ALTERNATIVE CALCULATION OF $\mathbb{I}_r(h_D)$

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1. Overview

The goal is to sketch the proof of Theorem 6.6, which was stated last time:

Theorem 1.1 (Theorem 6.6)**.** *Let D be an effective divisor on X, of degree $d \geq \max\{2g' - 1, 2g\}$. Then there exists $\zeta \in \mathrm{Ch}_{2d-g+1}(\mathrm{Hk}^{\mu}_{\mathcal{M},d})_{\mathbf{Q}}$ such that*

(1) $\zeta|_{\mathrm{Hk}^{\mu}_{\mathcal{M}^{\diamond},d}}$ *is the fundamental class, and*
(2) $\deg((\mathrm{Id}, \mathrm{Frob}^!)\zeta)_D = \mathbb{I}_r(h_D) = \langle [\mathrm{Sht}^{\mu}_T], h_D * [\mathrm{Sht}^{\mu}_T] \rangle_{\mathrm{Sht}^{\mu}_G}$.

The strategy of the proof is to:

(1) Give a formula for ζ.
(2) Prove that ζ satisfies properties (1), (2) of Theorem (1.1).

The basic idea is that Sht is the intersection of something with the graph of Frobenius. On the right hand side of Theorem 1.1(2), we are taking an intersection of objects obtained by intersecting with the graph of Frobenius. On the left hand side, we are intersecting first and then intersecting with the graph of Frobenius. That these coincide is the substance of the "octahedron lemma" from Rapoport's talk.

2. The fundamental diagram

Consider the commutative diagram of algebraic stacks

$$\begin{array}{ccccc} A_{11} & \xrightarrow{\alpha} & A_{12} & \longleftarrow & A_{13} \\ \downarrow & & \downarrow & & \downarrow \\ A_{21} & \longrightarrow & A_{22} & \longleftarrow & A_{23} \\ \uparrow & & \uparrow & & \uparrow{\scriptstyle\beta} \\ A_{31} & \longrightarrow & A_{32} & \longleftarrow & A_{33} \end{array}$$

Think of A_{13} as the "bad" object. Thus, also fibered products involving it are also "bad", while all objects not involving it are "good". We'll be more precise about this shortly. The point is that we can take the fiber product of the rows and then columns, or columns and then rows.

If we take the fibered products of rows first, then we get

$$\begin{array}{ccccccc}
A_{11} & \xrightarrow{\alpha} & A_{12} & \longleftarrow & A_{13} & & A_{1*} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A_{21} & \longrightarrow & A_{22} & \longleftarrow & A_{23} & & A_{2*} \\
\uparrow & & \uparrow & & \beta\uparrow & & \delta\uparrow \\
A_{31} & \longrightarrow & A_{32} & \longleftarrow & A_{33} & & A_{3*}
\end{array}$$

If we take the fibered products of columns first, then we get and

$$\begin{array}{ccccc}
A_{11} & \xrightarrow{\alpha} & A_{12} & \longleftarrow & A_{13} \\
\downarrow & & \downarrow & & \downarrow \\
A_{21} & \longrightarrow & A_{22} & \longleftarrow & A_{23} \\
\uparrow & & \uparrow & & \beta\uparrow \\
A_{31} & \longrightarrow & A_{32} & \longleftarrow & A_{33} \\
 & & & & \\
A_{*1} & \xrightarrow{\gamma} & A_{*2} & \longleftarrow & A_{*3}
\end{array}$$

Proposition 2.1. *The two fiber products $A_{1*} \times_{A_{2*}} A_{3*}$ and $A_{*1} \times_{A_{*2}} A_{*3}$ are canonically equivalent.*

From the diagrams we have refined Gysin maps

$$\mathrm{Ch}(A_{13}) \xrightarrow{\alpha^!} \mathrm{Ch}(A_{1*}) \xrightarrow{\delta^!} \mathrm{Ch}(A_{1*} \times_{A_{2*}} A_{3*})$$
$$\mathrm{Ch}(A_{13}) \xrightarrow{\beta^!} \mathrm{Ch}(A_{*3}) \xrightarrow{\gamma^!} \mathrm{Ch}(A_{*1} \times_{A_{*2}} A_{*3}).$$

Thanks to Proposition 2.2 it is meaningful to ask if they agree.

