Proof of Langlands for GL(2), II

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April 8, 2016

1 Overview

Let $X/\mathbb{F}_q$ be a smooth, projective, geometrically connected curve. The aim is to show that if E is a geometrically irreducible local system of rank 2 on X, then there is a Hecke eigensheaf $\mathrm{Aut}_E =: A_E$ on Bun_2 with eigenvalue E. Under the function-sheaf correspondence this Aut_E gives the automorphic function f_E.

Let us recall the strategy from the very beginning: the rank 1 case. If L is a local system of rank 1 then we knew how to construct a local system on the symmetric power

$$\begin{array}{c} X^{(d)} \\ \downarrow \\ \mathrm{Pic}^d(X) \end{array}$$

We can think of $X^{(d)}$ as the classifying space of line bundles of degree *d plus a section*. There we constructed the sheaf $L^{(d)}$, the symmetric products of L. The idea is that $X^{(d)} \to \mathrm{Pic}^d(X)$ is a fiber bundle with fibers being projective spaces (for large enough d), so any local system descends.

In Stefan's talk, we saw how to construct a candidate function/sheaf A'_E on a space Bun'_2 lying over Bun_2:

$$\begin{array}{c} \mathrm{Bun}'_2 = \{\Omega \hookrightarrow \mathcal{E}\}. \\ \downarrow \\ \mathrm{Bun}_2 \end{array}$$

Just as in the rank 1 case, over a large open subset the fibers are projective spaces, so if the A'_E were a local system we were be done by descent. Unfortunately it is not a local system, so we need to find some other way to descend A'_E to A_E on Bun_2.

Aims. The rest of the argument breaks up into three steps.

1. Show that A'_E is a perverse sheaf.

2. Descend A'_E to A_E on Bun_2.

3. Show that A_E is a Hecke eigensheaf. (This is sort of independent of the other two steps.)

Some of the constructions only work over large open subsets because the map is only a fibration on such. We will happily ignore these issues.

2 Construction of the Laumon sheaf

We briefly remind you about the construction of the Laumon sheaf. There is a moduli space Mod^d parametrizing degree d modifications of vector bundles:

$$\mathrm{Mod}^d = \left\{ \begin{matrix} \mathcal{E}, \mathcal{E}' \in \mathrm{Bun}_2 \\ \iota \colon \mathcal{E}' \hookrightarrow \mathcal{E} \\ \deg(\operatorname{coker} \iota) = d \end{matrix} \right\}.$$

We have a map $\mathrm{Mod}^d \to \mathrm{Coh}^d$ sending $(\mathcal{E}' \subset \mathcal{E}) \mapsto (\mathcal{E}/\mathcal{E}')$. This is a smooth map, since the fibers are parametrized by a choice of $\mathcal{E} \in \mathrm{Bun}_2$ plus choices of points in projective spaces specifying the modifications.

But while Mod^d is infinite-dimensional, the space Coh^d is a finite-dimensional space related to symmetric powers of the curve. So we can think of it as a finite-dimensional model for Mod^d. A resolution of singularities for Coh^d is given by specifying a "flag" of torsion sheaves with subquotients of length one (which will be uniquely determined for most torsion sheaves).

$$\begin{CD} \widetilde{\mathrm{Coh}}^d @. = \left\{ \begin{matrix} \mathcal{T}_1 \subset \ldots \subset \mathcal{T}_d \\ \deg \mathcal{T}_{i+1}/\mathcal{T}_i = 1 \end{matrix} \right\} \\ @V{\pi}VV @VVV \\ \mathrm{Coh}^d @. = \{T_d\} \end{CD}$$

Oh Coh^d, we defined the Laumon sheaf L_E^d as follows. We have a commutative diagram

$$\begin{CD} \widetilde{\mathrm{Coh}}_0^d @>{\mathrm{gr}}>> (\mathrm{Coh}_0^1)^d @>>> X^d \\ @V{\pi}VV @. @VVV \\ \mathrm{Coh}_0^d @>>> @>>> X^{(d)} \end{CD}$$

Then we defined

$$\mathcal{L}_E := R\pi_*(\mathrm{gr}^* E^{\boxtimes d})^{S_d}.$$

This is formally similar to what we did in the case of GL_1. Yesterday Stefan defined it slightly differently, as $\mathcal{L}_E = j_{!*} E^{(d)}|_{(X^{(d)} \setminus \Delta)}$. The two definitions turn out to coincnide, giving a way of computing this middle extension sheaf.

