AN OVERVIEW OF THE GROSS-ZAGIER AND WALDSPURGER FORMULAS

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1. The modular curve $X_0(N)$

1.1. **The open modular curve.** To state the Gross-Zagier formula, we need to introduce modular curves. We begin by defining the *open modular curve* $Y_0(N)$. Over a field of characteristic 0, it is the moduli space of pairs (E', C) where E' is an elliptic curve and C is a subgroup of E' isomorphic to $\mathbf{Z}/N\mathbf{Z}$.

The complex points $Y_0(N)(\mathbf{C})$ have the structure of the locally symmetric space $\Gamma_0(N)\backslash\mathbf{H}$, where

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \colon c \equiv 0 \pmod{N} \right\}.$$

The point $\tau \in \mathbf{H}$ parametrizes the curve $\mathbf{C}/\mathbf{Z} + \tau\mathbf{Z}$, with N-torsion point being $\frac{\tau}{N}$.

1.2. **Cusps.** The *cusps* of $\Gamma_0(N)$ are in bijection with the set

$$\Gamma_0(N)\backslash\mathbf{P}^1(\mathbf{Q}) = \bigsqcup_{d|N} (\mathbf{Z}/f_d\mathbf{Z})^\times \quad f_d = \gcd(d, N/d).$$

We define $X_0(N)$ as the compactification of $Y_0(N)$ obtained by adjoining a point for each cusp. There is a moduli interpretation of $X_0(N)$ as parametrizing isogenies of *generalized elliptic curves*

$$\phi\colon E' \to E''$$

such that $\ker\phi \cong \mathbf{Z}/N\mathbf{Z}$ and $\ker\phi$ meets every component of E'. A generalized elliptic curve is a family whose geometric fibers are either an elliptic curve or a "Néron n-gon" of $\mathbf{P}^1$'s.

There are two special cusps on $X_0(N)$:

- The cusp ∞ corresponds to the n-gon for $n = 1$, which is the nodal cubic.
- The cusp 0 corresponds to the N-gon.

1.3. **CM points.** In terms of the uniformization of $X_0(N)$ by $\mathbf{H}$, CM points correspond to $\tau \in \mathbf{H}$ such that there exist $a, b, c \in \mathbf{Z}$ such that

$$a\tau^2 + b\tau + c = 0.$$

We can assume that $\gcd(a, b, c) = 1$. With this assumption, the discriminant $D = b^2 - 4ac$ is the discriminant of $\mathrm{End}_{\mathbf{C}}(E_\tau) \cong \mathbf{Z} + \mathbf{Z}[\frac{D+\sqrt{D}}{2}]$.

1.4. **Heegner points.** Heegner points are a special type of CM points. Fix K to be an imaginary quadratic field of discriminant D over $\mathbf{Q}$. Assume D is odd. The *Heegner condition* says that for all $p \mid N$,

(1) p is split or ramified in K, and
(2) $p^2 \nmid N$.

Remark 1.1. These conditions are equivalent to saying that D is a square $\pmod{4N}$.

The Heegner condition is equivalent to the existence of a point $x := (\phi\colon E' \to E) \in X_0(N)(\overline{\mathbf{Q}})$ satisfying

$$\mathrm{End}_{\overline{\mathbf{Q}}}(E') = \mathrm{End}_{\overline{\mathbf{Q}}}(E'') = \mathcal{O}_K.$$

The theory of complex multiplication implies that Heegner points are defined over the Hilbert class field of K, which we denote by H. In terms of the complex uniformization, the Heegner point x corresponds to

$$x = [\mathbf{C}/\mathcal{O}_K \to \mathbf{C}/\mathcal{N}^{-1}\mathcal{O}_K]$$

where $\mathcal{N} \subset \mathcal{O}_K$ is an ideal of norm N. Its existence is guaranteed by the Heegner condition as follows. For every $p \mid N$ we can choose $\mathfrak{p} \subset \mathcal{O}_K$ such that $\mathrm{Nm}\,\mathfrak{p} = p$, and then set $\mathcal{N} = \prod_p \mathfrak{p}^{v_p(N)}$.

Finally, we can form a degree 0 divisor on $X_0(N)$ from the Heegner point, which will actually be defined over K, as follows: let

$$P := \sum_{\sigma \in \mathrm{Gal}(H/K)} (\sigma(x) - \infty).$$

2. Néron-Tate height

We now define the "Néron-Tate height". This construction can be done for any abelian variety, but we will only do it for Jacobians; this is all we need to state Gross-Zagier.