Proposition 2.2. *Assume that*

- *the A_{ij} are smooth equidimensional all $i, j \neq 1, 3$,*
- *the $A_{2*}, A_{3*}, A_{*1}, A_{*2}$ are smooth of expected dimension,*
- *the map $\alpha, \beta, \gamma, \delta$ satisfy assumptions (A) and (B) from Rapoport's talk.*

Then we have

$$\delta^! \alpha^! [A_{13}] = \gamma^! \beta^! [A_{13}] \in \mathrm{Ch}(A_{1*} \times_{A_{2*}} A_{3*}) = \mathrm{Ch}(A_{*1} \times_{A_{*2}} A_{*3}).$$

The content of Proposition 2.2, as compared to Proposition 2.1, is that Proposition 2.2 involves both "derived fiber products" and classical fiber products. Therefore, the content of Proposition 2.2 is that the "derived fiber product = usual fiber product". The assumptions are there to make that true.

3. Application to shtukas

3.1. **The fundamental diagram.** We are interested in the specialization of the fundamental diagram to shtukas:

$$\begin{array}{ccccc}
\mathrm{Hk}_T^{\mu} \times \mathrm{Hk}_T^{\mu} & \xrightarrow{\Pi^{\mu} \times \Pi^{\mu}} & \mathrm{Hk}_G'^{r} \times \mathrm{Hk}_G'^{r} & \longleftarrow & \mathrm{Hk}_{G,d}'^{r} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Bun}_T^2 \times \mathrm{Bun}_T^2 & \longrightarrow & \mathrm{Bun}_G^2 \times \mathrm{Bun}_G^2 & \longleftarrow & H_d \times H_d \\
{\scriptstyle \mathrm{Id} \times \mathrm{Frob}}\uparrow & & {\scriptstyle \mathrm{Id} \times \mathrm{Frob}}\uparrow & & {\scriptstyle \mathrm{Id} \times \mathrm{Frob}}\uparrow \\
\mathrm{Bun}_T \times \mathrm{Bun}_T & \xrightarrow{\Pi \times \Pi} & \mathrm{Bun}_G \times \mathrm{Bun}_G & \longleftarrow & H_d
\end{array}$$

Some of the objects of this diagram haven't even been defined yet; we will give the definitions (and recall old ones) shortly.

The fibered products column-wise will be

$$\begin{array}{ccccc}
\mathrm{Hk}_T^{\mu} \times \mathrm{Hk}_T^{\mu} & \xrightarrow{\Pi^{\mu} \times \Pi^{\mu}} & \mathrm{Hk}_G'^{r} \times \mathrm{Hk}_G'^{r} & \longleftarrow & \mathrm{Hk}_{G,d}'^{r} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Bun}_T^2 \times \mathrm{Bun}_T^2 & \longrightarrow & \mathrm{Bun}_G^2 \times \mathrm{Bun}_G^2 & \longleftarrow & H_d \times H_d \\
{\scriptstyle \mathrm{Id} \times \mathrm{Frob}}\uparrow & & {\scriptstyle \mathrm{Id} \times \mathrm{Frob}}\uparrow & & {\scriptstyle \mathrm{Id} \times \mathrm{Frob}}\uparrow \\
\mathrm{Bun}_T \times \mathrm{Bun}_T & \xrightarrow{\Pi \times \Pi} & \mathrm{Bun}_G \times \mathrm{Bun}_G & \longleftarrow & H_d
\end{array}$$

$$\mathrm{Sht}_T^{\mu} \times \mathrm{Sht}_T^{\mu} \xrightarrow{\theta^{\mu} \times \theta^{\mu}} \mathrm{Sht}_G'^{r} \times \mathrm{Sht}_G'^{r} \longleftarrow \mathrm{Sht}_{G,d}'$$

The fibered products row-wise will be

$$\begin{array}{cccccc}
\mathrm{Hk}_T^{\mu} \times \mathrm{Hk}_T^{\mu} & \xrightarrow{\Pi^{\mu} \times \Pi^{\mu}} & \mathrm{Hk}_G'^{r} \times \mathrm{Hk}_G'^{r} & \longleftarrow & \mathrm{Hk}_{G,d}'^{r} & \mathrm{Hk}_{G,d}'^{r} \\
\downarrow & & \downarrow & & \downarrow & \downarrow \\
\mathrm{Bun}_T^2 \times \mathrm{Bun}_T^2 & \longrightarrow & \mathrm{Bun}_G^2 \times \mathrm{Bun}_G^2 & \longleftarrow & H_d \times H_d & \mathcal{M}_d \times \mathcal{M}_d \\
{\scriptstyle \mathrm{Id} \times \mathrm{Frob}}\uparrow & & {\scriptstyle \mathrm{Id} \times \mathrm{Frob}}\uparrow & & {\scriptstyle \mathrm{Id} \times \mathrm{Frob}}\uparrow & {\scriptstyle \mathrm{Id} \times \mathrm{Frob}}\uparrow \\
\mathrm{Bun}_T \times \mathrm{Bun}_T & \xrightarrow{\Pi \times \Pi} & \mathrm{Bun}_G \times \mathrm{Bun}_G & \longleftarrow & H_d & \mathcal{M}_d
\end{array}$$

