Formulation of Fargues' Conjecture

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Let $E = \mathbb{F}_q((t))$ or a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$, and ϖ be a uniformizer for E. We fix an algebraic closure $\overline{\mathbb{F}}_q$. Let G/E be a quasi-split reductive group.

(Several of the assertions here are not yet proved.)

1 The stack Bun_G

1.1 Structure as a diamond

For a perfectoid space $S \in \mathrm{Perf}_{\mathbb{F}_p}$, we can associate the "relative Fargues-Fontaine curve" X_S, which is an adic space over E. Then Bun_G is the stack on $\mathrm{Perf}_{\mathbb{F}_q}$ for the pro-étale topology, with functor of points

$$\mathrm{Bun}_G(S) = \{G\text{-bundles}/X_S\}.$$

Theorem 1.1. *We have the following.*

1. *The diagonal Δ_{Bun_G} is represented by a diamond. (In equal characteristic, it is even a perfectoid space.)*

2. *For all vector bundles $\mathcal{E}$ on X_S, we define the sheaf*

$$\begin{array}{l} \mathrm{Quot}^{\lambda}_{\mathcal{E}/X/S} \;:\; T/S \to \{\textit{locally free quotients of } \mathcal{E}|_{X_T}\} \\ \quad\big\downarrow \\ \quad S \end{array}$$

   This is represented by a diamond over S. (Again, in equal characteristic it is even represented by a perfectoid space.)

Remark 1.2. The theorem reflects the general phenomenon that one doesn't need stacks in equal characteristic.

This gives a "smooth" presentation of Bun_G by perfectoid spaces. We want to use it later to give a more constructive proof that the B^+_{dR}-affine Grassmannian is a diamond.

1.2 Points

For $b \in G(\breve{E})$ we obtain a point

$$x_b\colon \operatorname{Spa}(\overline{\mathbb{F}}_q) \to \operatorname{Bun}_{G,\overline{\mathbb{F}}_q} \tag{1}$$

Here $\operatorname{Spa}(\overline{\mathbb{F}}_q)$ is the sheaf on $\operatorname{Perf}_{\overline{\mathbb{F}}_p}$ which assigns to each perfectoid space a point, so it is tautologically the final object. The map is given by $\mathcal{E}_b$, i.e. assigns to S over $\operatorname{Spa}(\overline{\mathbb{F}}_q)$ the G-bundle $\mathcal{E}_b/X_S$.

This induces a bijection

$$B(G) \xrightarrow{\sim} |\operatorname{Bun}_{G,\overline{\mathbb{F}}_q}|$$

(modulo a conjecture in the equal characteristic case $E = \mathbb{F}_q((\varpi))$). Since we have defined $\operatorname{Bun}_{G,\overline{\mathbb{F}}_q}$ as a sheaf it may not be clear what is meant by its points: that meaning is

$$|\operatorname{Bun}_{G,\overline{\mathbb{F}}_q}| = \left(\coprod_{F \text{ perfectoid}/\overline{\mathbb{F}}_q} \operatorname{Bun}(F) \right) / \sim .$$

1.3 Connected components

We now discuss the topology on Bun_G. Again, it is not obvious what this means. The answer is that the topology is determined by declaring the open sets to be those coming from open substacks. (Conjecturally the topology on $B(G)$ is such that the closure of a point is the set of points with HN polygon over it.)

We have seen that there is a Kottwitz map

$$\kappa\colon B(G) \to \pi_1(G)_\Gamma$$

where $\Gamma = \operatorname{Gal}(\overline{E}/E)$.

Theorem 1.3. *(Assume that $Z_{G_{sc}}$ is étale if $E = F_q((\varpi))$.) The map κ is locally constant on* Bun_G.

This gives a decomposition

$$\operatorname{Bun}_G = \coprod_{\alpha \in \pi_1(G)_\Gamma} \operatorname{Bun}^\alpha$$

where $\operatorname{Bun}^\alpha = \kappa^{-1}(\alpha)$, which is open and closed.

