Relation with Cohomology of Lubin-Tate Spaces

Notes by Tony Feng
for a talk by Jared Weinstein

April 8, 2016

The goal of this talk is to confirm Fargues' conjecture in the following (non-abelian!) case:

- $G = \mathrm{GL}_n / \mathbb{Q}_p$,
- $\mu(z) = \mathrm{diag}(z, 1, \ldots, 1)$,
- $b = \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ p^{-1} & & & \end{pmatrix} \in G(\breve{\mathbb{Q}}_p)$,
- $J_b(\mathbb{Q}_p) = D^*$ where $D/\mathbb{Q}_p$ is a division algebra of invariant $1/n$.

1 The Hecke stack

We have a Hecke stack

$$\begin{array}{ccc} & \mathrm{Hecke}^{\mu} & \\ \swarrow^{h^{\leftarrow}} & & \searrow^{h^{\rightarrow}} \\ \mathrm{Bun}_{G,\overline{\mathbb{F}}_p} & & \mathrm{Bun}_{G,\overline{\mathbb{F}}_p} \times \mathrm{Spa}\, \mathbb{Q}_p^{\diamond} \end{array}$$

where Hecke has functor of points

$$\mathrm{Hecke}^{\leq \mu}(S) = \left\{ (\mathcal{E}, \mathcal{E}', S^{\#}, u) \colon \begin{array}{c} \mathcal{E}, \mathcal{E}' = G\text{-bundles} \\ S^{\#} = \text{ untilt } \leftrightarrow i \colon D \hookrightarrow \mathrm{Div}^1_{X/S} \\ u \colon \mathcal{E} \overset{\leq \mu}{\dashrightarrow} \mathcal{E}' \text{ such that} \\ \mathrm{coker}\, \mu \text{ supported on } D \end{array} \right\}$$

We could (and usually would) write $\mathrm{Hecke}^{\leq \mu}$ but in this case there's no difference because μ is miniscule. The modification will be

$$0 \to \mathcal{E} \to \mathcal{E}' \to i_* W \to 0$$

where W is a rank 1 $S^{\#}$-module.

This should not be a perfectoid space but a "stack" because there are many automorphisms. We can address this by rigidifying, and that is how the Lubin-Tate tower shows up.

2 Rigidification: the Lubin-Tate tower at infinite level

Let $y_1\colon \operatorname{Spa}\overline{\mathbb{F}}_p \to \operatorname{Bun}_{G,\overline{\mathbb{F}}_p}$ and $y_b\colon \operatorname{Spa}\overline{\mathbb{F}}_p \to \operatorname{Bun}_{G,\overline{\mathbb{F}}_p}$ be two points. (We pass to the algebraic closure because we do not want to keep track of the Weil descent datum right now; one can always go back to this later.) We define a sheaf $\mathcal{M}_\infty$ on Perf by the cartesian diagram

$$\begin{array}{ccc} \mathcal{M}_\infty & \longrightarrow & \mathrm{Hecke}^\mu \\ \downarrow & & \downarrow {\scriptstyle h^{\leftarrow}\times h_0^{\rightarrow}} \\ \operatorname{Spa}\overline{\mathbb{F}}_p & \longrightarrow & \operatorname{Bun}_{G,\overline{\mathbb{F}}_p}\times\operatorname{Bun}_{G,\overline{\mathbb{F}}_p} \end{array}$$

where $h_0^{\rightarrow} = p_1 \circ h^{\rightarrow}$. Here since y_1 corresponds to the trivial bundle, $\mathcal{M}_\infty$ parametrizes modifications of the form

$$0 \to \mathcal{O}_X^{\oplus n} \xrightarrow{u} \mathcal{O}_X(1/n) \to i_*W \to 0.$$

Note that the only thing that varies in moduli is u.