Definition 3.1. The total fiber product is denoted $\mathrm{Sht}_{\mathcal{M},d}^{\mu}$. This can be viewed as the fiber product of the column-wise fiber products, or as the fiber product as the row-wise fiber products, thanks to Proposition 2.1.

3.2. **Explication of the terms.** Now we're going to tell you what these things are.

- H_d is the Hecke stack. We can view $H_d = \widetilde{H}_d / \operatorname{Pic}_X$, where $\widetilde{H}_d$ parametrizes modifications
$$\phi \colon \mathcal{E} \hookrightarrow \mathcal{E}'$$
with $\operatorname{coker} \phi$ is finite over S and flat of rank d.

- The map $\mathrm{Bun}_T \to \mathrm{Bun}_G$ sends $\mathcal{L} \mapsto \nu_* \mathcal{L}$.
- The stacks Hk_T^μ and $\mathrm{Hk}_G^\mu \cong \mathrm{Hk}_G^r$ parametrize modifications of type μ between chains of bundles. It is perhaps easier to phrase things in terms of Hk_2^μ, which parametrize chains of modifications
$$\mathcal{E}_0 \xrightarrow{\text{at } x_1} \mathcal{E}_1 \xrightarrow{\text{at } x_2} \dots \xrightarrow{\text{at } x_r} \mathcal{E}_r$$
Then we define $\mathrm{Hk}_G^r = \widetilde{\mathrm{Hk}}_G^r / \mathrm{Pic}_X$, and $\mathrm{Hk}'^r_{G,d} = \mathrm{Hk}_G^r \times_{X^r} (X')^r$.
- The stack $\widetilde{\mathrm{Hk}}^r_{G,d}$ parametrizes "modifications of (modifications of type μ) of degree d": that is, chains of rank 2 vector bundles
$$\begin{array}{ccccccc} \mathcal{E}_0 & \xrightarrow{\text{at } x_1} & \mathcal{E}_1 & \xrightarrow{\text{at } x_2} & \dots & \xrightarrow{\text{at } x_r} & \mathcal{E}_r \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{E}'_0 & \xrightarrow{\text{at } x_1} & \mathcal{E}'_1 & \xrightarrow{\text{at } x_2} & \dots & \xrightarrow{\text{at } x_r} & \mathcal{E}'_r \end{array}$$
such that the rows are in Hk_G^r and the columns are in H_d. As usual, we require the modifications of both at $x_1, \dots, x_r$. We define $\mathrm{Hk}_G^r = (X')^r \times_{X^r} \mathrm{Hk}'^r_G$.
- Finally we set $\mathrm{Hk}^r_{G,d} = \widetilde{\mathrm{Hk}}^r_{G,d} / \mathrm{Pic}_X$ and $\mathrm{Hk}'^r_{G,d} = \widetilde{\mathrm{Hk}'}^r_{G,d} / \mathrm{Pic}_X$.

The potentially bad objects in the diagram are $\mathrm{Hk}'^r_{G,d}$ and everything that involves it: $\mathrm{Sht}'_{G,d}$ and $\mathrm{Hk}'^r_{\mathcal{M},d}$. Everything else is smooth. To show this, you first study the objects in the big diagram. For example, the map $\mathrm{Hk} \xrightarrow{\mathrm{pr}_1} \mathrm{Bun}_G$ is smooth of relative dimension $2d$. (We've already seen this for $d = 1$: there is one dimension for the choice of point and one dimension coming from the $\mathbf{P}^1$ parametrizing the choice of modification type at that point.)

