# DEFINITION AND DESCRIPTION OF $\mathrm{Hk}^{\mu}_{\mathcal{M},d}$: EXPRESSING $\mathbb{I}_r(h_D)$ AS A TRACE

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## 1. New moduli spaces

1.1. **Goal.** Recall that $\nu\colon X' \to X$ is an étale (geometrically connected) double cover. Let $D$ be an effective divisor on $X$ of degree $d$. We have constructed a map

$$\theta^{\mu}\colon \mathrm{Sht}^{\mu}_{T} \to \mathrm{Sht}'^{\mu}_{G} := \mathrm{Sht}^{\mu}_{G} \times_{X^r} (X')^r.$$

The goal is to understand the intersection number

$$\mathbb{I}_r(h_D) := \langle \theta^{\mu}_{*}[\mathrm{Sht}^{\mu}_{T}], h_D * \theta^{\mu}_{*}[\mathrm{Sht}^{\mu}_{T}] \rangle_{\mathrm{Sht}'^{\mu}_{G}} \in \mathbf{Q}.$$

1.2. **The stack** $\mathrm{Sht}^{\mu}_{\mathcal{M},D}$**.** For formal reasons, $\mathbb{I}_r(h_D)$ coincides with the intersection number in the product:

$$\begin{array}{ccc} \mathrm{Sht}^{\mu}_{\mathcal{M},D} & \longrightarrow & \mathrm{Sht}'^{\mu}_{G}(h_D) \\ \downarrow & & \downarrow \\ \mathrm{Sht}^{\mu}_{T} \times \mathrm{Sht}^{\mu}_{T} & \longrightarrow & \mathrm{Sht}'^{\mu}_{G} \times \mathrm{Sht}'^{\mu}_{G} \end{array}$$

To be clear, let us flesh out the definition of $\mathrm{Sht}^{\mu}_{\mathcal{M},D}$.

**Definition 1.1.** We first define the moduli stack $\widetilde{\mathrm{Sht}}^{\mu}_{\mathcal{M},D}$ parametrizing

(1) Modifications of line bundles

$$\mathcal{L}_0 \overset{f_0}{\dashrightarrow} \mathcal{L}_1 \overset{f_1}{\dashrightarrow} \dots \overset{f_r}{\dashrightarrow} \mathcal{L}_r \overset{\tau}{\cong} \mathcal{L}_0,$$

with modification points at $x'_1, \dots, x'_r$.

(2) Modifications of line bundles

$$\mathcal{L}'_0 \overset{f'_0}{\dashrightarrow} \mathcal{L}'_1 \overset{f'_1}{\dashrightarrow} \dots \overset{f'_r}{\dashrightarrow} \mathcal{L}'_r \overset{\tau}{\cong} \mathcal{L}'_0,$$

with modifications points also at the same $x'_1, \dots, x'_r$ as above, because by definition the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{Sht}^{\mu}_{\mathcal{M},D} & \longrightarrow & \mathrm{Sht}'^{\mu}_{G}(h_D) & \longrightarrow & (X')^r \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Sht}^{\mu}_{T} \times \mathrm{Sht}^{\mu}_{T} & \longrightarrow & \mathrm{Sht}'^{\mu}_{G} \times \mathrm{Sht}'^{\mu}_{G} & \longrightarrow & (X')^r \times (X')^r \end{array}$$

(3) Compatible modifications

$$c_i \colon \nu_* \mathcal{L}_i \hookrightarrow \nu_* \mathcal{L}'_i$$

such that $\det(\nu_* \mathcal{L}'_i / c(\nu_* \mathcal{L}_i))$ is an invertible sheaf on $D \times S$.

The stack $\widetilde{\mathrm{Sht}}^{\mu}_{\mathcal{M},D} / \operatorname{Pic}_X(k)$ has an action of $\operatorname{Pic}_X(k)$ as usual. Finally, we have

$$\mathrm{Sht}^{\mu}_{\mathcal{M},D} = \widetilde{\mathrm{Sht}}^{\mu}_{\mathcal{M},D} / \operatorname{Pic}_X(k).$$

As we saw yesterday, datum (3) in Definition **??** is equivalent to the data of

$$(\alpha_\bullet, \beta_\bullet) \colon \mathcal{L}_\bullet \oplus \sigma^* \mathcal{L}_\bullet \to \mathcal{L}'_\bullet.$$

The central object of this talk is a "Hecke version" of this moduli space.