Theorem 2.1 (Scholze-Weinstein)**.** *Let $H_0/\overline{\mathbb{F}}_p$ be the p-divisible group which is connected of dimension* 1 *and height n (exactly the one corresponding to the isocrystal b).*

1. *We have*

$$\mathcal{M}_\infty(R, R^+/\mathbb{Q}_p) = \left\{ (H,\iota,\alpha)\colon \begin{array}{c} H = \text{p-div group}\,/\,R^+ \\ \alpha = \text{ quasi-isog. } : H\otimes_{R^+} R^+/p \sim H_0 \otimes_{\overline{\mathbb{F}}_p} R^+/p \\ \iota\colon T_pH\otimes\mathbb{Q}_p \cong \mathbb{Q}_p^{\oplus n} \end{array} \right\}$$

   This has an action of $\mathrm{GL}_n(\mathbb{Q}_p)\times J_b(\mathbb{Q}_p)$*, with* $\mathrm{GL}_n(\mathbb{Q}_p)$ *acting on* ι *and* $J_b(\mathbb{Q}_p)$ *acting on* α*.*

2. *$\mathcal{M}_\infty$ is a perfectoid space.*

Remark 2.2. The $\mathrm{GL}_n(\mathbb{Q}_p)\times J_b$-action is also clear from the description of $\mathcal{M}_\infty$ as parametrizing extensions

$$0 \to \mathcal{O}_X^{\oplus n} \xrightarrow{u} \mathcal{O}_X(1/n) \to i_*W \to 0.$$

because $\mathrm{GL}_n(\mathbb{Q}_p)$ is automorphism group of $\mathcal{E}_1 = \mathcal{O}_X^n$ and J_b is automorphism group of $\mathcal{E}_b = \mathcal{O}_X(1/n)$.

Remark 2.3. $\mathcal{M}_\infty$ comes equipped with a map to $\mathbb{Q}_p$ because it's fibered over Hecke^μ, which has such a map, because anything over $\operatorname{Spa}\mathbb{Q}_p^\diamond$ has a map to $\operatorname{Spa}\mathbb{Q}_p$.

Proof Sketch. 2. How do we parametrize these morphisms u? Well, u is a map of vector bundles $\mathcal{O}_X^n \to \mathcal{O}_X(1/n)$, which is the same as giving n global sections of $\mathcal{O}_X(1/n)$. So that gives a map

$$\mathcal{M}_\infty \mapsto H^0(X, \mathcal{O}(1/n))^{\oplus n}.$$

(For clarity, we spell out that $H^0(X, \mathcal{O}(1/n))^{\oplus n}$ is the sheaf that assigns to $S \in \mathrm{Perf}_{\mathbb{F}_p}$ n sections in $H^0(X_S, \mathcal{O}(1/n))$.) As covered in the discussion session on p-divisible groups, the sheaf $H^0(X, \mathcal{O}(1/n))$ is the same as $\widetilde{H}$, the universal cover of any lift $H/W(\overline{\mathbb{F}}_p)$ of H_0. (We have $H_0 \leftrightarrow b \leftrightarrow \mathcal{E}_b$, and the general theorem is that $H^0(X, \mathcal{E}_b) = \widetilde{H}$). Scholze-Weinstein proves that this map is a locally closed embedding, from which it follows that $\mathcal{M}_\infty$ is a perfectoid space.

□

3 Another Rigidification

We just related the Hecke stack to a perfectoid space at infinite level. This is a little overkill. What if we rigidify at just one vector and not the other? Suppose we just fix $\mathcal{E}' = \mathcal{O}_X(1/n)$. Then we are considering

$$0 \to \mathcal{E} \to \mathcal{O}_X(1/n) \to i_* W \to 0.$$

This is easy to parametrize because we just have to say what W is. It is a rank 1 quotient of the fiber of $\mathcal{O}_X(1/n)$ at the point D, so it's parametrized by $\mathbb{P}^{n-1}$.