3 Construction of the sheaf A'_E

As Dennis discussed yesterday, one wants to consider another moduli space

$$Q = \left\{ \begin{matrix} \mathcal{E} \in \mathrm{Bun}_2 \\ \mathcal{J} \in \mathrm{Ext}^1(\mathcal{O}, \Omega) \\ \mathcal{J} \hookrightarrow \mathcal{E} \end{matrix} \right\}$$

Recall that Bun'_2 is the moduli space parametrizing $\{(\mathcal{E} \in \mathrm{Bun}_2, \Omega \hookrightarrow \mathcal{E})\}$. There is a map

$$\nu \colon Q \to \mathrm{Bun}'_2$$

by sending $(\mathcal{J} \hookrightarrow \mathcal{E})$ to $(\mathcal{E}, \Omega \hookrightarrow \mathcal{J} \hookrightarrow \mathcal{E})$.

In addition, we have maps

1. $\mathrm{ext} \colon Q \to \mathbb{A}^1$, sending the datum $(\mathcal{J} \hookrightarrow \mathcal{E})$ to the class of $\mathcal{J}$ in $\mathrm{Ext}^1(\mathcal{O}, \Omega) = H^1(\Omega) \cong H^0(\mathcal{O})$.
2. $q \colon Q \to \mathrm{Coh}$ sending the datum $(\mathcal{J} \hookrightarrow \mathcal{E})$ to the torsion sheaf $\mathcal{E}/\mathcal{J}$.

This all fits together in the following diagram.

$$\begin{array}{ccc} Q & \xrightarrow{\mathrm{ext} \times q} & \mathbb{A}^1 \times \mathrm{Coh} \\ \downarrow{\scriptstyle \nu} & & \\ \mathrm{Bun}'_2 & & \\ \downarrow & & \\ \mathrm{Bun}_2 & & \end{array}$$

Definition 3.1. Let $\mathcal{L}_\chi$ be an Artin-Schreier sheaf on $\mathbb{A}^1$. We define

$$A'_E = \nu_!(\mathrm{ext}^*(AS_{\mathbb{A}^1}) \otimes q^*(\mathcal{L}_E))$$

where $\mathcal{L}_E$ is the Laumon sheaf on Coh.

4 Perversity

We want to convine you that A'_E is a perverse sheaf. To do this, it would suffice to rewrite it as an iterated sequence of Laumon's Fourier transforms, since we know that Fourier transforms latter preserves perversity.

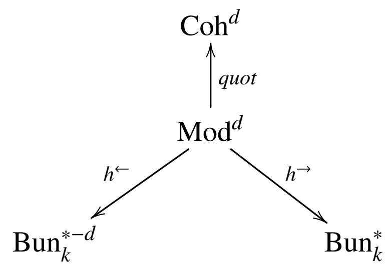
We need the following basic vanishing result.

Lemma 4.1. *For all $k < n$ and all bundles $\mathcal{E}, \mathcal{E}' \in \mathrm{Bun}_k$ with $\deg \mathcal{E}' \leq \deg \mathcal{E} - d$ with $d > kn(2g-2)$, we have*

$$H^*(\mathrm{Hom}^{inj}(\mathcal{E}', \mathcal{E}) \subset \mathrm{Mod}, q^*\mathcal{L}_E) = 0.$$

Example 4.2. Let's unravel the statement of the lemma for $n = 2$. By twisting, we can assume that $\mathcal{E}'$ is trivial. Then the claim is that on the space $\mathcal{H}^0(\mathcal{E}) - 0$, the cohomology of the Laumon sheaf vanishes.

Here is an equivalent formulation. Consider the diagram



Define the averaging functor

$$Av_E\colon D^b(\mathrm{Bun}_k) \to D^b(\mathrm{Bun}_k)$$

by

$$K \mapsto h^{\to}_!(h^{\leftarrow *}(K) \otimes quot^* \mathcal{L}_E).$$

Then the claim is that this is identically 0 for $d > kn(2g-2)$ and E irreducible of rank $n > k$. This goes back to Dennis's "formula" for producing Hecke eigensheaves from yesterday: the formula is to pull back, convolve with the Laumon sheaf, and push forward. This is telling us that if you perform this process when the rank is too small, so that you don't expect to get any eigensheaves, then you'll get 0.

