COMPARISON OF $\mathcal{M}_d$ AND $\mathcal{N}_d$; THE WEIGHT FACTORS

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1. Review

1.1. **Goal.** Let D be an effective divisor on X of degree $d \geq \max\{2g'-1, 2g\}$. We have an associated Hecke function h_D. The goal is to prove the *key identity* (Theorem 8.1 in the paper)

$$(\log q)^{-r}\mathbb{J}_r(h_D) = \mathbb{I}_r(h_D)$$

We'll use the notation in the paper [YZ]:

$$\mathcal{A}_d = (\widehat{X}_d \times_{\mathrm{Pic}^d_X} \widehat{X}_d) \setminus \text{both sections vanish.}$$

(This is consistent with talks up to Wednesday morning, but inconsistent with the ones afterwards.)

To prove this we'll use the geometrization of both sides that we have been developing, and which we now review.

1.2. **The analytic side.** On the analytic side, the geometrization takes the form

$$(\log q)^{-r}\mathbb{J}_r(h_D) = \sum_{\underline{d}\in\Sigma_d}\sum_{a\in\mathcal{A}_D(k)} (2d_{12}-d)^r \cdot \mathrm{Tr}(\mathrm{Frob}_a, (Rf_{\mathcal{N}_{\underline{d}}*}L_{\underline{d}})_{\overline{a}}). \tag{1.1}$$

Remark 1.1. We obtained this formula from geometrization of $\mathrm{Tr}(u, h_D)$ by taking the sum over invariants $u \in \mathbf{P}^1(F) - \{1\}$

The analytic side was geometrized by the moduli space $\mathcal{N}_{\underline{d}}$. Recall that we defined a map

$$Rf_{\mathcal{N}_{\underline{d}}*}\colon \mathcal{N}_{\underline{d}} \to \mathcal{A}_d$$

in the following way: it is the restriction to an open substack of the "addition" map

$$\mathrm{add}_{d_{11},d_{22}} \times \mathrm{add}_{d_{12},d_{21}}\colon (\widehat{X}_{d_{11}} \times \widehat{X}_{d_{22}}) \times_{\mathrm{Pic}^d_X} (\widehat{X}_{d_{12}} \times \widehat{X}_{d_{21}}) \to \widehat{X}_d \times_{\mathrm{Pic}^d_X} \widehat{X}_d.$$

Also, recall that the local system $L_{\underline{d}}$ from (1.1) was the restriction to $\mathcal{N}_{\underline{d}}$ from $(\widehat{X}_{d_{11}} \times \widehat{X}_{d_{22}}) \times_{\mathrm{Pic}^d_X} (\widehat{X}_{d_{12}} \times \widehat{X}_{d_{21}})$ of

$$(L_{d_{11}} \boxtimes \mathbf{Q}_\ell) \boxtimes (L_{d_{12}} \boxtimes \mathbf{Q}_\ell)$$

where for $d' \geq 0$, $L_{d'}$ is the local system on $\widehat{X}$ pulled back from the local system on $\mathrm{Pic}^{d'}_X$ corresponding to

$$L := (\nu_*\mathbf{Q}_\ell)^{\sigma=-1}, \quad \nu\colon X' \to X.$$

Note that $L_d|_{X_{d'} \subset \widehat{X}_{d'}}$ is descended from $L^{\boxtimes d'}$ on $X^{d'}$. The upshot is that on the open substack $X_{d'} \subset \widehat{X}_{d'}$ we understand the local system L_d very concretely, so we have a chance of computing $H^*(X_{d'}, L_{d'})$.