1.4 Harder-Narasimhan filtration

Fix as usual a triplet $(A \subset T \subset B)$ where B is a Borel, T is a maximal torus inside B, and A is a maximal split torus inside T. There is a "Harder-Narasimhan polygon map"

$$HN\colon |\operatorname{Bun}_G| = B(G) \to X_*(A)^+_{\mathbb{Q}}$$

and a theorem of Kedlaya-Liu implies that this is semi-continuous. What is important is that this implies Bun^{ss} is open. Moreover,

1. each $|\mathrm{Bun}_G^{\alpha,ss}|$ is a single point, represented by a basic element of $B(G)_{\mathrm{basic}}$ via

$$\kappa\colon B(G)_{\mathrm{basic}} \xrightarrow{\sim} \pi_1(G)_\Gamma.$$

In other words, there is a unique semi-stable point in each component, which is the image of the basic locus.

2. For all $\nu \in X_*(A)^+_{\mathbb{Q}}$, $|\mathrm{Bun}_G^{\alpha,HN=\nu}|$ is either empty or a singleton.

1.5 Uniformization

When b is basic, the map (1) giving the point $\mathcal{E}_b$ descends through the quotient by $J_b(E) = \mathrm{Aut}(\mathcal{E}_b)$:

$$x_b\colon [\mathrm{Spec}\,(\overline{\mathbb{F}}_q)/\underline{J_b(E)}] \xrightarrow{\sim} \mathrm{Bun}_{G,\overline{\mathbb{F}}_q}^{\kappa(b),ss}$$

where the left side is the classifying stack of pro-étale $\underline{J_b(E)}$-torsors. These J_b are extended pure inner forms, as in Ana's talk.

What is the meaning of this map? An S-point of $[\mathrm{Spec}\,(\overline{\mathbb{F}}_q)/\underline{J_b(E)}]$ is a $J_b(E)$-torsor over S. What torsor is it? The automorphism torsor of the bundle $\mathcal{E}_b$ on X_S.

Remark 1.4. The dimension of the non-semistable Harder-Narasimhan strata goes to $-\infty$ when you go deeper in the Weyl chamber.

2 Through the looking glass

2.1 The mirror curve

The moduli space of effective degree 1 Cartier divisors on the curve is not itself a curve (unlike in the classical setting). We call it the "mirror curve". To describe it, recall the diamond formula for the Fargues-Fontaine curve:

$$\begin{array}{ccc} X_S^\diamond & = & (S \times \mathrm{Spa}(E)^\diamond)/\varphi_S^{\mathbb{Z}} \\ \downarrow & & \\ \mathrm{Spa}(E)^\diamond & & \end{array}$$

The mirror curve is a characteristic p version which sits over S:

$$\begin{array}{c} (S \times \mathrm{Spa}(E)^\diamond)/\varphi_{E^\diamond}^{\mathbb{Z}} \\ \downarrow \\ S \end{array}$$

These two diamonds have the same topological space, the same étale site, and are locally isomorphic, but they are not isomorphic (for instance, $X_S^\diamond$ has no natural map to S). The point is that $\varphi_S \circ \varphi_{E^\diamond}$ is the identity on $S \times \mathrm{Spa}(E)^\diamond$; therefore, quotienting by one or the other gives us something interesting, while quotienting by both at the same time does nothing.

Let us carefully highlight the difference between φ_S and $\varphi_{E^\diamond}$. Consider a T-valued point of $S \times \mathrm{Spa}(E)^\diamond$ (so T is a perfectoid space over S).

$$\begin{array}{ccc} T & \longrightarrow & S \times \mathrm{Spa}(E)^\diamond \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

The Frobenius φ_S acts on the S coordinate, translating the structure morphism $T \to S$ by φ_S.