To understand what $\mathcal{E}$ is, we note that $\mathcal{O}_X(1/n)$ has rank n and degree 1 while $i_* W$ has rank 0 and degree 1. By the additivity of rank and degree, we deduce that $\mathcal{E}$ has rank n and degree 0. We also know that $\mathcal{O}_X(1/n)$ is semistable. So what could a slope of $\mathcal{E}$ be? There cannot be a slope $> 1/n$ by the semistability of $\mathcal{O}_X(1/n)$. However, any other positive slope would have a larger denominator, hence larger rank. So we conclude that $\mathcal{E}$ must be semistable of slope 0. It's then proven in Kedlaya-Liu that there's some pro-étale cover trivializing it.

Remark 3.1. This is a really special feature of the Lubin-Tate situation.

As we said, specifying W is picking a line, i.e. 1-dimensional quotient of an n-dimensional space. So we have

$$\mathbb{P}^{n-1,\diamond}_{\breve{\mathbb{Q}}_p} \to \mathrm{Hecke}^\mu \xrightarrow{h^{\leftarrow}} \mathrm{Bun}_{G,\overline{\mathbb{F}}_p}$$

The preceding discussion showed that $\mathcal{E}$ is always pro-étale locally the trivial bundle, so the composite map factors through

$$y_1 \colon [\operatorname{Spa}\overline{\mathbb{F}}_p / \mathrm{GL}_n(\mathbb{Q}_p)] \to \mathrm{Bun}_{G,\overline{\mathbb{F}}_p}.$$

Thus we get a diagram

$$\begin{array}{ccccc} & & & & [\operatorname{Spa}\overline{\mathbb{F}}_p/\operatorname{GL}_n(\mathbb{Q}_p)] \\ & & \nearrow^{r} & & \downarrow \\ \mathbb{P}^{n-1,\diamond}_{\breve{\mathbb{Q}}_p} & \longrightarrow & \mathrm{Hecke}^{\mu} & \xrightarrow{h^{\leftarrow}} & \mathrm{Bun}_{G,\overline{\mathbb{F}}_p} \end{array} \tag{1}$$

The map $r\colon \mathbb{P}^{n-1,\diamond}_{\breve{\mathbb{Q}}_p} \to [\operatorname{Spa}\overline{\mathbb{F}}_p/\operatorname{GL}_n(\mathbb{Q}_p)]$ corresponds by a definition to a $\operatorname{GL}_n(\mathbb{Q}_p)$-torsor on $\mathbb{P}^{n-1,\diamond}_{\breve{\mathbb{Q}}_p}$, and *it turns out to be* $\mathcal{M}_\infty$. The map to $\mathbb{P}^{n-1,\diamond}$ factors through some finite layer, i.e. we have a diagram

$$\begin{array}{ccc} \mathcal{M}_{\infty,\diamond} & \longrightarrow & \mathbb{P}^{n-1,\diamond} \\ & \searrow \quad \nearrow & \\ & \mathcal{M}_K^{\diamond} & \end{array}$$

where $K \subset \operatorname{GL}_n(\mathbb{Q}_p)$ is a compact open subgroup.

In order to match things up with the Hecke correspondence, we now base change to $\mathbb{Q}_p$ (because one of the maps of Hecke^{μ} is to $\mathrm{Bun}_{G,\overline{\mathbb{F}}_p} \times (\operatorname{Spa}\mathbb{Q}_p)^{\diamond}$).

$$[\operatorname{Spa}\breve{\mathbb{Q}}_p^{\diamond}/J_b(\mathbb{Q}_p)] \xrightarrow{(x_b,1)} \mathrm{Bun}_{G,\overline{\mathbb{F}}_p} \times \operatorname{Spa}\mathbb{Q}_p^{\diamond}.$$

