HOROCYCLES

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1. Outlook

Let $G = \mathrm{PGL}_2$, $B \subset G$ be the Borel, and $H \subset B$ be the torus. (Actually, it is better to regard H as a quotient of B.)

Consider the diagram

$$\begin{array}{ccccc} & & \mathrm{Sht}_B & & \\ & \swarrow & & \searrow & \\ \mathrm{Sht}_G & & & & \mathrm{Sht}_H \\ & \searrow & & \swarrow & \\ & & X^r & & \end{array}$$

We basically want to prove

$$\mathbb{J}_r(\pi) = \mathbb{I}_r(\pi).$$

What we have is $\mathbb{J}_r = \mathbb{I}_r$. So we need to have some spectral decomposition. This has been done for the analytic side. The geometric side rests on spectral decomposition of the cohomology of shtukas, $H^{2r}(\mathrm{Sht}_G)$. This is achieved by an analysis of the Hecke action.

2. Hecke action

Let $D \hookrightarrow X$ be a divisor. We have defined a correspondence

$$\begin{array}{ccccc} & & \mathrm{Sht}_G(h_D) & & \\ & \swarrow & & \searrow & \\ \mathrm{Sht}_G & & & & \mathrm{Sht}_G \end{array}$$

The stack $\mathrm{Sht}_G(h_D)$ parametrizes modifications of iterated shtukas

$$(\mathcal{E}_. \hookrightarrow \mathcal{E}'_.).$$

In fact, for every $g = \bigotimes g_v \in G(\mathbf{A}_F)$, we get a correspondence $\mathrm{Sht}_G(g)$. This defines an algebra homomorphism

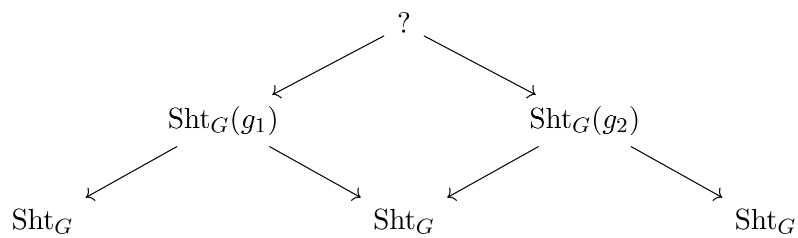
$$\mathcal{H}_G \to \mathrm{End}\, H^i(\mathrm{Sht}_G).$$

We sketch why this is the case.

Recall that the ring structure on the Hecke algebra is defined by convolution:

$$\mathbf{1}_{Kg_1K} * \mathbf{1}_{Kg_2K} = \sum_{g_3 \in K\backslash G/K} [g_3^{-1}Kg_1K \cap Kg_2^{-1}K : K] \cdot \mathbf{1}_{Kg_3K}$$

The fibered product of $\mathrm{Sht}_G(g_1)$ and $\mathrm{Sht}_G(g_2)$ is basically several copies of $\mathrm{Sht}_G(g_3)$:



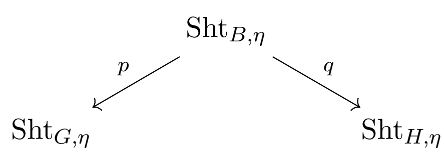
In fact the number $[g_3^{-1}Kg_1K \cap Kg_2^{-1}K : K]$ is the number of copies of $\mathrm{Sht}_G(g_3)$ appearing in the fiber product.

3. The constant term map

We are going to define a constant term map

$$H_c^r(\mathrm{Sht}_G) \to H_c^0(\mathrm{Sht}_H).$$

We begin by considering the diagram



where η is the generic point of X^r, and the subscript η denotes restriction to the generic fiber. Since we have restricted everything to the generic fiber, we have $\dim \mathrm{Sht}_{G,\eta} = r$, $\dim \mathrm{Sht}_{B,\eta} = r/2$, and $\dim \mathrm{Sht}_{H,\eta} = 0$.

Theorem 3.1 (Drinfeld-Varshavsky)**.** *The map* $\mathrm{Sht}_{B,\eta} \to \mathrm{Sht}_{G,\eta}$ *is finite unramified.*

We need this properness, because to define a map on cohomology from a cohomological correspondence requires properness of the first map.

Definition 3.2. The *constant term map* is the composition

$$CT\colon H_c^r(\mathrm{Sht}_G) \xrightarrow{p^*} H_c^r(\mathrm{Sht}_B) \xrightarrow{q_*} H_c^0(\mathrm{Sht}_H).$$

The key point is that the constant term map is compatible with the Satake homomorphism.