Suppose we have a line bundle $\mathcal{L}$ on $J_0(N)$, corresponding to twice a theta divisor Θ. (More This is ample, so we can use it to define a height. Namely, we can pick a large power of n and use $\mathcal{L}^{\otimes n}$ to embed

$$\mathcal{L}^{\otimes n}\colon J_0(N) \hookrightarrow \mathbf{P}^m.$$

On projective space we have the standard height function due to Weil, which we can restrict to $J_0(N)$ to obtain a height function $\frac{1}{n} h^K_{\mathcal{L}^{\otimes n}}$. To make this well defined, we normalize: define $h^K_{\mathcal{L}}$ on $J_0(N)(K)$ by $\frac{1}{n} h^K_{\mathcal{L}^{\otimes n}}$.

Definition 2.1. The *Néron-Tate height* for $J_0(N)$ is defined to be

$$\hat{h} := \lim_{n \to \infty} \frac{h^K_{\mathcal{L}}(2^n x)}{4^n}.$$

This satisfies

$$\hat{h}(2x) = 4\hat{h}(x).$$

Remark 2.2. The Néron-Tate height can be decomposed into a sum of local terms, which is used in the original proof of the Gross-Zagier formula.

3. L-functions

Let f be a weight 2 newform for $\Gamma_0(N)$. (This means that f is a cuspidal Hecke eigenform, orthogonal to modular forms coming from smaller level.) We have a Fourier expansion

$$f = \sum_{n \geq 1} a_n q^n.$$

If $a_n \in \mathbf{Z}$ for all n, then by Eichler-Shimura we have an elliptic curve $E/\mathbf{Q}$ with conductor N. Conversely, for an elliptic curve $E/\mathbf{Q}$ the modularity theorem (Wiles, Taylor-Wiles, Breuil-Conrad-Diamond-Taylor) produces a modular form with the same L-function.

The modular form f can be viewed as an automorphic form for $\mathrm{GL}_2/\mathbf{Q}$. If f_k denotes its base change to K, then

$$L(f_K, s) = L(f, s)L(f \otimes \eta_{K/\mathbf{Q}}, s) \tag{3.1}$$

where $\eta_{K/\mathbf{Q}}$ is the quadratic character associated to $K/\mathbf{Q}$ by class field theory. Explicitly, we can write

$$L(f, s) = \sum a_n q^n$$
$$L(f \otimes \eta_{K/\mathbf{Q}}, s) = \sum \eta(n) a_n q^n$$

Remark 3.1. The base change for automorphic forms can be understood concretely in terms of elliptic curve. If f corresponds to the elliptic curve E under Eichler-Shimura, then

$$L(f_K, s) = L(E_K, s).$$

Thus (3.1) becomes

$$L(E_K, s) = L(E, s)L(E^D, s)$$

where E^D is the quadratic twist of E by D. This has an Euler product

$$L(E_K, s) = \prod_{v \text{ finite place of } K} L_v$$

where for good reduction v,

$$L_v = (1 - a_v q_v^{-s} + q_v^{1-2s})^{-1}, \quad a_v = q_v + 1 - \#E(\mathbf{F}_v),$$

and in the bad reduction case,

$$L_v = (1 - a_v q_v^{-s})^{-1}$$

where $a_v = 1$ for split multiplicative reduction, $a_v = -1$ for a nonsplit multiplicative reduction, and $a_v = 0$ for additive reduction. (This can again be phrased in terms of a point count for the non-singular locus of the reduction.)

The Heegner condition implies that

$$\epsilon(L(f_K, s)) = -1 \implies L(f_K, 1) = 0.$$

4. Gross-Zagier

4.1. **The elliptic curve case.** Let $\phi\colon X_0(N) \to E$ be the modular parametrization, sending $\infty \mapsto e$. Thanks to the modularity theorem of Wiles, this parametrization is induced by a modular form f. We define

$$P(\phi) := \sum_{\sigma \in \mathrm{Gal}(H/K)} \phi(\sigma(x)) \in E(K).$$

Theorem 4.1 (Gross-Zagier)**.** *We have*

$$\hat{h}(P(\phi)) = \frac{\deg \phi \cdot u^2 \cdot |D|^{1/2}}{8\pi^2 ||f||_{\mathrm{Pet}}} L'(E_K, 1)$$

where $u = |\mathcal{O}_K^\times|$, *and*

$$||f||_{\mathrm{Pet}} := \int_{\Gamma_0(N)\backslash \mathbf{H}} f(z)\overline{f(z)}\, dxdy$$

We can rewrite this in terms of modular forms, which fits better with the generalization to automorphic forms.

