Relation to Classical Local Langlands

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The goal is to recall the local Langlands correspondence and its refined form for quasi-split groups, and then move towards non-quasisplit groups. Finally, we'll explain the connection to the upcoming conjecture.

1 The quasisplit case

1.1 The basic conjecture

Let E be a local field of characteristic 0 and G a connected reductive group over E. The basic problem is to understand irreducible admissible representations of $G(E)$. The Langlands correspondence reduces us to understanding the tempered representations $\Pi_{\text{temp}}(G)$.

Definition 1.1. Let $\Gamma = \text{Gal}(\overline{E}/E)$. We denote by L_E the *local Langlands group of E*, which is

$$L_E := \begin{cases} W_E & \text{archimedean} \\ W_E \times \text{SU}_2(\mathbb{R}) & \text{non-arch} \end{cases}$$

Definition 1.2. The *L-group* of G is is

$$^L G := \widehat{G} \rtimes \Gamma$$

with Weil form $\widehat{G} \rtimes W_E$.

Example 1.3. For $G = \text{GL}_n$, $\widehat{G} = \text{GL}_n$. For $G = \text{SL}_n$, $\widehat{G} = \text{PGL}_n$.

Example 1.4. For split groups the L-group splits as a direct product. Inner forms of the same group have the same L-group.

Definition 1.5. We define $\Phi_{\text{temp}}(G)$ to be the set of *tempered L-homomorphisms*

$$\phi \colon L_E \to {}^L G.$$

which are homomorphisms ϕ as above such that

- ϕ commutes with projection to the Weil group,

$$\begin{array}{ccc} L_E & \xrightarrow{\phi} & {}^L G \\ & \searrow \quad \swarrow & \\ & W_E & \end{array}$$

- (*tempered*) ϕ has bounded image in $\widehat{G}$, and
- (*admissible*) ϕ maps the Weil group to semisimple elements in $\widehat{G}$.

Conjecture 1.6 (Conjecture A)**.** *There exists a map*

$$LL\colon \Pi_{\text{temp}}(G) \to \Phi_{\text{temp}}(G)$$

with finite fibers $\Pi_\phi(G) := LL^{-1}(\phi)$*, which are called L-packets for the tempered parameter* ϕ*.*

Example 1.7. For GL_n this is a bijection (each Π_ϕ is a singleton). But SL_n already has two elements in its discrete series L-packets.

Remark 1.8. The map has nice properties.

- We understand the image (it should be those ϕ factoring through parabolic subgroups relevant to G).
- In the unramified case the correspondence is controlled by the Satake isomorphism.
- We understand how this behaves with respect to parabolic induction.

Main question: How can we address representations in $\Pi_\phi(G)$ individually?

1.2 Refined Langlands conjectures

Langlands realized the importance of the group

$$S_\phi = \{g \in \widehat{G} \mid g\phi(L_E)g^{-1} = \phi\}$$

i.e. the centralizer of the L-parameter. Kottwitz showed that S_ϕ^0 is a reductive group (this uses the admissibility condition). We have $Z(\widehat{G})^\Gamma \subset S_\phi$, and we define

$$\overline{S}_\phi = S_\phi / Z(\widehat{G})^\Gamma.$$

We now assume that G is quasi-split, which allows us to choose a *Whittaker datum* $\omega = (B, \psi)$ where $\psi\colon U \to \mathbb{C}^*$ is a non-degenerate character. The parametrization of the L-packet depends on the choice of Whittaker datum.

Conjecture 1.9 (Conjecture B)**.** *There is an injective map* $\iota_\omega \colon \Pi_\phi(G) \hookrightarrow \mathrm{Irr}(\pi_0(\overline{S}_\phi))$ *which is bijective if* E *is p-adic.*

The importance of the centralizer S_ϕ comes up in connection to global computations, using the trace formula. That the Whittaker datum is necessary comes from a conjecture of Shahidi that there is a unique generic constituent of the L-packet and it should correspond under i_ω to the trivial representation.

Conjecture 1.10 (Conjecture C)**.** *There is a unique generic constituent of* $\Pi_\phi(G)$ *corresponding to the trivial representation under* ι_ω*.*

There is another part of the conjecture that we're not going to say anything about. Part of the motivation for why we want to access member of the L-packet individually is to make sense of some calculations coming out of the trace formula, namely stabilization and what happens on the spectral side. Because of that, there should be some character relations that are encoded in this map.

