Adic Spaces

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1 Huber rings

The basic building blocks of adic spaces are *Huber rings*.

Definition 1.1. A *Huber ring* is a topological ring A, such that there exists an open subring $A_0 \subset A$ and a *finitely generated* ideal $I \subset A_0$ such A has the I-adic topology. We call A_0 the *ring of definition* and I the *ideal of definition*.

Definition 1.2. We denote by A^{00} the set of topologically nilpotent elements of A and A^0 the set of power-bounded elements. $A^0 \subset A$ is an open subring and A^{00} is an ideal in A^0.

Definition 1.3. A *Tate ring* is a Huber ring A with a topologically nilpotent element ϖ, which we call a *pseudo-uniformizer*. Equivalently, a Tate ring is a Huber ring A such that $A^{00} \cap A^\times \neq \emptyset$.

Remark 1.4. If A is a Tate ring and ϖ is a pseudo-uniformizer, then

1. If $\varphi \colon A \to B$ is a continuous homomorphism of Huber rings then B is a Tate ring and $\varphi(\varpi)$ is a pseudo-uniformizing unit of B, so B is automatically a Tate ring.

2. If A is a Tate ring then we may assume that A_0 contains ϖ and $I = \varpi A_0$. It follows that $A = A_0[1/\varpi]$.

Example 1.5. Let k be a non-archimedean field, i.e. a topological field whose topology is given by a non-archimedean absolute value of height 1:

$$|\cdot| \colon k \to \mathbb{R}_{\geq 0}.$$

Then k is a Tate ring, with $k^0 = \mathcal{O}_k$. We take k^0 to be the ring of definition. Any $\varpi \in k$ with $|\varpi| < 1$ is a pseudo-uniformizer.

Example 1.6. The *Tate algebra*

$$A = k\langle X_1, \ldots, X_n \rangle = \{\sum a_I x^I \mid a_I \to 0 \text{ for } I \to \infty\}$$

is a Huber ring, with ring of definition

$$A_0 := A^0 = \mathcal{O}_k\langle X_1, \ldots, X_n \rangle.$$

This is a Tate ring, and a uniformizer is again any ϖ with $|\varpi| < 1$.

2 Affinoid adic spaces

2.1 Underlying set

Definition 2.1. A *Huber pair* (affinoid ring) is a pair (A, A^+) where A is a Huber ring and $A^+ \subset A^0$ is an open subring which is integrally closed.

To such a pair we will define an *affinoid adic space*. We begin by describing the underlying set:

Definition 2.2. For a Huber pair (A, A^+) we define the *adic spectrum* (just a set for now) to be

$$X = \mathrm{Spa}(A, A^+) := \left\{ |\cdot| \colon A \to \Gamma \cup \{0\} \mid \substack{\text{continuous, multiplicative, non-arch.} \\ |f| \le 1 \text{ for all } f \in A^+} \right\}.$$

where Γ is a totally ordered abelian group. The meaning of continuity is that for all γ,

$$\{a \in A \mid |a| < \gamma\} \subset A \text{ is open.}$$

Remark 2.3. It is equivalent to demand that $\{a \in A \colon |a| \le \gamma\}$ is open for all γ. Indeed, the non-archimedean property implies that both versions (with strict or non-strict inequalities) are groups, and any group containing an open subset is open.

For $x \in X$ and $f \in A$, we denote

$$|f(x)| := x(f).$$

2.2 Topology of rational subsets

Let $T \subset A$ be a finite subset such that $T \cdot A$ generates an open ideal. (If A is a Tate ring then this is equivalent to $TA = A$.)

Definition 2.4. We define a *rational open subset* of $X := \mathrm{Spa}(A, A^+)$ to be a subset of the form

$$X\left(\frac{T}{s}\right) = \{x \in X \mid \forall t, |t(x)| \le |s(x)| \ne 0\}.$$

Theorem 2.5. *There is a unique topology on X in which $X\left(\frac{T}{s}\right)$ forms a basis consisting of quasicompact open subsets such that the system of rational subsets stable under finite intersections. With this topology, X is a spectral space. (i.e. homemorphic to* $\mathrm{Spa}(R)$ *for some ring R).*

This gives a functor

$$(\text{Huber pairs}) \to (\text{spectral spaces}).$$

Lemma 2.6. $\mathrm{Spa}(\widehat{A}, \widehat{A}^+) \to \mathrm{Spa}(A, A^+)$ *is a homeomorphism preserving rational subsets.*

This means that we can always pass to the completion.

Proposition 2.7. *Let (A, A^+) be a Huber pair, and assume A is complete. Then*

1. $\mathrm{Spa}(A, A^+) = \emptyset \iff A = 0$,
2. $f \in A$ *is invertible if and only if* $|f(x)| \neq 0$ *for all* x.
3. $f \in A^+ \iff |f(x)| \leq 1$ *for all* x.

Remark 2.8. Where do we need the fact that A^+ is integrally closed? It is used in the third part of the preceding proposition.

This is everything that we need to say about the topological space underlying an affinoid adic space. Next we describe the structure sheaf.

2.3 Structure (pre)sheaf

From now on, we abbreviate Huber pairs (A, A^+) by A.

Theorem 2.9 (Localization). *Let T, s be as above. Then there exists a morphism of Huber pairs $A \to A\langle\frac{T}{s}\rangle$ which is universal for morphisms of Huber pairs $\varphi\colon A \to B$ with B complete such that $\varphi(s) \in B^\times$ and for all $t \in T$ we have $\varphi(t)\varphi(s)^{-1} \in B^+$. (This implies that $A\langle\frac{T}{s}\rangle$ is a complete ring.)*

Remark 2.10. By the preceding proposition, the property we are asking for is exactly that the induced morphism of adic spectra factors through the open subset $X\left(\frac{T}{s}\right)$.