### 1.3. The stack $\mathrm{Hk}^{\mu}_{\mathcal{M},d}$.

**Definition 1.2.** Define $\widetilde{\mathrm{Hk}}^{\mu}_{\mathcal{M},d}$ whose $S$-point are:

(1) $x'_1, \dots, x'_r \in X'(S)$,
(2) $\mathcal{L}_0 \xrightarrow{f_0} \mathcal{L}_1 \xrightarrow{f_1} \dots \xrightarrow{f_r} \mathcal{L}_r$,
(3) $\mathcal{L}'_0 \xrightarrow{f'_0} \mathcal{L}'_1 \xrightarrow{f'_1} \dots \xrightarrow{f'_r} \mathcal{L}'_r$,
(4) A commutative diagram

$$\begin{array}{ccccccc}
\mathcal{L}'_0 & \xrightarrow{f'_1} & \mathcal{L}'_1 & \xrightarrow{f'_2} & \dots & \xrightarrow{f'_r} & \mathcal{L}'_r \\
\uparrow{\scriptstyle \alpha_0} & & \uparrow{\scriptstyle \alpha_1} & & \uparrow & & \uparrow{\scriptstyle \alpha_r} \\
\mathcal{L}_0 & \xrightarrow{f_1} & \mathcal{L}_1 & \xrightarrow{f_2} & \dots & \xrightarrow{f_r} & \mathcal{L}_r \\
\downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle \beta_1} & & \downarrow & & \downarrow{\scriptstyle \beta_r} \\
\sigma^*\mathcal{L}'_0 & \xrightarrow{f'_1} & \sigma^*\mathcal{L}'_1 & \xrightarrow{f'_2} & \dots & \xrightarrow{f'_r} & \sigma^*\mathcal{L}'_r
\end{array} \tag{1.1}$$

such that each row in (**??**) gives a point of $\widetilde{\mathrm{Hk}}^{\mu}_{T}$ over $x'_1, \dots, x'_r$, and each column

$$\begin{array}{c}
\mathcal{L}'_i \\
\uparrow{\scriptstyle \alpha_i} \\
\mathcal{L}_i \\
\downarrow{\scriptstyle \beta_i} \\
\sigma^*\mathcal{L}'_i
\end{array}$$

gives a point of $\widetilde{\mathcal{M}}_d$, which really just means that

$$\deg \mathcal{L}'_i - \deg \mathcal{L} = d$$

and

$$\mathrm{Nm}(\alpha_i) \neq \mathrm{Nm}(\beta_i)$$

(this is the $\heartsuit$ condition in the paper).

Finally, we define

$$\mathrm{Hk}^{\mu}_{\mathcal{M},d} := \widetilde{\mathrm{Hk}}^{\mu}_{\mathcal{M},d} / \mathrm{Pic}_X .$$

There is a map

$$\begin{array}{ccc} \mathrm{Hk}^{\mu}_{\mathcal{M},d} & & (x', \mathcal{L}'_\bullet \xleftarrow{\alpha_\bullet} \mathcal{L}_\bullet \xrightarrow{\beta_\bullet} \sigma^*\mathcal{L}'_\bullet) \\ \downarrow & & \downarrow \\ \mathcal{M}_d & & (\mathcal{L}'_\bullet \xleftarrow{\alpha_\bullet} \mathcal{L}_\bullet \xrightarrow{\beta_\bullet} \sigma^*\mathcal{L}'_\bullet) \end{array}$$

**Remark 1.3.** We have a cartesian diagram

$$\begin{array}{ccc} \mathrm{Sht}^{\mu}_{\mathcal{M},d} & \longrightarrow & \mathrm{Hk}^{\mu}_{\mathcal{M},d} \\ \downarrow & & \downarrow \gamma_0 \times \gamma_r \\ \mathcal{M}_d & \xrightarrow{\mathrm{Id} \times \mathrm{Frob}} & \mathcal{M}_d \times \mathcal{M}_d \end{array}$$

1.4. **Relation to Hitchin fibration.** Set $\mathcal{H} := \mathrm{Hk}^1_{\mathcal{M},d}$. Then we have

$$\text{(??)}\ \mathrm{Hk}^{\mu}_{\mathcal{M},d} = \underbrace{\mathcal{H} \times_{\mathcal{M}_d} \mathcal{H} \times_{\mathcal{M}_d} \ldots \times_{\mathcal{M}_d} \mathcal{H}}_{r \text{ terms}} \tag{1.2}$$

where the maps $\mathcal{H} \to \mathcal{M}_d$ are $\gamma_1$, and the maps $\mathcal{M}_d \leftarrow \mathcal{H}$ are $\gamma_0$.