Conjecture 1.11 (Conjecture D)**.** *If* E *is non-archimedean, then the bijection*

$$\Pi_\phi(G) \cong \mathrm{Irr}(\pi_0(\overline{S_\phi}))$$

can be re-interpreted as a "perfect pairing" (one has to be careful about what this means in the non-abelian case)

$$\langle \cdot, \cdot \rangle \colon \Pi_\phi(G) \times \pi_0(\overline{S_\phi}) \to \mathbb{C}$$

which defines a virtual character attached to an L*-homomorphism* ϕ *and* $s \in \pi_0(\overline{S_\phi})$*:*

$$\Theta_\phi^s = \sum_{\pi \in \Pi_\phi(G)} \langle \pi, s \rangle \Theta_\pi$$

where Θ_π *is the Harish-Chandra character. This pairing should satisfy certain endoscopic relations.*

We won't say more other than that these endoscopic relations come from global motivations.

2 The non-quasisplit case

2.1 Problems for non-quasisplit groups

Now suppose G is not quasisplit. Obviously Conjecture C doesn't make sense because it depends on having a Whittaker datum. Although it is not clear from our brief discussion, Conjecture D also cannot be formulated because the endoscopy relations depend on transfer factors which require the quasi-splitness to normalize. Even if you had a way of looking at transfer factors "up to scalars", it turns out that the map giving the right character relations doesn't exist, because Conjecture 1.9 is false.

Example 2.1. Let E be a p-adic field and F/E a quadratic extension. Let $G = \mathrm{SL}_2/E$ and $G' = (D^\times)^{\mathrm{Nm}=1}$, where D is a quaternion algebra over E (so G' is an inner form of SL_2). (F is a maximal commutative subalgebra of D, since any quadratic extension embeds into any quaternion algebra.) Let $\sigma \in \mathrm{Gal}(F/E)$ be the non-trivial element. Choose a character

$$\theta\colon F^\times \to \mathbb{C}^\times$$

such that $\theta^{-1} \cdot (\theta \circ \sigma)$ is *non-trivial* of order 2. We define an L-parameter

$$\phi\colon W_{F/E} \to \mathrm{PGL}_2(\mathbb{C})$$

by

$$f \mapsto \begin{pmatrix} \theta(f) & \\ & \theta(\sigma(f)) \end{pmatrix}$$

Then it turns out that

$$\overline{S_\phi} = S_\phi = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

Langlands-Labesse found $\#\Pi_\phi(G) = 4$ but $\#\Pi_\phi(G') = 1$. For the $\pi \in \Pi_\phi(G')$ the hypothetical character relations would imply that $\langle \pi, 1 \rangle = 2$ and $\langle \pi, s \rangle = 0$ for $s \in S_\phi \setminus \{1\}$, which is not a character even up to scalars.

2.2 Inner twists

The fundamental idea is that you shouldn't work with a single inner form of a quasi-split group, but rather *treat all of them together at once*. Numerical computations for unitary groups that suggest that this is reasonable.

Example 2.2. Let $E = \mathbb{R}$. The unitary groups $U(p,q)$ for $p + q = n$ constitute a clas of inner forms. If ϕ is a discrete parameter for $U(n)$ (a quasisplit group), then $\#S_\phi = 2^n$ and $\#\overline{S}_\phi = 2^{n-1}$. For $G = U(p,q)$ we have $\#\Pi_\phi(G) = \binom{p+q}{q}$. So if you add up the contributions for all inner forms then you cover the size of the $\overline{S_\phi}$. This suggests that we should treat $U(p,q)$ and $U(q,p)$ as being distinct, even though they have the same L-group.

Vogan introduced the notion of inner twists to codify this phenomenon.

Definition 2.3. For G^* a quasisplit form over E, an *inner twist* is an isomorphism class of maps

$$\xi\colon G^*_{\overline{E}} \xrightarrow{\sim} G$$

such that $\xi^{-1} \circ \sigma(\xi)$ is an inner automorphism of $G^*_{\overline{E}}$ for every $\sigma \in \Gamma$, and isomorphisms are diagram isomorphisms.

Example 2.4. Contrast this with the notion of *inner form*, which is an inner twist (G, ξ) but forgetting ξ. This is badly behaved; for instance, you can think of GL_n as an inner twist in two ways: via identity or transpose maps. If you didn't consider both you wouldn't have a chance of parametrizing the L-packet in a canonical way.

Unfortunately, the refinement of inner twists is also not quite enough.