We have a commutative diagram

$$\begin{array}{ccc} [\mathbb{P}^{n-1,\diamond}_{\breve{\mathbb{Q}}_p}/J_b(\mathbb{Q}_p)] & \xrightarrow{i} & \mathrm{Hecke}^{\mu} \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle h^{\rightarrow}} \\ [\operatorname{Spa}\breve{\mathbb{Q}}_p{}^{\diamond}/J_b(\mathbb{Q}_p)] & \xrightarrow{(x_b,1)} & \mathrm{Bun}_{G,\overline{\mathbb{F}}_p} \times \operatorname{Spa}\mathbb{Q}_p^{\diamond} \end{array}$$

(We have written down this diagram before without modding out be J_b on the left side.) The map $i\colon [\mathbb{P}^{n-1,\diamond}_{\breve{\mathbb{Q}}_p}/J_b(\mathbb{Q}_p)] \to \mathrm{Hecke}^{\mu}$ is an open embedding. Indeed, as Peter mentioned in his talk, there is a theorem that

$$\coprod_{b \text{ basic}} \left[\frac{\operatorname{Spa}\overline{\mathbb{F}}_p}{\underline{J_b(\mathbb{Q}_p)}}\right] = \mathrm{Bun}_G^{ss}$$

and i is a base change of this map.

To summarize, we have the commutative diagram

$$\begin{tikzcd}
 & & {[\mathrm{Spa}\,\overline{\mathbb{F}}_p /\, \mathrm{GL}_n\, \mathbb{Q}_p]} \arrow[d, "x_1"] \\
{[\mathbb{P}^{n-1,\diamond}_{\breve{\mathbb{Q}}_p} / J_b(\mathbb{Q}_p)]} \arrow[urr, "r"] \arrow[r, "i"] \arrow[d, "j"] & \mathrm{Hecke}^{\mu} \arrow[r, "h^{\leftarrow}"] \arrow[d, "h^{\rightarrow}"] & \mathrm{Bun}_G \\
{[\mathrm{Spa}\,\breve{\mathbb{Q}}_p^{\diamond} / J_b(\mathbb{Q}_p)]} \arrow[r, "{(x_b,1)}"] & \mathrm{Bun}_{G,\overline{\mathbb{F}}_p} \times \mathrm{Spa}\,\mathbb{Q}_p^{\diamond} &
\end{tikzcd}$$

4 Fargues' conjecture

Let $\phi\colon W_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ be a discrete Weil parameter. What does Fargues's conjecture say in this case? (The situation here is a little simplified by the fact that S_ϕ is trivial.) It predicts that there exists $\mathcal{F}_\phi$ on $\mathrm{Bun}_{G,\overline{\mathbb{F}}_p}$ such that (up to shifts and twists)

1. We have
$$h^{\rightarrow}_! h^{\leftarrow *} \mathcal{F}_\phi = \mathcal{F}_\phi \boxtimes \phi. \tag{2}$$
(This is simpler than in general because IC sheaf is constant up to shifts and twists, and also it is unnecessary to write r_μ because it is the standard representation of GL_n.)

2. We have $x_1^* \mathcal{F}_\phi = \pi$ and $x_b^* \mathcal{F}_\phi = \rho$ where π and ρ correspond to ϕ under the local Langlands correspondence.

Consequences of the conjecture. Pulling back (2) through $(x_b, 1)^*$ gives

$$(x_b, 1)^* h^{\rightarrow}_! h^{\leftarrow *} \mathcal{F}_\phi = (x_b, 1)^* \mathcal{F}_\phi \boxtimes \phi. \tag{3}$$

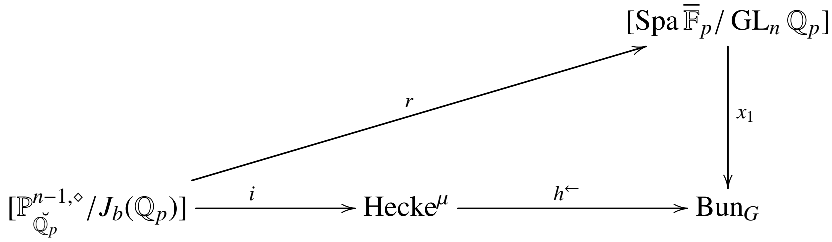
On the left side we get $\rho \otimes \phi$ by the second part of the conjecture. On the right side, first apply proper base change to j from the earlier diagram