**Lemma 1.4.** *The composition*

$$\begin{array}{ccc} \mathrm{Hk}^{\mu}_{\mathcal{M},d} & & (x', \mathcal{L}'_\bullet \xleftarrow{\alpha_\bullet} \mathcal{L}_\bullet \xrightarrow{\beta_\bullet} \sigma^*\mathcal{L}'_\bullet) \\ \downarrow \gamma_i & & \downarrow \\ \mathcal{M}_d & & (\alpha_i \colon \mathcal{L}_i \to \mathcal{L}'_i; \beta_i \colon \mathcal{L}_i \to \sigma^*\mathcal{L}'_i) \\ \downarrow f_{\mathcal{M}} & & \downarrow \\ \mathcal{A}_d & & (\Delta := \mathrm{Nm}(\mathcal{L}'_i) \otimes \mathrm{Nm}(\mathcal{L}_i)^{-1}, \mathrm{Nm}(\alpha_i), \mathrm{Nm}(\beta_i)) \end{array}$$

*is independent of $i$.*

*Proof.* We have $\mathcal{A}_d \subset \widehat{X}_d \times_{\mathrm{Pic}^d_X} \widehat{X}_d$, included as the open locus where the sections take distinct values. Consider

$$\begin{array}{ccccc} \mathcal{L}'_i & \xleftarrow[\alpha_i]{} & \mathcal{L}_i & \xrightarrow{\beta_i} & \sigma^*\mathcal{L}'_i \\ \downarrow \text{at } x' & & \downarrow \text{at } x' & & \downarrow \text{at } x' \\ \mathcal{L}'_{i+1} & \xleftarrow[\alpha_{i+1}]{} & \mathcal{L}_{i+1} & \xrightarrow{\beta_{i+1}} & \sigma^*\mathcal{L}'_{i+1} \end{array}$$

so

$$\mathcal{L}'_{i+1} \otimes \mathcal{L}_{i+1}^{-1} \cong \mathcal{L}'_i(x') \otimes (\mathcal{L}_i(x'))^{-1} \cong \mathcal{L}'_i \otimes \mathcal{L}_i^{-1}$$

and $\alpha = \alpha_{i+1}$ under this identification, and $\mathrm{Nm}(\beta_i) = \mathrm{Nm}(\beta_{i+1})$. (So in fact, all the maps agree to a slightly more refined space, $\widehat{X'}_d \times_{\mathrm{Pic}^d_X} \widehat{X}_d$.) ☐

1.5. **The $\diamond$ locus.** Consider the following "nice locus".

$$\begin{array}{ccccc} & & \mathrm{Hk}^{\mu}_{\mathcal{M},d} & & \\ & \swarrow^{\gamma_0} & & \searrow^{\gamma_r} & \\ \mathcal{M}_d & & & & \mathcal{M}_d \\ & \searrow_{f_{\mathcal{M}}} & & \swarrow_{f_{\mathcal{M}}} & \\ & & \mathcal{A}_d & & \end{array}$$

We denote by $\mathcal{A}_d^{\diamond} \subset \mathcal{A}_d$ the open substack $(\Delta, a, b)$ where $b \neq 0$, and for our other moduli spaces we use $\diamond$ to denote the full pre-image of $\mathcal{A}_d^{\diamond}$. Thus we have the commutative diagram:

$$\begin{array}{ccccc} & & \mathrm{Hk}^{\mu}_{\mathcal{M}^{\diamond},d} & & \\ & \swarrow^{\gamma_0} & & \searrow^{\gamma_r} & \\ \mathcal{M}_d^{\diamond} & & & & \mathcal{M}_d^{\diamond} \\ & \searrow_{f_{\mathcal{M}}} & & \swarrow_{f_{\mathcal{M}}} & \\ & & \mathcal{A}_d^{\diamond} & & \end{array}$$