Definition 4.2. The *Hecke algebra* is the algebra of correspondences on $X_0(N)$ generated by

$$T_m \colon [E \xrightarrow{\phi} E'] \mapsto \sum_{\substack{C \subset C: \\ \#C = m \\ C \cap \ker \phi = e}} [E/C \to E'/C].$$

It acts on $X_0(N)$, hence also on $J_0(N)$. Let $P(f)$ be the isotypic component of $J_0(N) \otimes \mathbf{Q}$, where we need to extend scalars because the idempotent has denominators. Then the reformulation of Gross-Zagier is:

$$\hat{h}(P(f)) = \frac{u^2 \cdot |D|^{1/2}}{8\pi^2} \frac{L'(F_K, 1)}{||f||_{Pet}}.$$

Remark 4.3. The proof considers the height pairing

$$\langle (x - \infty), T_M(\sigma(x) - \infty) \rangle_{NT}$$

for $X_0(N)$. This is the Fourier coefficient of a cusp form of weight 2 on $X_0(N)$. It is part of a general philosopy of Kudla that the generating series for special cycles is a modular form. The L-function is also associated to a modular form. The proof goes by arguing that these two forms coincide, up to an old form. The higher Gross-Zagier also has to do with this.

5. Generalized Heegner conditions

We now explain a generalization of of Heegner points, following work of Zhang and Yuan-Zhang-Zhang.

Let $(N, D) = 1$. Assume $N = N^+ N^-$ where N^- is squarefree and its number of prime factors is even. In this case we can have a quaternion algebra B ramified at N^-, giving rise to a Shimura curve

$$X = B^\times(\mathbf{Q}) \backslash \mathbf{H}^\pm \times B^\times(\mathbf{A}_f)/U.$$

From an elliptic curve $E/\mathbf{Q}$ we get a modular form f. By Jacquet-Langlands, we get a modular parametrization $X \to E$. For an embedding $K \to B(\mathbf{Q})$ of an imaginary quadratic field K, we get a Heegner point $x \in X(H)$, where H is the Hilbert class field of K. (The Shimura curve parametrizes abelian surfaces with real multiplication, while the CM point parametrizes things with endomorphism by $\mathcal{O}_K$. The Heegner condition forces endomorphisms by the maximal order. In particular, this implies that the CM point is defined over H.)

Definition 5.1. We define the generalized Heegner point

$$P(\phi) := \sum_{\sigma \in \mathrm{Gal}(H/K)} \phi(\sigma(x)) \in E(K).$$

Theorem 5.2 (Zhang, YZZ)**.** *We have*

$$\hat{h}(P(\phi)) = \frac{L'(E/K,1)}{||f||_{\mathrm{Pet}}}.$$

6. Waldspurger formula

We normalize so that the center of the L-function is $1/2$.

Let F be a number field and $\mathbf{A} = \mathbf{A}_F$. Let B be a quaternion algebra over F, and G the algebraic group associated to $B^\times$. Denote the center of G by $Z_G = F^\times$. Let K/F be a quadratic extension with a given embedding $K \hookrightarrow B$. Let $T = \mathrm{Res}_{K/F}\,\mathbf{G}_m$; note that we can naturally view $T \subset G$. Let η be the quadratic Hecke character associated to K/F.

Let π be an irreducible cuspidal automorphic representation of G, and ω_π the central character. Let π_K denote the base change of π to K. Let

$$\chi \colon T(F)\backslash T(\mathbf{A}) \to \mathbf{C}^\times$$

be a unitary character, such that $\omega_\pi \cdot \chi|_{\mathbf{A}^\times} = 1$. (The purpose of χ is to get a trivial central character.)

The Waldspurger formula concerns a *period integral*. We define

$$P_\chi \colon \pi \to \mathbf{C}$$

by

$$f \mapsto P_\chi(f) = \int_{T(F)\backslash T(\mathbf{A})/\mathbf{A}^\times} f(t)\chi(t)\,dt.$$

Theorem 6.1 (Waldspurger)**.** *For $f_1 \in \pi$ and $f_2 \in \widetilde{\pi}$ (the contragredient representation), we have*

$$P_\chi(f_1)P_\chi(f_2) \sim \frac{L(\pi_K \otimes \chi, 1/2)}{L(\pi, \mathrm{Ad}, 1)} \alpha(f_1 \otimes f_2)$$

where $\alpha = \prod_v \alpha_v$ is a product of local terms

$$\alpha_v \in \mathrm{Hom}_{K_v^\times}(\pi_v \otimes \chi_v, \mathbf{C}) \otimes \mathrm{Hom}_{K_v^\times}(\widetilde{\pi}_v \otimes \chi_v^{-1}, \mathbf{C}),$$

normalized by Waldspurger (so in particular, they are 1 in the spherical case).