1.3. **The geometric side.**

On the geometric side, the geometrization takes the form

$$\mathbb{I}_r(h_D) = \sum_{a \in \mathcal{A}_D(k)} \mathrm{Tr}((f_{\mathcal{M}!}[\mathcal{H}^\diamond]_a)^r \circ \mathrm{Frob}_a, (Rf_{\mathcal{M}!}\mathbf{Q}_\ell)_{\overline{a}}).$$

Here the point is that one can understand $[\mathcal{H}^\diamond]$ as the fundamental class over the "nice" locus $\diamond$. The map $f_{\mathcal{M}} \colon \mathcal{M}_d \to \mathcal{A}_d$ was the norm map

$$\widehat{X'}_d \times_{\mathrm{Pic}^d_X} \widehat{X'}_d \xrightarrow{\widehat{\nu}_d \times \widehat{\nu}_d} \widehat{X}_d \times_{\mathrm{Pic}^d_X} \widehat{X}_d.$$

1.4. **The comparison.**

We have reduced to a comparison of traces on cohomology:

$$\sum_{\underline{d} \in \Sigma_d} \sum_{a \in \mathcal{A}_D(k)} (2d_{12} - d)^r \cdot \mathrm{Tr}(\mathrm{Frob}_a, (Rf_{\mathcal{N}_{\underline{d}}*} L_{\underline{d}})_{\overline{a}}) \sim \sum_{a \in \mathcal{A}_D(k)} \mathrm{Tr}((f_{\mathcal{M}!}[\mathcal{H}^\diamond]_a)^r \circ \mathrm{Frob}_a, (Rf_{\mathcal{M}!}\mathbf{Q}_\ell)_{\overline{a}}).$$

To tackle this, we'll first compare the local systems. So the strategy is:

(1) Compute $Rf_{\mathcal{M}!}\mathbf{Q}_\ell$.
(2) Compute $Rf_{\mathcal{N}_{\underline{d}}!} L_{\underline{d}}$.
(3) Compute the action of $f_{\mathcal{M}!}[\mathcal{H}^\diamond]$.

The idea is to use the "perverse continuation principle". This tells us that if we know that $Rf_{\mathcal{M}!}\mathbf{Q}_\ell$ and $Rf_{\mathcal{N}_{\underline{d}}!} L_{\underline{d}}$ satisfy certain special properties, then we can establish an "identity" between them globally if we can do it over a "nice" open set. This is important technically because the geometrization process was that we understood the relevant moduli spaces well over a nice open subset, but not everywhere.

For this we need to show that the sheaves are (shifted) perverse, and moreover that they are the middle extension of their restriction to a "nice" open subset of $\mathcal{A}_d$. The poins is that middle extensions are completely determined by their restriction to the open subset. So we'll compare $Rf_{\mathcal{M}!}\mathbf{Q}_\ell$ and $Rf_{\mathcal{N}_{\underline{d}}!} L_{\underline{d}}$ on an open subset, using the representation theory of finite groups.

We probably won't have time to do everything, so we'll focus on (1).

2. The geometric side

We want to compute $Rf_{\mathcal{M}!}\mathbf{Q}_\ell$. Let

$$j \colon X_d^\circ \hookrightarrow X_d \hookrightarrow \widehat{X}_d$$

to be the locus of multiplicity-free divisors.

Taking pre-images of X_d°, we get a étale Galois covers

$$\underbrace{(X')^{d,\circ} \xrightarrow{\{\pm 1\}^d} X^{d,\circ} \xrightarrow{S_d} X_d^\circ}_{\mathrm{Gal} = \{\pm 1\}^d \rtimes S_d}.$$

Let $\chi_i \colon \{\pm 1\}^d \to \{\pm 1\}$ be the character which is non-trivial on the first i factors, and trivial on the last $d - i$ factors.

Let $S_{i,d-i} \cong S_i \times S_{d-i}$ be the stabilizer of the first i elements. Let $\Gamma_d(i) = \{\pm 1\}^i \rtimes S_{i,d-i} \subset \Gamma_d := \{\pm 1\}^d \rtimes S_d$. Since the $S_{i,d-i}$-action on $\{\pm 1\}^i$ stabilizes the character χ_i, we can inflate χ_i to $\Gamma_d(i)$ as $\chi_i \boxtimes \mathbf{1}$, and then we set

$$\rho(i) = \mathrm{Ind}_{\Gamma_d(i)}^{\Gamma_d}(\chi_i \boxtimes \mathbf{1}).$$

Note that this has dimension $\binom{d}{i}$. It determines an irreducible local system $L(\rho_i)$ on X_d°. We want to extend this to a (shifted) perverse sheaf $K_i = (j_{!*}L(\rho_i)[d])[-d]$. This is called the *middle extension*: it is a perverse extension of $L(\rho_i)[d]$ to $\widehat{X}_d$, which is characterized by the following property:

> If $Z := \widehat{X}_d - X_d^\circ$, and $i\colon Z \hookrightarrow \widehat{X}_d$, then $(j_{!*}L(\rho_i)[d])[-d]$ is the unique (shifted) perverse extension of $L(\rho_i)[d]$ such that it has no subobjects or quotients of the form i_*M where M is perverse on Z.