For $n = 2, k = 1$ the statement is easy: for $E^{(d)}$ on $X^{(d)} \xrightarrow{AJ} \mathrm{Pic}^d$, we have

$$R(AJ_*)E^{(d)} = 0 \text{ if } E \text{ is irreducible.}$$

This is a result of Deligne.

Proof sketch. First check that this is a local system by checking that the map is locally acyclic. Then since Pic has abelian fundamental group, if there is a non-trivial cohomology sheaf we can tensor with a local system of rank 1 to make one summand trivial, so in particular it has sections. But then one would find a non-trivial cohomology group upstairs on $X^{(d)}$ by the Leray spectral sequence.

On the other hand, one can compute the cohomology using the Künneth formula on X. By duality the only non-zero group must be in H^1, and the cohomology of the symmetric power is the exterior power of the cohomology, which vanishes for dimension reasons. □

♠♠♠ TONY: [there was then a discussion of the Fourier transform, but I could not follow it]

5 The descent step

An irreducible perverse sheaf is a local system over an open subset. Then we can apply this theorem if we know that the open subset contains one pulled back from downstairs. How do we prove something like this?

The general trick is that if you have a perverse sheaf A' on a smooth space, it is locally constant if and only if the Euler characteristics of stalks are constants. Why? Assume that A' is irreducible (hence an IC sheaf); it's easy to reduce to this case. Then over an open set it's a local system. We know that A' is a middle extension (since it's an IC sheaf), so think about what happens when we form the middle extension. Since we're on a smooth space, the only action happens at the codimension one stalks, and *there you take invariants under inertia*, so the Euler characteristic can only stay constant if it extends to a local system in codimension one.

So we want the Euler characteristics to be constant along fibers of the map $\mathrm{Bun}' \to \mathrm{Bun}$. In the Frenkel-Gaitsgory-Vilonen article this is argued as follows. If we hadn't constructed our sheaf by Fourier transform, but instead by a procedure that *only uses pushforward from proper maps*, then we would know that the Euler characteristic only depends on the local isomorphism classes of the input sheaves. (In characteristic 0, this is clear from the topology perspective because, since the sheaves are locally isomorphic, we can take a small triangulation in which they are isomorphic, and then the Euler characteristics would coincide because cohomology can be computed locally. In characteristic p, it follows by reduction to the case of curves and using the Grothendieck-Ogg-Shafarevich formula.)

To summarize, the idea is to show that this is independent of E! This holds because we can rewrite the construction using proper maps only. That's where the Drinfeld compactification is useful. Recall that this was a compactification of Q by cutting up $\mathcal{E}$ into a flag with subquotients being powers of Ω; in terms of Plücker coordinates it could be described (for GL_2) as

$$\overline{Q}^d = \left\{ \substack{\Omega \hookrightarrow \mathcal{E} \\ \Omega \otimes \mathcal{O} \hookrightarrow \wedge^2 \mathcal{E}} \right\}$$

$$\begin{array}{ccc} & & \{\mathcal{J} \subset \mathcal{E}\} \\ & \swarrow & \downarrow \\ \overline{Q}^d = \left\{ \substack{\Omega \hookrightarrow \mathcal{E} \\ \Omega \otimes \mathcal{O} \hookrightarrow \wedge^2 \mathcal{E}} \right\} & & \mathrm{Mod}^d \\ \downarrow & \swarrow & \\ \mathrm{Bun} & & \end{array}$$

where the left down map is proper after dividing by $\mathbb{G}_m$s. You then rewrite the construction in terms of $\overline{Q}^d$.

The upshot of this discussion is that *we only need the result for one E* (irreducible or not). Now there are several options. For example, we could try the trivial bundle. Then we would get a purely geometric statement, which is unclear how to prove. In the paper, Frenkel-Gaitsgory-Vilonen give the following argument instead. We took the Fourier transform, so we know that if we start with a pure local system then we end up with a pure perverse sheaf. So to get constancy along fibers *it suffices to show that the trace function is constant along fibers*. So we need a local system E such that f'_E comes from f_E downstairs. That is, all we need an automorphic function for *one* local system. We could try to construct this by cyclic base change, which is what they do, but it is hard!

