla_A^*\nabla_A\psi + \frac{s}{4}\psi + \frac{1}{2}\left(\psi\otimes\psi^* - \frac{|\psi|^2}{2}\,\mathrm{Id}\right)\psi \\ &= \nabla_A^*\nabla_A\psi + \frac{s}{4}\psi + \frac{|\psi|^2}{4}\psi. \end{aligned}$$

Taking the inner product with ψ gives the result.

♠♠♠ TONY: [We skipped a computation that $F_A = \frac{1}{2}\left(\psi\otimes\psi^* - \frac{|\psi|^2}{2}\,\mathrm{Id}\right)\psi$?] □

Corollary 15.11. *Setting* $s_X^- = \max_{x\in X}\{0, -s(x)\}$, *we have*

$$s_X^- ||\psi||_{L^2}^2 \le ||\psi||_{L^4}^4.$$

Proof. Immediately from applying the above because $||\nabla_A\psi||_{L^2}^2 \ge 0$. □

Corollary 15.12. *If X has non-negative scalar curvature, then all solutions to the SW equation have* $\psi = 0$.

Remark 15.13. Every solution to the SW equations is gauge equivalent to a C^∞ solution. (There was some completion that we brushed under the rug when discussing Banach manifolds, etc. which left the smooth world.)

Corollary 15.14. *Let X^4 be compact and oriented and (A,ψ) a solution to the SW equations. Then* $|\psi(x)|^2 \le s_X^-$.

Proof. Let x_0 be the point in X where $|\psi(x_0)|^2$ is maximal. We'll show that at x_0,

$$|\psi(x_0)^2| \le s_X^-.$$

In the proof of Lemma 15.10 we found that

$$\langle \nabla_A^*\nabla_A\psi(x), \psi(x)\rangle + \frac{s(x)}{4}|\psi(x)|^2 + \frac{|\psi(x)|^4}{4} = 0.$$

Since the first term is non-negative,

$$\frac{s(x_0)}{4}|\psi(x_0)|^2 + \frac{|\psi(x_0)|^4}{4} \le 0.$$

Either $\psi(x_0) = 0$, hence $\psi(x)\equiv 0$, or $|\psi(x_0)|^2 \le s^-(x_0)$ where $s^-(x_0) = \max\{0, -s(x_0)\}$. □

Corollary 15.15. *Let (A,ψ) be a solution to the SW equations. Then*

$$|F_A^+(x)| \le \frac{s_X^-}{2}.$$

Proof. We have $F_A^+ = \psi \otimes \psi^* - \frac{|\psi|^2}{2}\,\mathrm{Id}$, so

$$|F_A^+(x)| = \frac{|\psi(x)|^2}{2}.$$

Then we apply the bound from Corollary 15.15. $\square$