**Lemma 1.5** (Description of $\mathcal{H}^{\diamond}$)**.** *We have a diagram of cartesian squares*

$$\begin{array}{ccccc} \mathcal{M}_d^{\diamond} & \xleftarrow{\gamma_0} & \mathcal{H}^{\diamond} & \xrightarrow{\gamma_r} & \mathcal{M}_d^{\diamond} \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{X}'_d \times_{\mathrm{Pic}_X^d} X'_d & \longleftarrow & \widehat{X}'_d \times_{\mathrm{Pic}_X^d} I_d & \longrightarrow & \widehat{X}'_d \times_{\mathrm{Pic}_X^d} X'_d \\ \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ X'_d & \longleftarrow & I'_d & \longrightarrow & X'_d \end{array}$$

*where* $I'_d = \{(D,x) \in X'_d \times X' \mid x' \in D\}$ *and the maps are*

$$\begin{array}{ccccc} X'_d & \longleftarrow & I'_d & \longrightarrow & X'_d \\ \\ D & \longleftarrow & (D,x') & \longrightarrow & D - x' + \sigma(x') \end{array}$$

*Proof.* A point of $\mathcal{H}^{\diamond}$ is a diagram

$$\begin{array}{ccccc} \mathcal{L}'_i & \xleftarrow[\alpha_i]{} & \mathcal{L}_i & \xrightarrow[\beta_i]{} & \sigma^*\mathcal{L}'_i \\ \downarrow{\scriptstyle \text{at } x'} & & \downarrow{\scriptstyle \text{at } x'} & & \downarrow{\scriptstyle \text{at } \sigma(x')} \\ \mathcal{L}'_{i+1} & \xleftarrow[\alpha_{i+1}]{} & \mathcal{L}_{i+1} & \xrightarrow[\beta_{i+1}]{} & \sigma^*\mathcal{L}'_{i+1} \end{array}$$

The content of the statement is that we can resconstruct this diagram from the top row, plus the point $x'$. If we know where $x'$ is. Indeed, from this information we get $\alpha_i$ for free, as $\mathcal{L}_{i+1} = \mathcal{L}_i(x')$. The map $\beta_i$ is also unique if it exists, but we do not get its existence for free, since we need it to be a *regular* (rational than rational) map. Its divisor is determined by the condition

$$\mathrm{Div}(\beta_i) + \sigma(x') = \mathrm{Div}(\beta_{i+1}) + x'$$

and we need $\mathrm{Div}(\beta_1)$ to be effective, so since the $\diamond$ locus forces $\sigma(x') \neq x'$, the requirement for $\beta_i$ to exist is

$$x' \in \mathrm{Div}(\beta_i).$$

□

**Corollary 1.6.** *The map*

$$\gamma = \gamma_i \colon \mathrm{Hk}^{\mu}_{\mathcal{M}^{\diamond},d} \to \mathcal{M}^{\diamond}_d$$

*is finite surjective (because $I'_d \to X'_d$ is). Therefore*

$$\dim \mathrm{Hk}^{\mu}_{\mathcal{M},d} = \dim \mathcal{M}^{\diamond}_d = 2d - (g-1).$$

## 2. Trace formula for intersection number

Let $[\mathcal{H}^{\diamond}]$ be the class of the Zarisi closure of $\mathcal{H}^{\diamond}$ in $\mathrm{Ch}_{2d-g+1}(\mathcal{H})_{\mathbf{Q}}$. As we have seen, this gives a cohomological correspondence in $\mathrm{Corr}(\mathbf{Q}_{\ell,\mathcal{M}_d}, \mathbf{Q}_{\ell,\mathcal{M}_d})$ via the diagram

$$\begin{array}{ccc} & \mathcal{H} & \\ \swarrow & & \searrow \\ \mathcal{M}_d & & \mathcal{M}_d \end{array}$$

We can then push this down via

$$\begin{array}{ccccc} & & \mathcal{H} & & \\ & \swarrow & \downarrow & \searrow & \\ \mathcal{M}_d & & & & \mathcal{M}_d \\ \downarrow f_{\mathcal{M}} & & \downarrow & & \downarrow f_{\mathcal{M}} \\ \mathcal{A}_d & = & \mathcal{A}_d & = & \mathcal{A}_d \end{array}$$

to obtain a cohomological correspondence on the Hitchin space

$$f_{\mathcal{M}!}[\mathcal{H}^{\diamond}] \colon Rf_{\mathcal{M}!}\mathbf{Q}_{\ell} \to Rf_{\mathcal{M}!}\mathbf{Q}_{\ell}.$$

We have a map $\delta \colon \mathcal{A}_d \to X_d$ sending $(\Delta, a, b) \mapsto (\Delta, a-b)$. The preimage of a divisor $D \in X_d$ will be denoted $\mathcal{A}_D$.