This condition can be rephrased in terms of "support and co-support" conditions. Perverse sheaves form an abelian subcategory of $D = D_c^b(\widehat{X}_d)$. They are defined by support and co-support conditions. The support condition cuts out a subcategory ${}^pD^{\leq 0} \subset D$, and the co-support condition is the Verdier dual of the support condition, cutting out ${}^pD^{\geq 0} \subset D$.

The *middle extension*

$$K_i = j_{!*}(L(\rho_i)[d])$$

is the unique perverse extension such that $i^*K \in {}^pD^{\leq -1}(Z)$, and $i^!K \in {}^pD^{\geq 1}(Z)$.

Proposition 2.1. *Assume that $d \geq 2g' - 1$. Then we have a canonical isomorphism of shifted perverse sheaves on $\mathcal{A}_d$:*

$$Rf_{\mathcal{M}!}\mathbf{Q}_\ell \cong \bigoplus_{i,j=0}^{d} (K_i \boxtimes K_j)|_{\mathcal{A}_d}$$

Proof. Recall that $f_{\mathcal{M}}$ is the restriction of

$$\widehat{\nu}_d \times \widehat{\nu}_d\colon \widehat{X'}_d \times \widehat{X'}_d \to \widehat{X}_d \times \widehat{X}_d.$$

Since these maps are proper, we can use base change and the Künneth formula to reduce to showing:

$$R\widehat{\nu}_{d*}\mathbf{Q}_\ell \cong \bigoplus_{i=0}^{d} K_i.$$

We argue this by showing that $R\widehat{\nu}_{d*}\mathbf{Q}_\ell$ is a middle extension (up to shift) and then computing over the "good" open set.

Why is $R\widehat{\nu}_{d*}\mathbf{Q}_\ell$ a middle extension? Actually this follows from a general principle: $\widehat{\nu}_d\colon \widehat{X'}_d \to \widehat{X}_d$ is a *small map*, which means that

$$\mathrm{codim}\{y \in \widehat{X}_d \mid \dim \widehat{\nu}_d^{-1}(y) \geq r\} > 2r \text{ for } r \geq 1.$$

We can check this explicitly: the map $\widehat{\nu}_d$ is a union of $\nu_d\colon X'_d \to X_d$ and $\mathrm{Nm}\colon \mathrm{Pic}^d_{X'} \to \mathrm{Pic}^d_X$. Since $\nu_d\colon X'_d \to X_d$ is finite, the only positive dimensional fibers live over $\mathrm{Pic}^d_X \hookrightarrow \widehat{X}_d$. The codimension here is $d - g + 1$, while the dimension of the fibers in this locus is $g - 1$. So the map will clearly be small for large enough d.

Now, it is formal that if $\widehat{\nu}_d$ is proper and and the source $\widehat{X'}_d$ is smooth and geometrically irreducible, then

$$R\widehat{\nu}_{d*}\mathbf{Q}_\ell = j_{!*}(R\nu^\circ_{d*}\mathbf{Q}_\ell)$$

where $\nu^\circ_d \colon (X')^\circ_d \to X^\circ_d$ (Why? You check the support condition by bounding the cohomological dimension of fibers, and the complex is automatically self-dual because the map is proper, so you get the cosupport condition for free. The strictness of the inequality gets you the middle extension property).