Example 2.5. Let $E = \mathbb{R}$ and $G = \mathrm{SL}_2 / \mathbb{R}$ (viewed as an inner twist of itself via the identity map). There is a discrete series L-packet $\{\pi^+, \pi^-\}$ such that for $g = \begin{pmatrix} i & \\ & -i \end{pmatrix}$, the automorphism $\mathrm{Ad}(g)$ preserves the inner twist but acts on $\mathrm{SO}_2(\mathbb{R})$ as $x \mapsto x^{-1}$. One can show that $\mathrm{Ad}(g)$ exchanges π^+ and π^-, which shows that inner twists are also not sufficiently rigid to provide a canonical parametrization of L-packets.

2.3 Extended pure inner forms

Definition 2.6. We need to refine further: a *pure inner twist* is the isomorphism class of a pair (ξ, z) where

- ξ is an inner twist of G^*, and
- $z \in Z^1(\Gamma, G^*)$ is such that $\xi^{-1} \circ \sigma(\xi) = \mathrm{Ad}(z(\sigma))$ for all $\sigma \in \Gamma$. (This adds an extra rigidification.)

These are parametrized by $H^1(\Gamma, G^*)$. Pure inner forms are defined analogously.

Conjecture 2.7. *Let G^* be a quasisplit connected reductive group over E. We have a commutative diagram*

$$\begin{array}{ccc} \coprod_{(\xi,z)} \Pi_\phi(\xi, z) & \xrightarrow{\sim} & \mathrm{Irr}(\pi_0(S_\phi)) \\ \Big\downarrow{\scriptstyle (\xi,z)\mapsto z} & & \Big\downarrow \\ H^1(\Gamma, G^*) & \xrightarrow[\sim]{} & \pi_0(Z(\widehat{G})^\Gamma)^{*=dual} \end{array}$$

Here $\Pi_\phi(\xi, z)$ is the L-packet corresponding to inner forms determined by (ξ, z).

Recall that inner forms for G are parametrized by $H^1(\Gamma, G^*_{\mathrm{ad}})$. Here we see a problem: the conjecture only gives access to the image of $H^1(\Gamma, G^*) \to H^1(\Gamma, G^*_{\mathrm{ad}})$. So this conjecture doesn't reach all inner forms. Therefore, we need to expand the notion of pure inner forms.

Definition 2.8. Recall the set

$$B(G^*)_{\mathrm{basic}} := G^*(\breve{E})/\text{conjugacy}$$

It turns out that any $b \in B(G^*)_{\mathrm{basic}}$ determines an inner twist (b, ξ) with corresponding inner form J_b. This pair (b, ξ) is called an *extended pure inner twist*.

As was discussed in Rapoport's talk, Kottwitz defined a map

$$\kappa \colon B(G^*)_{\mathrm{basic}} \to \pi_1(G)_\Gamma \cong X^*(Z(\widehat{G})^\Gamma).$$

♠♠♠ TONY: [isomorphism seems hard to believe] This is compatible with the map from Conjecture 2.7 for the inclusion $H^1(\Gamma, G^*) \hookrightarrow B(G^*)$:

$$\begin{array}{ccc} H^1(\Gamma, G^*) & \hookrightarrow & B(G^*)_{\text{basic}} \\ \downarrow \sim & & \sim \downarrow \kappa \\ \pi_0(Z(\widehat{G})^\Gamma)^* & \hookrightarrow & X^*(Z(\widehat{G})^\Gamma) \cong \pi_1(G)_\Gamma \end{array}$$

Conjecture 2.9 (Conjecture F)**.** *Assume ϕ is a discrete parameter, i.e. $S_\phi/Z(\widehat{G})^\Gamma$ is finite. Then there exists a unique bijection*

$$\iota_\omega \colon \coprod_{(\xi,b)} \Pi_\phi(\xi, b) \xrightarrow{\sim} \mathrm{Irr}(S_\phi)$$

such that the following diagram commutes:

$$\begin{array}{ccc} \coprod_{(\xi,b)} \Pi_\phi((\xi, b)) & \xrightarrow[\sim]{\iota_\omega} & \mathrm{Irr}(S_\phi) \\ \downarrow {\scriptstyle (\xi,b)\mapsto b} & & \downarrow {\scriptstyle \textit{restriction}} \\ B(G^*)_{\text{basic}} & \xrightarrow[\kappa]{\sim} & X^*(Z(\widehat{G})^\Gamma) \end{array}$$