$$\begin{tikzcd}
{[\mathbb{P}^{n-1,\diamond}_{\breve{\mathbb{Q}}_p} / J_b(\mathbb{Q}_p)]} \arrow[r, "i"] \arrow[d, "j"] & \mathrm{Hecke}^{\mu} \arrow[d, "h^{\rightarrow}"] \\
{[\mathrm{Spa}\,\breve{\mathbb{Q}}_p^{\diamond} / J_b(\mathbb{Q}_p)]} \arrow[r, "{(x_b,1)}"] & \mathrm{Bun}_{G,\overline{\mathbb{F}}_p} \times \mathrm{Spa}\,\mathbb{Q}_p^{\diamond}
\end{tikzcd}$$

to deduce that

$$\rho \otimes \phi = (x_b, 1)^* h_!^{\rightarrow} h^{\leftarrow *} \mathcal{F}_\phi = (x_b, 1)^* \mathcal{F}_\phi \boxtimes \phi = j_! i^* h^{\leftarrow *} \mathcal{F}_\phi. \tag{4}$$

Now we use the top part of the diagram



to deduce that

$$j_! i^* h^{\leftarrow *} \mathcal{F}_\phi = j_! r^* x_1^* \mathcal{F}_\phi.$$

Then part 2 of the conjecture implies that this is $j_! r^* \pi$, so combining this with (4) gives

$$\rho \otimes \phi = j_! r^* x_1^* \mathcal{F}_\phi.$$

Recall that r corresponds to a $\mathrm{GL}_n(\mathbb{Q}_p)$-torsor on $\mathbb{P}^{n-1,\diamond}$. We can compose this with the representation associated to π to obtain a sheaf $r^*\pi$ on $[\mathbb{P}^{n-1,\diamond}_{\breve{\mathbb{Q}}_p} / J_b(\mathbb{Q}_p)]$ (recall that $\mathcal{M}_\infty \to \mathbb{P}^{n-1,\diamond}_{\breve{\mathbb{Q}}_p}$ is a $\mathrm{GL}_n(\mathbb{Q}_p)$-torsor).

Now we apply $j_!$ to get

$$\rho \otimes \phi = H_c^*(\mathbb{P}^{n-1}_{\mathbb{C}_p}, r^*\pi). \tag{5}$$

Here we have base-changed to $\mathbb{C}_p$ and gotten rid of J_b quotient at the cost of remembering the action of Galois and J_b. (You can get rid of quotients in your sheaves at the cost of remembering the action). So the above isomorphism is equivariant for the action of $J_b(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$.)

We ignored shifts and twists; if you keep track of them then (assuming that π is cuspidal) you get

$$\boxed{\rho \otimes \phi = H_c^{n-1}(\mathcal{M}_\infty, \mathbb{Q}_\ell)[\pi^\vee](\frac{1-n}{2}).} \tag{6}$$

This is a very deep theorem of Harris-Taylor. How did we get from (5) to (6)? The Hoschild-Serre spectral sequence for the fibration

$$\begin{array}{c} \mathcal{M}_{\infty, \mathbb{C}_p} \\ \downarrow \\ \mathbb{P}^{n-1}_{\mathbb{C}_p} \\ \downarrow \\ \mathbb{C}_p \end{array}$$

converges as

$$H_i(\mathrm{GL}_n(\mathbb{Q}_p), H_c^j(\mathcal{M}_{\infty,\mathbb{C}_p}, \overline{\mathbb{Q}}_\ell) \otimes \pi) \implies H^{-i+j}(\mathbb{P}^{n-1}, r^*\pi).$$

In the supercuspidal case there is no higher group cohomology, so you take the invariants in this tensor product, which gives what we claim.