Proposition 3.3. *For* $h \in \mathcal{H}_G$*, we have*

$$CT \circ h = \mathrm{Sat}(h) \circ CT.$$

Proof. Consider the diagram

$$\begin{array}{ccccc} \mathrm{Sht}_H & \longleftarrow & \mathrm{Sht}_H(h_x) & \longrightarrow & \mathrm{Sht}_H \\ \uparrow & & & & \uparrow \\ \mathrm{Sht}_B & & & & \mathrm{Sht}_B \\ \downarrow & & & & \downarrow \\ \mathrm{Sht}_G & \longleftarrow & \mathrm{Sht}_G(h_x) & \longrightarrow & \mathrm{Sht}_G \end{array}$$

Define a middle term $\mathrm{Sht}_B(h_x)$ to make the bottom right square cartesian. We claim that it automatically makes the bottom left square cartesian.

$$\begin{array}{ccccc} \mathrm{Sht}_H & \longleftarrow & \mathrm{Sht}_H(h_x) & \longrightarrow & \mathrm{Sht}_H \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Sht}_B & \longleftarrow & \mathrm{Sht}_B(h_x) & \longrightarrow & \mathrm{Sht}_B \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Sht}_G & \longleftarrow & \mathrm{Sht}_G(h_x) & \longrightarrow & \mathrm{Sht}_G \end{array}$$

Let's unravel the claim. The stack $\mathrm{Sht}_B(h_x)$ parametrizes modifications

$$\begin{array}{ccc} \mathcal{L} & \longrightarrow & \mathcal{L}' \\ \downarrow & & \downarrow \\ \mathcal{E} & \xhookrightarrow{\text{at } x} & \mathcal{E}' \end{array}$$

The fact that both diagrams are cartesian amounts to saying that given the datum

$$\begin{array}{ccc} \mathcal{L} & & \\ \downarrow & & \\ \mathcal{E} & \xhookrightarrow{\text{at } x} & \mathcal{E}' \end{array}$$

we can fill it in to

$$\begin{array}{ccc} \mathcal{L} & \dashrightarrow & \mathcal{L}' \\ \downarrow & & \downarrow \\ \mathcal{E} & \xhookrightarrow{\text{at } x} & \mathcal{E}' \end{array}$$

Indeed, we take $\mathcal{L}'$ to be the saturation of the image of $\mathcal{L}$ in $\mathcal{E}'$.

Thanks to the cartesian-ness, base change applies to the squares in the diagram. Therefore, we get an obvious compatibility relation by following two maps $H_c^*(\mathrm{Sht}_G) \to H_c^*(\mathrm{Sht}_H)$.

To finish, we recall that the constant term

$$\mathcal{H}_G \to \mathcal{H}_H$$

sends

$$h_x \mapsto t_x + q_x t_x^{-1}.$$

The middle row

$$\mathrm{Sht}_B \dashleftarrow S \dashrightarrow \mathrm{Sht}_B \tag{3.1}$$

maps points as in

$$\begin{array}{ccccccc} \mathcal{L} & \dashleftarrow & \mathcal{L} & \hookrightarrow & \mathcal{L}' & \dashrightarrow & \mathcal{L}' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{E} & \dashleftarrow & \mathcal{E} & \hookrightarrow & \mathcal{E}' & \dashrightarrow & \mathcal{E}' \end{array}$$

The middle object $\mathrm{Sht}_B(h_x)$ is a disjoint union of two things. One is where the modification occurs in the sub, and one is where it doesn't. In that latter case, it occurs in the quotient. Write $\mathrm{Sht}_B(h_x) = S_1 \sqcup S_2$, where S_1 parametrizes the modifications with $\mathcal{L} \cong \mathcal{L}'$ and S_2 parametrizes the modifications with $\mathcal{M} \cong \mathcal{M}'$. Then we can separate the correspondence (3.1) into two ones:

$$\mathrm{Sht}_B \overset{\sim}{\leftarrow} S_2 \xrightarrow{q_x:1 \text{ étale}} \mathrm{Sht}_B \tag{3.2}$$

and

$$\mathrm{Sht}_B \xleftarrow{1:q_x \text{ étale}} S_1 \overset{\sim}{\rightarrow} \mathrm{Sht}_B \tag{3.3}$$

□

4. Statement of the main theorem

There's a finite type substack of Sht_G outside of which the map from Sht_B is an isomorphism. Thus Sht_B is the "infinite part" of Sht_G. So the cohomology of Sht_G on this infinite part is the same as on the corresponding part of Sht_B, which can then be calculated by pushforward to Sht_H. The Sht_H is a $\mathrm{Pic}^0(\mathbf{F}_q)$-torsor over X, which we understand well. So the issue is in understanding the fibers of $\mathrm{Sht}_B \to \mathrm{Sht}_H$.

Theorem 4.1. *For large enough degrees, fibers of $\mathrm{Sht}_B^d \to \mathrm{Sht}_H^d$ are isomorphic to an affine space $\mathbf{G}_a^{r/2}$ divided by a finite étale group scheme Z.*

Corollary 4.2. *Let $\pi_G \colon \mathrm{Sht}_G \to X^r$. For large d, the cone of*

$$R\pi_{G!}(\mathrm{Sht}_G^{<d}) \to R\pi_{G!}(\mathrm{Sht}_G^{\leq d})$$

has cone some locally constant sheaf on X^r, concentrated in degree r.