There's also a different argument by Gaitsgory, which goes by comparing the construction with Eisenstein series for a generic $E = \bigoplus L_i$. If you look at how people construct geometric Eisenstein series, then you see that they also use bundles with flags, so one could expect a comparison. The first problem with this approach is that the identity $j_! = j_*$ no longer holds for reducible local systems. But recall how we proved this in rank 2: it was again a computation that something was locally acyclic. So again the we can try to prove that the Euler characteristic doesn't depend on the local system. In the end, it comes down to comparing the intermediate extension and extension-by-zero using the Euler characteristic.

Anyway, this proves that A'_E descends.

6 The Hecke eigensheaf property

We first reduce to checking the eigensheaf property for the first Hecke operator T_1, which comes from the correspondence

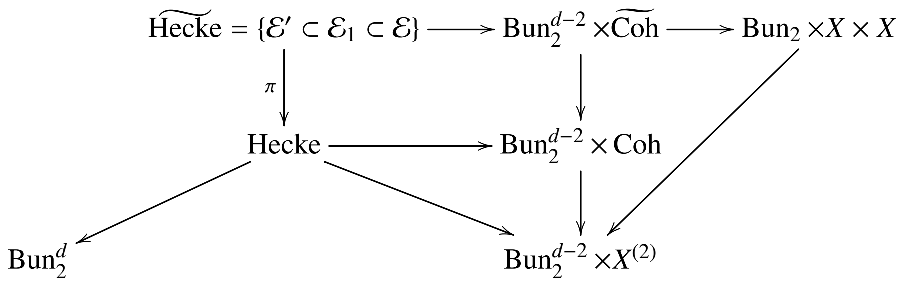
$$\begin{array}{ccc} & \{(x, \mathcal{E}' \subset \mathcal{E}) \colon \mathcal{E}/\mathcal{E}' \cong k(x)\} & \\ \swarrow & & \searrow \\ \mathrm{Bun}_n^d & & \mathrm{Bun}_n^{d-1} \times X \end{array}$$

The point is that the S_2 symmetry implies that this is an eigensheaf for all Hecke operators.

Example 6.1. Consider modifications of length 2 for rank 2 bundles.

$$\begin{array}{ccc} & \mathrm{Hecke} = \left\{ {\mathcal{E}' \hookrightarrow \mathcal{E} \atop \text{length } \mathcal{E}/\mathcal{E}' = 2} \right\} & \\ \swarrow & & \searrow \\ \mathrm{Bun}_2^d & & \mathrm{Bun}_2^{d-2} \times X^{(2)} \end{array}$$

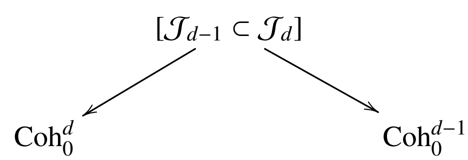
The Hecke eigensheaf property describes what happens if we start with A_E viewed as a perverse sheaf on Bun_2, pull it back to Hecke and convolve with the IC sheaf, and then apply proper pushforward. Consider the diagram



Instead of pulling and pushing, consider going around the top of the diagram. The only difference is that we get $A_E \boxtimes E \boxtimes E$ pushing up through the top of the diagram. The map $\widetilde{\text{Hecke}} \to \text{Hecke}$ is a small resolution, which is locally modelled by $\widetilde{\text{Coh}} \to \text{Coh}$. Therefore, for $n = 2$ the fibers of π are finite except over the diagonal points $k(x)^{\oplus 2}$, where they are $\mathbb{P}^1$. So we know that the pushforward is a sum of perverse sheaves (which are their own middle extensions). Then $R\pi_*\mathbb{Q}$ has an action of S_2, and $(R\pi_*\mathbb{Q})^{S_2}$ gives the Hecke operator T^2 supported on the diagonal.

♠♠♠ TONY: [I didn't understand this example.]

So how do you check the Hecke eigensheaf property for T_1? The easiest thing to say in 2 minutes is that we just compute. Perhaps we should also say the Laumon sheaf has a Hecke property. You also check this by computation. A useful approach is to use the diagram



and check that it transform $\mathcal{L}_E^d$ to $\mathcal{L}_E^{d-1} \boxtimes L_E^1 = E$.