On the other hand, by definition $\mathrm{Spa}(E)^\diamond(T)$ is the set of untilts $(T^\#, \iota)$ of T over E. The Frobenius φ_E acts by translating the untilt isomorphism $\iota \colon T \cong T^{\#,\flat}$ by φ_T.

Example 2.1. If $E = \mathbb{F}_q((\varpi))$, then $Y_S = \mathbb{D}_S^*$. This has two maps

$$\begin{array}{ccc} & Y_S = D_S^* & \\ \swarrow & & \searrow \\ S & & \mathbb{D}_{\mathbb{F}_q}^* = \mathrm{Spa}(E) \end{array}$$

We have $X_S = \mathbb{D}_S^* / \varphi_S$. In this case the mirror curve is

$$\mathrm{Div}^1_{X/S} = \mathbb{D}_S^{*,1/p^\infty} / \varphi_E^{\mathbb{Z}}.$$

2.2 The mirror curve as the moduli space of divisors

The remarkable fact is that the mirror curve can be identified with the moduli space of degree 1 divisors on X:

$$\boxed{S \times \mathrm{Spa}(E)^\diamond / \varphi_{E^\diamond}^{\mathbb{Z}} \xrightarrow{\sim} \mathrm{Div}^1_{X/S} = \left\{ \begin{matrix} \mathcal{L} = \deg 1 \text{ line bundle on } X_S \\ f \in H^0(X_S, \mathcal{L}) \text{ fiberwise non-zero}/S \end{matrix} \right\}}$$

Another way to say this is that

$$\mathrm{Div}^1_{X/S} \cong (B_S^{\varphi = \varpi} \setminus \{0\}) / \underline{E}^*,$$

where the right hand side is viewed as a projective space over a Banach-Colmez space.

More generally, for all $d \geq 1$ we define the moduli space

$$\begin{array}{l} \mathrm{Div}^d_{X/S} = \{\deg d \text{ effective Cartier divisors on } X_S\} \\ \quad \Big\downarrow \\ \quad S \end{array}$$

and

$$\mathrm{Div}^d_{X/S} = \mathrm{Div}^d_X \times_{\mathbb{F}_q} S.$$

The Div^d_X is not a diamond, but it is an "absolute diamond". This just means that it is not representable by a diamond, but its pullback to any perfectoid space is a repesentable by a diamond.

Example 2.2. The situation is similar for $\mathrm{Spa}(\overline{\mathbb{F}}_q)$: it is not a diamond, but the "diagonal is a diamond", i.e. the pullback to any perfectoid space is a diamond. This is just the statement that the final object, which takes every perfectoid space to a point, is not a diamond; but after base-changing to a perfectoid space S, obviously S is the final object in the category of perfectoid spaces over S.

Theorem 2.3. *We have an isomorphism*

$$\frac{\mathrm{Spa}(E)^\diamond \times \ldots \times \mathrm{Spa}(E)^\diamond}{\varphi^{\mathbb{Z}}_{E^\diamond} \times \ldots \times \varphi_{E^\diamond} \rtimes S_d} \xrightarrow{\sim} \mathrm{Div}^d_X \cong (B_S^{\varphi = \varpi^d} \setminus \{0\})/\underline{E}^*$$

as absolute diamonds.

We have a map $\mathrm{Div}^d_X \to \mathrm{Pic}^d_X := [\mathrm{Spa}\, \mathbb{F}_q / \underline{E}^*]$

$$\begin{array}{ccccc} \frac{\mathrm{Spa}(E)^\diamond \times \ldots \times \mathrm{Spa}(E)^\diamond}{\varphi^{\mathbb{Z}}_{E^\diamond} \times \ldots \times \varphi_{E^\diamond} \rtimes S_d} & \xrightarrow{\sim} & \mathrm{Div}^d_X & \xrightarrow{\sim} & (B_S^{\varphi = \varpi^d} \setminus \{0\})/\underline{E}^* \\ & \searrow^{AJ^d} & & & \downarrow \\ & & & & \mathrm{Pic}^d_X := [\mathrm{Spa}\, \mathbb{F}_q / \underline{E}^*] \end{array}$$

which plays the role of the *Abel-Jacobi map* AJ^d. This looks like it's over a point; but when you pull back to any S you see the Abel-Jacobi map for the relative curve.