3.3. **Intersection numbers.** Specializing Propositions 2.1, 2.2 we get:

Corollary 3.2. *We have*
$$(\mathrm{Id}, \mathrm{Frob})^! (\Pi^\mu \times \Pi^\mu)^! ([\mathrm{Hk}'^r_{G,d}]) = (\theta^\mu \times \theta^\mu)^! (\mathrm{Id} \times \mathrm{Frob})^! ([\mathrm{Hk}'^r_{G,d}]) \in \mathrm{Ch}_0(\mathrm{Sht}^\mu_{\mathcal{M},d})$$

Definition 3.3. We now define $\zeta := (\Pi^\mu \times \Pi^\mu)^! (\mathrm{Hk}'^r_G) \in \mathrm{Ch}_{2d-g+1}(\mathrm{Hk}'^r_{\mathcal{M}_d})$.

We then need to check that ζ enjoys the following two properties promised in Theorem 1.1.

(1) Consider the fiber product
$$\begin{array}{ccc} \mathrm{Hk}^\mu_{\mathcal{M},d} & \longrightarrow & \mathrm{Hk}'^r_{G,d} \\ \downarrow & & \downarrow \\ \mathrm{Hk}^\mu_T \times \mathrm{Hk}^\mu_T & \longrightarrow & \mathrm{Hk}'^r_G \times \mathrm{Hk}'^r_G \end{array}$$

The total space $\mathrm{Hk}_{\mathcal{M},d}$ is bad, and hard to understand. However, we claim that the open substack $\mathrm{Hk}_{\mathcal{M}^\diamond,d}$ has the expected dimension. Then the claim is that $[\zeta] \in \mathrm{Ch}(\mathrm{Hk}_{\mathcal{M}^\diamond,d})$ is the fundamental class.

(2) We know that we can write

$$\mathrm{Sht}^{\mu}_{\mathcal{M}_d} = \coprod_{D \in X_d(k)} \mathrm{Sht}^{\mu}_{\mathcal{M},D},$$

which implies that $\mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M}_d})_{\mathbf{Q}} = \bigoplus \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},D})_{\mathbf{Q}}$. Using Corollary 3.2, we have

$$\begin{aligned}\deg((\mathrm{Id}, \mathrm{Frob}^!)\zeta)_D &= \deg[(\mathrm{Id}, \mathrm{Frob})^!(\Pi^{\mu} \times \Pi^{\mu})^!([\mathrm{Hk}'^{r}_{G,d}]) \mid_{\mathrm{Sht}^{\mu}_{\mathcal{M},D}}] \\ &= (\theta^{\mu} \times \theta^{\mu})^!(\mathrm{Id} \times \mathrm{Frob})^!([\mathrm{Hk}'^{r}_{G,d}]) \in \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},d})\end{aligned}$$

To establish Theorem 1.1 (2), we need to show that

$$(\theta^{\mu} \times \theta^{\mu})^!(\mathrm{Id} \times \mathrm{Frob})^!([\mathrm{Hk}'^{r}_{G,d}]) \in \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},d}) = \mathbb{I}_r(h_D).$$

Note that $(\theta^{\mu} \times \theta^{\mu})^!(\mathrm{Id} \times \mathrm{Frob})^! = [\mathrm{Sht}'^{r}_{G}]$. We use that we have the cartesian diagram

$$\begin{array}{ccc} \mathrm{Sht}_{\mathcal{M}_D} & \longrightarrow & \mathrm{Sht}_G(h_D) \\ \downarrow & & \downarrow \\ \mathrm{Sht}^{\mu}_T \times \mathrm{Sht}^{\mu}_T & \longrightarrow & \mathrm{Sht}'^{r}_G \times \mathrm{Sht}'^{r}_G \end{array}$$

We have the compatibility relation "pullback then restrict to DS is the same as restrict to D and then pullback", so

$$\begin{aligned}\mathbb{I}_r(h_D) &:= \langle \theta^{\mu}_*[\mathrm{Sht}^{\mu}_T], h_D * \theta^{\mu}_*[\mathrm{Sht}^{\mu}_T] \rangle \\ &= \langle (\theta^{\mu} \times \theta^{\mu})_*([\mathrm{Sht}^{\mu}_T] \times [\mathrm{Sht}^{\mu}_T]), \mathrm{pr}_*[\mathrm{Sht}'_G(h_D)] \rangle_{\mathrm{Sht}'^{r}_G \times \mathrm{Sht}'^{r}_G} \\ &= \deg(\theta^{\mu} \times \theta^{\mu})^![\mathrm{Sht}'^{r}_G]\end{aligned}$$

as desired.