COHOMOLOGICAL SPECTRAL DECOMPOSITION AND FINISHING THE PROOF

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1. Overview

The goal is to prove the main theorem:

Theorem 1.1. *For all $f \in \mathcal{H}_G$,*

$$(\log q)^{-r}\mathbb{J}_r(f) = \mathbb{I}_r(f).$$

If we have proved this for all f, then we can apply spectral decomposition. On the left side, we use the analytic spectral decomposition to extract the term

$$\lambda_\pi(f)\mathcal{L}^{(r)}(\pi_{F'}, 1/2).$$

On the right side we use the cohomological spectral decomposition to extract the term

$$\langle [\mathrm{Sht}_T]_\pi, f * [\mathrm{Sht}_T]_\pi \rangle.$$

If we then take $f = \mathbf{1}_K \in \mathcal{H}$ then we get

$$\mathcal{L}^{(r)}(\pi_{F'}, 1/2) = \langle [\mathrm{Sht}_T]_\pi, [\mathrm{Sht}_T]_\pi \rangle.$$

So we arrive at the desired higher Gross-Zagier formula.

The most difficult intersection is the self-intersection. Allowing the choice of h_D for large D lets us take the intersection in easier cases, and then deduce it for the one we really want.

The strategy is to show that validity of the identity for h_D for all D with $d := \deg D \gg 0$ (in fact, $d \geq \max\{2g' - 1, 2g\}$) is enough to deduce it for all $f \in \mathcal{H}$. This implication uses only elementary commutative algebra. What makes it possible is finiteness properties of the action of $\mathcal{H}$ on the middle-dimensional cohomology of shtukas.

2. Key finiteness theorems

Let $\mathcal{H}_G = \bigotimes_{x \in |X|} \mathcal{H}_x$ be the Hecke algebra. It acts on the vector space $V := H_c^{2r}(\mathrm{Sht}_G, \mathbf{Q}_\ell)$. The difficulty is that V is infinite-dimensional, because Sht_G is only locally of finite type.

Example 2.1. For $r = 0$, $V = \mathcal{A} = C_c^\infty(G(F)\backslash G(\mathbf{A})/K, \mathbf{Q}_\ell) = \mathcal{A}_{\mathrm{Eis}} \oplus \mathcal{A}_{\mathrm{cusp}}$. The cuspidal part is finite-dimensional, but the Eisenstein part is infinite-dimensional.

Therefore, we bring in the Eisenstein ideal to kill the Eisenstein part. Recall we have the Satake transform

$$a_{\mathrm{Eis}} \colon \mathcal{H}_G \xrightarrow{\mathrm{Sat}} \mathcal{H}_{\mathbf{G}_m} = \mathbf{Q}_\ell[\mathrm{Div}_X(k)] \twoheadrightarrow \mathbf{Q}_\ell[\mathrm{Pic}_X(k)].$$

This is compatible with the local Satake transforms

$$\begin{array}{ccc} \mathcal{H}_G & \xrightarrow{\mathrm{Sat}} & \mathcal{H}_{\mathbf{G}_m} \\ \uparrow & & \uparrow \\ \mathcal{H}_{G,x} & \xrightarrow{\mathrm{Sat}} & \mathcal{H}_{\mathbf{G}_m,x} \end{array}$$

$$h_x \longrightarrow t_x + q_x t_x^{-1}.$$

Definition 2.2. We define

$$\mathcal{I}_{\mathrm{Eis}} := \ker(\mathcal{H}_G \to \mathbf{Q}_\ell[\mathrm{Pic}_X(k)]).$$

The map a_{Eis} is surjective:

$$\mathcal{H}_G/\mathcal{I}_{\mathrm{Eis}} \cong \mathbf{Q}_\ell[\mathrm{Pic}_X(k)]^\iota.$$

Thus $Z_{\mathrm{Eis}} := \mathrm{Spec}\ \mathcal{H}/\mathcal{I}_{\mathrm{Eis}} \hookrightarrow \mathrm{Spec}\ \mathcal{H}$ is a reduced, 1-dimensional subvariety.

Remark 2.3. A $\overline{\mathbf{Q}}_\ell$-point of Spec $\mathcal{H}$ is a map

$$\mathcal{H}_G \xrightarrow{s} \overline{\mathbf{Q}}_\ell.$$

The condition that this is in Z_{Eis} says that it factors through

$$\begin{array}{ccccc} \mathcal{H}_G & \longrightarrow & \mathcal{H}_G/\mathcal{I}_{\mathrm{Eis}} & \longrightarrow & \overline{\mathbf{Q}}_\ell \\ & \searrow & \| & \nearrow & \\ & & \mathbf{Q}_\ell[\mathrm{Pic}_X(k)]^\iota & & \end{array}$$

So it factors through a character $\chi\colon \mathrm{Pic}_X(k) \to \overline{\mathbf{Q}}_\ell^*$, and the definition of the Eisenstein ideal implies that

$$s(h_x) = \chi(t_x) + q_x\chi(t_x^{-1}).$$

Example 2.4. For $\chi = 1$, we see that $s(h_x) = 1 + q_x$. This is analogous to Mazur's Eisenstein ideal, $T_p \mapsto 1 + p$.