**Theorem 2.1.** *Suppose $D$ is an effective divisor of degree $d \geq \max\{4g-3, 2g\}$. Then*

$$\mathbb{I}_r(h_D) = \sum_{a \in \mathcal{A}_D(k)} \mathrm{Tr}((f_{\mathcal{M}!} \circ [\mathcal{H}^{\diamond}]_a)^r \,\mathrm{Frob}_a, (Rf_{\mathcal{M}!}\mathbf{Q}_{\ell})_{\overline{a}})$$

*Proof.* Recall the diagram

$$\begin{array}{ccccc}
\mathrm{Sht}^{\mu}_{\mathcal{M},d} & \longrightarrow & \mathrm{Hk}^{\mu}_{\mathcal{M},d} & & \\
\downarrow & & \downarrow{\scriptstyle \gamma_0\times\gamma_r} & \searrow & \\
\mathcal{M}_d & \xrightarrow{\mathrm{Id}\times\mathrm{Frob}} & \mathcal{M}_d\times\mathcal{M}_d & & \mathcal{A}_d \\
\downarrow{\scriptstyle f_{\mathcal{M}}} & & \downarrow{\scriptstyle f_{\mathcal{M}}\times f_{\mathcal{M}}} & \swarrow{\scriptstyle \Delta} & \\
\mathcal{A}_d & \xrightarrow{\mathrm{Id}\times\mathrm{Frob}} & \mathcal{A}_d\times\mathcal{A}_d & &
\end{array}$$

The map from $\mathrm{Hk}^{\mu}_{\mathcal{M},d}$ factors through the diagonal of $\mathcal{A}_d \times \mathcal{A}_d$, which implies that $\mathrm{Sht}^{\mu}_{\mathcal{M},d}$ is fibered over $\mathcal{A}_d(k)$:

$$\mathrm{Sht}^{\mu}_{\mathcal{M},d} \cong \coprod_{a\in\mathcal{A}_d(k)} \mathrm{Sht}^{\mu}_{\mathcal{M},d}(a).$$

So we have a map

$$\begin{array}{ccc}
\bigoplus_{D\in X_d(k)} \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},D})_{\mathbf{Q}} \cong \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},d})_{\mathbf{Q}} & \longleftarrow & \mathrm{Ch}_{2d-g+1}(\mathrm{Hk}^{\mu}_{\mathcal{M},d})_{\mathbf{Q}} \\
(\mathrm{Id},\mathrm{Frob})^{!}\zeta & \longleftarrow & \zeta
\end{array}$$

$\square$

In the next talk the proof of the following theorem will be sketched:

**Theorem 2.2** (Theorem 6.6). *There exists $\zeta \in \mathrm{Ch}_{2d-g+1}(\mathcal{H})$ such that $\zeta|_{\mathcal{H}^\diamond}$ is the fundamental cycle, and*

$$\mathbb{I}_r(h_D) = \deg(\mathrm{Id},\mathrm{Frob})^{!}\zeta$$

*Then it follows from the trace formula that*

$$\begin{aligned}
\mathbb{I}_r(h_D) &= \sum_{a\in\mathcal{A}_D(k)} \mathrm{Tr}((f_{\mathcal{M}!}\mathrm{cl}(\zeta))_a \circ \mathrm{Frob}_a, (Rf_{\mathcal{M}!}\mathbf{Q}_\ell)_{\overline{a}}) \\
&= \sum_{a\in\mathcal{A}_D(k)} \mathrm{Tr}((f_{\mathcal{M}!}\mathrm{cl}([\mathcal{H}^\diamond]))_a \circ \mathrm{Frob}_a, (Rf_{\mathcal{M}!}\mathbf{Q}_\ell)_{\overline{a}})
\end{aligned}$$

**Remark 2.3.** There's a technical issue that $\zeta$ and $[\mathcal{H}^\diamond]$ aren't the same, but at least they're the same on the $\diamond$ locus. You can show by dimension estimate that the difference on the boundary doesn't contribute.