Lemma 2.11. *The natural map*

$$\mathrm{Spa}(A\langle\frac{T}{s}\rangle) \to \mathrm{Spa}(A)$$

is an open embedding, with image $X\left(\frac{T}{s}\right)$.

Let $X = \mathrm{Spa}(A)$. We now define the structure presheaf $(\mathcal{O}_X, \mathcal{O}_X^+)$ by

$$\mathcal{O}_X\left(X\left(\frac{T}{s}\right)\right) = A\langle\frac{T}{s}\rangle$$

and

$$\mathcal{O}_X^+\left(X\left(\frac{T}{s}\right)\right) = A\langle\frac{T}{s}\rangle^+.$$

In particular, $\mathcal{O}_X(X) = \widehat{A}$. This is a presheaf of complete topological rings with basis $X\left(\frac{T}{s}\right)$.

Definition 2.12. We call A *sheafy* if $\mathcal{O}_X$ is a sheaf (which automatically implies that $\mathcal{O}_X^+$ is a sheaf).

Any point $x \in X$ is a valuation, and induces a valuation on $\mathcal{O}_{X,x}$ (the usual stalk in the category of ringed spaces). You can check that $\mathfrak{m}_x = v_x^{-1}(0)$ is the unique maximal ideal in $\mathcal{O}_{X,x}$, so the latter is a local ring.

Definition 2.13. We define the category $\mathcal{V}$ to have objects tuples $(X, \mathcal{O}_X, \{v_x\}_{x\in X})$ where

- X is a topological space,
- $\mathcal{O}_X$ is a sheaf of complete topological rings such that $\mathcal{O}_{X,x}$ is local, and
- v_x a valuation on $\kappa(x)$.

Morphisms are the natural morphisms of such data.

Proposition 2.14. *The functor*

$$\text{(sheafy Huber pairs)} \to \mathcal{V}$$

sending $A \mapsto \mathrm{Spa}(A)$ *is fully faithful.*

The image of this functor are the *affinoid adic spaces*.

3 Adic spaces

Definition 3.1. A *adic space* is an object of V which is locally isomorphic to $\mathrm{Spa}(A)$ where A is a sheafy Huber ring.

It is annoying that A is not always sheafy. However, here are some conditions that guarantee the sheafiness.

Theorem 3.2. *Let* (A, A^+) *be a complete Huber pair. It is sheafy if any of the following are satisfied:*

1. *A has a Noetherian ring of definition.*
2. *A is a Tate ring and $A\langle X_1, \ldots, X_n\rangle$ is noetherian for all $n \geq 0$.*
3. *A is a Tate ring and for every rational subset $U \subset \mathrm{Spa}(A, A^+)$ the ring of power-bounded elements $\mathcal{O}_X(U)^0$ is a ring of definition.*

Example 3.3. There is a fully faithful embedding

$$\text{(locally noetherian formal schemes)} \hookrightarrow \text{(adic spaces)}$$

sending

$$\mathrm{Spf}(A) \mapsto \mathrm{Spa}(A, A).$$

In fact, (A, A^+) is also sheafy if A has the discrete topology, so $A \mapsto \mathrm{Spa}(A, A)$ embeds the full category of schemes into the category of adic spaces.

Example 3.4. If k is a non-archimedean field, then there is a fully faithful embedding

$$\text{(rigid analytic spaces}/k) \hookrightarrow \text{(adic spaces)}$$

sending

$$\mathrm{Spm}(A) \mapsto \mathrm{Spa}(A, A^0).$$

We haven't yet said what perfectoid spaces are, but they form a subcategory of adic spaces.

Example 3.5. Let k be a non-archimedean field with absolute value $|\cdot|$ and $k^0 = O_k$. Then we have an embedding

$$\mathrm{Spa}(k, k^0) = \{|\cdot|\} \hookrightarrow \mathrm{Spa}(k^0, k^0).$$

This is not surjective: $\mathrm{Spa}(k^0, k^0)$ has the valuation

$$\begin{cases} (k^0)^\times \mapsto 1 \\ k^{00} \mapsto 0 \end{cases}$$

which obviously does not extend to k.

Suppose O_k is a DVR. Then we have a fully faithful functor

$$(\text{formal schemes l.f.t. } / \, O_k,) \hookrightarrow (\text{adic spaces/ } \mathrm{Spa}(O_k, O_k)).$$

The local finite type hypothesis on a formal scheme X means that $X = \mathrm{Spf}(A)$ where there exists a surjection $O_k[[T_1, \ldots, T_n]]\langle X_1, \ldots, X_n\rangle \twoheadrightarrow A$. The theory of Raynaud/Bertholot attaches to such a scheme its generic fiber, which is a rigid analytic space over k. This also embeds fully faithfully into adic spaces over $\mathrm{Spa}(k, O_k)$ via "taking the generic fiber" (or more precisely base chang against $\mathrm{Spa}(k, O_k) \to \mathrm{Spa}(O_k, O_k)$, and we have the following commutative diagram:

$$\begin{array}{ccc} (\text{formal schemes l.f.t. } / \, O_k,) & \hookrightarrow & (\text{adic spaces/ } \mathrm{Spa}(O_k, O_k)) \\ \downarrow{\scriptstyle \text{Raynaud-Bertholot}} & & \downarrow \\ (\text{rigid analytic space } / \, k) & \hookrightarrow & (\text{adic spaces/ } \mathrm{Spa}(k, O_k)) \end{array}$$