Theorem 2.5. *$\mathcal{I}_{\mathrm{Eis}} \cdot V$ is finite-dimensional over $\mathbf{Q}_\ell$.*

Proof. We have a stratification

$$\mathrm{Sht}_G = \bigcup_d \mathrm{Sht}_G^{\leq d}$$

where each $\mathrm{Sht}_G^{\leq d}$ is an open substack of finite type. Then

$$V = \varinjlim_d H_c^{2r}(\mathrm{Sht}_G^{\leq d}).$$

The difference between the cohomology of $\mathrm{Sht}_G^{<d}$ and $\mathrm{Sht}_G^{\leq d}$ can be explicitly understood in terms of horocycles when $d > 2g-2$. By the discussion of horocycles, for $\pi_G \colon \mathrm{Sht}_G \to X^r$,

$$\mathrm{Cone}(R\pi_{G!}^{<d}\mathbf{Q}_\ell \to R\pi_{G!}^{\leq d}\mathbf{Q}_\ell) = R\pi_{\mathbf{G}_m!}^{d}\mathbf{Q}_\ell[-r],$$

which is moreover a local system concentrated in degree r.

So when $d \gg 0$, we understand how the cohomology grows, and we also understand the Hecke action. By the local constancy, it then suffices to show that on the geometric generic fiber $\overline{\eta}$ (so the middle dimension is r) the vector space

$$\mathcal{I}_{\mathrm{Eis}} \cdot H_c^r(\mathrm{Sht}_{G,\overline{\eta}})$$

is finite-dimensional over $\mathbf{Q}_\ell$.

For any finite-type substack of $\mathrm{Sht}_{G,\overline{\eta}}$, we get finiteness of cohomology for free. So it would be great to show that $\mathcal{I}_{\mathrm{Eis}} \cdot V$ lies in $H_c^r(U)$ for some finite type $U \subset \mathrm{Sht}_{G,\overline{\eta}}$. To this end it suffices to show that if we take $U = \mathrm{Sht}_G^{\leq d}$ for sufficiently large d, then for all $f \in \mathcal{I}_{\mathrm{Eis}}$ the composition

$$H_c^r(\mathrm{Sht}_G) \xrightarrow{f*} H_c^r(\mathrm{Sht}_G) \to H_c^r(\mathrm{Sht}_G)/\operatorname{Im} H_c^r(U)$$

is 0, as this implies that $\mathcal{I}_{\mathrm{Eis}} \cdot V$ is in the finite-dimensional vector space $\operatorname{Im} H_c^r(U)$.

We can extend the map through the injection (for large enough d)

$$H_c^r(\mathrm{Sht}_G)/\operatorname{Im} H_c^r(U) \hookrightarrow \prod_{d' > 2g-2} H_0(\mathrm{Sht}_{\mathbf{G}_m}^{d'})$$

which comes from the study of horocycles

$$\begin{array}{ccccc} H_c^r(\mathrm{Sht}_G) & \xrightarrow{f*} & H_c^r(\mathrm{Sht}_G) & \longrightarrow & H_c^r(\mathrm{Sht}_G)/\operatorname{Im} H_c^r(U) \\ & \searrow{\scriptstyle (\text{dashed})} & & & \big\downarrow \\ & & & & \prod_{d'>2g-2} H_0(\mathrm{Sht}_{\mathbf{G}_m}^{d'}) \end{array}$$

Since the last map in the composition is an injection, it suffices to show that the dashed arrow is 0. To this end, we extend the diagram

$$\begin{array}{ccccc} H_c^r(\mathrm{Sht}_G) & \longrightarrow & H_c^r(\mathrm{Sht}_G) & \longrightarrow & H_c^r(\mathrm{Sht}_G)/\operatorname{Im} H_c^r(U) \\ \big\downarrow & \searrow{\scriptstyle (\text{dashed})} & \big\downarrow & & \big\downarrow \\ \prod_d H_0(\mathrm{Sht}_{\mathbf{G}_m}^d) & \longrightarrow & \prod H_0(\mathrm{Sht}_{\mathbf{G}_m}^d) & \longrightarrow & \prod_{d'>2g-2} H_0(\mathrm{Sht}_{\mathbf{G}_m}^{d'}) \end{array}$$

If $f \in \mathcal{I}_{\mathrm{Eis}}$, then $\mathrm{Sat}(f)* = 0$ by the definition of $\mathcal{I}_{\mathrm{Eis}}$ and the compatibility of the Satake homomorphism with the constant term map, so

$$f * (H_c^r(\mathrm{Sht}_G)) \subset \operatorname{Im}(H_c^r(U) \to H_c^r(\mathrm{Sht}_G))$$

is finite-dimensional. □

Here is another important theorem, which we don't have time to prove.