Now it's enough to show that

$$R\nu^\circ_{d*}\mathbf{Q}_\ell \cong \bigoplus_{i=0}^{d} L(\rho_i)$$

This is just an equality of local systems, so it follows from a purely representation-theoretic fact:

$$\operatorname{Ind}_{S_d}^{\Gamma_d} \mathbf{1} = \bigoplus_{i=0}^{d} (\operatorname{Ind}_{\Gamma_d(i)}^{\Gamma_d} \chi_i \boxtimes \mathbf{1}).$$

To prove this, make a dimension count and show that there is a $\Gamma_d(i)$-equivariant embedding $\chi_i \boxtimes \mathbf{1} \hookrightarrow \mathbf{Q}_\ell[\Gamma_d/S_d]$. This can be done explicitly: send

$$\chi_i \boxtimes 1 \mapsto \mathbf{1}_{\chi_i} = \sum_{\varepsilon \in \Gamma_d/S_d} \chi(\varepsilon)\varepsilon.$$

□

3. The analytic side

Next we want to compute $Rf_{\mathcal{N}_{\underline{d}}!}L_{\underline{d}}$ on $\mathcal{A}_d$. Here one wrinkle is that the map $f_{\mathcal{N}_{\underline{d}}} \to \mathcal{A}_d$ is actually not small. It is obviously finite on $\mathcal{B} := X_d \times_{\operatorname{Pic}^d_X} X_d \subset \mathcal{A}_d$. So the problem occurs on

$$A_d \setminus \mathcal{B} = \underbrace{(\{0\} \times X_d)}_{=:\mathcal{C}} \sqcup \underbrace{(X_d \times \{0\})}_{=:\mathcal{C}'}.$$

Let's think about what the fibers look like over $\mathcal{C}$. A point of $\mathcal{C}$ is just a divisor, say $(0, D)$. Assume $d_{11} < d_{22}$; then by the definition of $\mathcal{N}_d$ (which we admittedly skated over) $\varphi_{11} \neq 0$ and $\varphi_{22} = 0$. So the fiber is

$$f^{-1}_{\mathcal{N}_{\underline{d}}}(0, D) = X_{d_{11}} \times \operatorname{add}^{-1}_{d_{12},d_{21}}(D). \tag{3.1}$$

This has dimension d_{11}, which can go up to about $d/2$. You can check that the smallness just fails by a constant factor of about g for all d.

However, we don't really need smallness. If you think about the argument we just made, you'll see that it's enough to bound the *cohomological dimension* (as opposed to dimension) of fibers of $f_{\mathcal{N}_{\underline{d}}}$ to conclude that the pushforward is a middle extension.

By (3.1) the cohomology of the geometric fiber over $(0, D)$ is then $H^*(X_{d_{11}} \otimes_k \overline{k}, L_{d_{11}}) \otimes H^0(\mathrm{add}^{-1}_{d_{12},d_{21}}(D) \otimes_k \overline{k}, L_{d_{12}})$. We can ignore the second factor, since it doesn't effect the cohomological dimension. Since $L_{d_{11}}$ is a non-trivial local system,

$$H^*(X_{d_{11}} \otimes_k \overline{k}, L_{d_{11}}) \cong \bigwedge^d (H^1(X, L_{d_{11}})[-d_{11}]).$$

which vanishes if $d_{11} > 2g - 2$.

The upshot is that $Rf_{\mathcal{N}_{\underline{d}}*} L_{\underline{d}}$ is the middle extension for large enough d. Then you want to show that

$$\mathrm{add}^{\circ}_{j,n-j*}(L_j \boxtimes \mathbf{Q}_\ell) \cong L(\rho_j) \text{ on } X_d^{\circ}.$$

where

$$\mathrm{add}^{\circ}_{j,n-j} \colon (X_j \times X_{d-j})^{\circ} \to X_d^{\circ}.$$

The local system $L_j \boxtimes \mathbf{Q}_\ell$ corresponds to $\chi_j \boxtimes \mathbf{1}$ on $(X_j \times X_{d-j})^{\circ}$.

4. Weight factors

We've completely run out of time to discuss the weight factors. The punchline is that, by using perversity, one can compute the Hecke action on $\mathcal{A}_d^{\diamond}$ using Lemma 6.3 of Liang Xiao's talk. This ends up giving the weight factors $d - 2j$ on $K_i \boxtimes K_j$.