Remark 2.4. The identification $\mathrm{Pic}^d_X := [\mathrm{Spa}\, \mathbb{F}_q / \underline{E}^*]$ depends on a choice of $\mathcal{O}(d)$, which depends on a choice of uniformizing element.

3 Hecke correspondences

For $\mu \in X_*(T)/\Gamma$ (just assume G is split), we have a Hecke correspondence $\mathrm{Hecke}^{\leq \mu}$

$$\begin{array}{ccc} & \mathrm{Hecke}^{\leq \mu} & \\ {}^{h^{\leftarrow}}\swarrow & & \searrow^{h^{\rightarrow}} \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times_{\mathbb{F}_q} \mathrm{Div}^1_X \end{array}$$

where $\mathrm{Hecke}^{\leq \mu}$ is the moduli stack

$$\mathrm{Hecke}^{\leq \mu}(S) = \left\{ \begin{array}{c} \mathcal{E}_1, \mathcal{E}_2 = G\text{-bundles} \\ D \in \mathrm{Div}^1_{X/S} \\ u \colon \mathcal{E}_1 \overset{\leq \mu}{\dashrightarrow} \mathcal{E}_2 \text{ such that} \\ \mathrm{coker}\, \mu \text{ supported on } D \end{array} \right\}$$

The map $h^{\leftarrow}$ takes $(\mathcal{E}_1, \mathcal{E}_2, D, \mu)$ to $\mathcal{E}_2$ and $h^{\rightarrow}$ takes it to $(\mathcal{E}_1, D)$. The map $h^{\rightarrow}$ is locally (in the pro-étale topology) a fibration in $\mathrm{Gr}_G^{B_{\mathrm{dR}}, \leq \mu} / \varphi_{E^\diamond}^{\mathbb{Z}}$.

Example 3.1. If $E = \mathbb{F}_q((\varpi))$ and G is a reductive group over E, then the affine Grassmannian Gr is an ind-scheme over E, whose functor of points is the sheafification of the presheaf $R \mapsto G(R((T)))/G(R[[T]])$. For R an E-algebra,

$$\mathrm{Gr}^{B_{\mathrm{dR}}} = \varprojlim_{\mathrm{Frob}} \mathrm{Gr}^{\mathrm{ad}}$$

and ind-perfectoid space. We know geometric Satake for IC_μ.

4 The conjecture

4.1 Setup

Assume $l \neq p$.

- Set ${}^L G = \widehat{G} \rtimes W_E$, where $\widehat{G}$ is the $\overline{\mathbb{Q}}_\ell$-Langlands dual of G.
- Let $\phi \colon W_E \to {}^L G$ be a Langlands parameter and

  $$S_\phi = \mathrm{Aut}(\phi) = \{g \in \widehat{G} \mid g\phi g^{-1} = \phi\}.$$

  We have $Z(\widehat{G})^\Gamma \subset S_\phi$. Suppose ϕ is discrete, so $S_\phi / Z(\widehat{G})^\Gamma$ is finite.
- Fix a Whittaker datum (B, ψ).

4.2 The conjecture

There exists a "perverse" Weil sheaf $\mathcal{F}_\phi$ on $\mathrm{Bun}_{G, \overline{\mathbb{F}}_q}$ (with "Weil" structure coming from $\overline{\mathbb{F}}_q$) equipped with an action of S_ϕ, satisfying the following properties.