Theorem 2.6. *V is a finitely generated module over $\mathcal{H}_x$ for all $x \in |X|$.*

3. Cohomological spectral decomposition

We're only interested in the Hecke action on V and through the Satake transform, so we make the following definition:

Definition 3.1. We define $\overline{\mathcal{H}} = \mathrm{Im}\,(\mathcal{H}_G \to \mathrm{End}(V) \times \mathbf{Q}_\ell[\mathrm{Pic}_X(k)])$.

Corollary 3.2. *$\overline{\mathcal{H}}$ is a finitely generated algebra over $\mathbf{Q}_\ell$.*

Proof. Since $\overline{\mathcal{H}} \hookrightarrow \mathrm{End}_{\mathcal{H}_x}(V \oplus \mathbf{Q}_\ell[\mathrm{Pic}_X(k)])$ and V and $\mathbf{Q}_\ell[\mathrm{Pic}_X(k)]$ are finitely generated $\mathcal{H}_x$-modules, we deduce that $\overline{\mathcal{H}}$ is a finitely generated module over $\mathcal{H}_x$. Therefore, it is a finitely generated algebra over $\mathbf{Q}_\ell$. □

Theorem 3.3. *We have*

(1) Spec $\overline{\mathcal{H}^{\mathrm{red}}} = Z_{\mathrm{Eis}} \cup Z_0^r$ where Z_0^r is finite. Here Z_{Eis} is 1-dimensional and Z_0^r is a finite set of closed points.

(2) $V = V_{\mathrm{Eis}} \oplus V_0$ as $\mathcal{H}_G$-modules, with

$$\mathrm{supp}\, V_{\mathrm{Eis}} \subset Z_{\mathrm{Eis}}$$

and

$$\mathrm{supp}\, V_0 = Z_0^r.$$

4. Proof of the main identity

Lemma 4.1. *Let $I \subset \mathcal{H}_x$ be a non-zero ideal. Then for any $m \geq 1$,*

$$I + \{h_{nx}, n \geq m\} = \mathcal{H}_x.$$

Proof. We can identify $\mathcal{H}_x \cong \mathbf{Q}_\ell[t, t^{-1}]^\iota \cong \mathbf{Q}_\ell[t + t^{-1}]$. Under this identification,

$$h_{nx} = t^n + t^{n-2} + \ldots + t^{-n}.$$

So the proof reduces to showing that

$$I + \{t^n + t^{-n} \colon n \geq m\} = \mathbf{Q}_\ell[t + t^{-1}].$$

This is an elementary exercise in algebra. □

Note that the validity of $\mathbb{I}(f) = \mathbb{J}(f)$ only depends on the image of f inside $\overline{\mathcal{H}} := \mathrm{Im}\,(\mathcal{H} \hookrightarrow \mathrm{End}(V) \oplus \mathrm{End}(\mathcal{A}))$ which is a finitely generated $\mathbf{Q}_\ell$-algebra by Corollary 3.2.

Definition 4.2. Let $\mathcal{H}'_{d_0} \subset \mathcal{H}_G$ be the subalgebra of $\mathcal{H}$ generated by the elements h_D for all $\deg D \geq d_0$.

Lemma 4.3. *For any $d_0 \geq 1$, there exists an ideal $I \subset \overline{\mathcal{H}}$ such that*

(1) $I \subset \mathrm{Im}\,(\mathcal{H}'_{d_0} \to \overline{\mathcal{H}})$, and

(2) $\overline{\mathcal{H}}/I$ is finite dimensional.

Proof. Commutative algebra using the key finiteness theorems. □

Using Lemmas 4.1 and 4.3, we deduce the result needed for the main identity.

Corollary 4.4. *For any $d_0 \geq 1$, the composition*

$$\mathcal{H}'_{d_0} \hookrightarrow \mathcal{H} \to \overline{\mathcal{H}}$$

is surjective.

Proof. Take I as in Lemma 4.3. Since I is generated by h_D for $\deg D \geq d_0$, it suffices to show that the composition

$$\mathcal{H}'_{d_0} \hookrightarrow \mathcal{H} \to \overline{\mathcal{H}} \to \overline{\mathcal{H}}/I$$

is surjective.

Consider the corresponding local statement: for all $x \in |X|$, we have a map

$$\mathcal{H}_x \cap \mathcal{H}' \hookrightarrow \mathcal{H}_x \to \mathrm{Im}\,(\mathcal{H}_x) \subset \overline{\mathcal{H}}/I.$$

Lemma 4.3 (2) tells us that $\overline{\mathcal{H}}/I$ is finite-dimensional. Therefore $\mathrm{Im}\,(\mathcal{H}_x)$ is finite-dimensional. By Lemma 4.1, noting that the image of I in $\mathcal{H}_x$ is non-zero because $\mathrm{Im}\,(\mathcal{H}_x)$ is finite-dimensional, $\mathcal{H}_x \cap \mathcal{H}' \twoheadrightarrow \mathrm{Im}\,(\mathcal{H}_x)$.

Since this works for every x, we get the surjectivity of the global map

$$\mathcal{H}' \hookrightarrow \mathcal{H} \twoheadrightarrow \overline{\mathcal{H}} \twoheadrightarrow \overline{\mathcal{H}}/I$$

as desired.

□