1. For all $\alpha \in \pi_1(G)_\Gamma$, the action of $Z(\widehat{G})^\Gamma$ on $\mathcal{F}_\phi|_{\mathrm{Bun}^\alpha}$ is given by α via the identification $\pi_1(G)_\Gamma = X^*(Z(\widehat{G})^\Gamma)$.
2. Suppose that ϕ is moreover cuspidal, meaning that the composite map

   $$\begin{array}{ccc} I_E & \longrightarrow & \widehat{G} \\ & {\scriptstyle \phi|_{I_E}} \searrow \quad \nearrow & \\ & {}^L G & \end{array}$$

   has finite image. Then $\mathcal{F}_\phi$ is cuspidal, meaning that

   $$\mathcal{F}_\phi = j_! j^* \mathcal{F}_\phi \quad \text{for} \quad j \colon \mathrm{Bun}^{ss} \hookrightarrow \mathrm{Bun}\,.$$

3. For all $b \in G(\breve{E})_{\text{basic}}$, consider the map:

$$x_b \colon [\operatorname{Spa} \overline{\mathbb{F}}_q / \underline{J_b}(E)] \hookrightarrow \operatorname{Bun}_{G, \overline{\mathbb{F}}_q}$$

The pullback $x_b^* \mathcal{F}_\phi$ has an action of $J_b(E) \times S_\phi$ (and the action of $J_b(E)$ is smooth because $\ell \neq p$). We conjecture that

$$x_b^* \mathcal{F}_\phi \cong \bigoplus_{\substack{\rho \in \widehat{S}_\phi \\ \rho|_{Z(\widehat{G})^\Gamma} = \kappa(b)}} \rho \otimes \pi_{\phi,b,\rho}$$

where $\pi_{\phi,b,\rho}$ is a representation of $J_b(E)$. (The direct sum is finite since ϕ is discrete.) Whatever "perverse" means, it should imply $\pi_{\phi,b,\rho}$ is admissible.

We also predict that $\{\pi_{\phi,b,\rho}\}_\rho$ is an L-packet of a local Langlands correspondence for the extended pure inner form J_b of G. Moreover (which is why we need to fix the Whittaker datum) $\pi_{\phi,1,1}$ is the unique generic representation of its L-packet.

4. (Hecke eigensheaf property) For $\mu \in X_*(T)/\Gamma$, there exists $r_\mu \in \operatorname{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L G)$ with the following "eigenvalue" property. For the Weil sheaf $r_\mu \circ \phi \colon W_E \to \operatorname{GL}_n(\overline{\mathbb{Q}}_\ell)$ on $\operatorname{Spa}(E)^\diamond / \varphi_E^{\mathbb{Z}} \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q = \operatorname{Div}^1_{\overline{\mathbb{F}}_q}$, which is equipped with an action of S_ϕ, we have an isomorphism

$$h_!^{\rightarrow}(h^{\leftarrow *}\mathcal{F}_\phi \otimes IC_\mu) \xrightarrow{\sim} \mathcal{F}_\phi \boxtimes r_u \circ \phi$$

as Weil sheaves enriched with S_ϕ-action.

5. ("naïve" character sheaf property) For elliptic $\delta \in G(E)$, which implies that $\delta \in G(\breve{E})$ is basic, we get a map

$$\{G(E)\}_{\text{ellip}} \to B(G)_{\text{basic}}$$

which induces

$$x_\delta \colon \operatorname{Spa}(\overline{\mathbb{F}}_q) \to \operatorname{Bun}_{G, \overline{\mathbb{F}}_q}.$$

Then $x_\delta^* \mathcal{F}_\phi$ has Frobenius and Weil structure on S_ϕ. We ask that Frob act like $\delta \in J_\delta(E)$, meaning that if

$$T_\phi \colon \{G(E)\}_{\text{ellip}} \to \overline{\mathbb{Q}}_\ell$$

is the stable distribution over $G(E)$ attached to ϕ, then

$$\delta \mapsto \operatorname{Tr}(\operatorname{Frob}, x_\delta^* \mathcal{F}_\phi).$$

6. (local/global compatibility) "The Caraiani-Scholze sheaf is purely local linked to $\mathcal{F}_\phi$".