Beauville-Laszlo Uniformization for the Fargues-Fontaine Curve

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1 The classical affine Grasmannian

Let me first recall the classical story. Let X be a smooth projective connected curve over an algebraically closed field $k = \overline{k}$, $x \in X(k)$, and G/k a semisimple group.

We consider the *affine Grassmannian* Gr_G, which is an ind-(projective scheme). Recall that Gr_G parametrizes G-torsors $\mathcal{F}/X$ plus a trivialization over the punctured curve:

$$\mathcal{F}|_{X \setminus \{x\}} \cong G \times (X \setminus \{x\}).$$

Forgetting the trivialization induces a map

$$\mathrm{Gr}_G \to \mathrm{Bun}_G$$

where Bun_G is the (Artin) moduli stack of G-bundles. We have only defined the map on objects, but we all know how to relativize it in this case. In the case of the Fargues-Fontaine curve, it will be more subtle.

Theorem 1.1 (Drinfeld-Simpson)**.** *This map is surjective in the fppf topology.*

Remark 1.2. If $X = \mathbb{P}^1$ then we can replace "semisimple" by "reductive". This may be useful for understanding the behavior for the Fargues-Fontaine curve, which behaves like a mix between genus 0 and genus 1 curves.

2 The B_{dR}^+-affine Grassmannian

2.1 The ring B_{dR}^+

Let R be any perfectoid algebra. Fix $R^+ \subset R$ and a pseudo-uniformizer $\varpi^\flat$. (Ultimately everything will be independent of these choices.) Then we have the map

$$\theta \colon W(R^{\flat +}) \to R^+$$

with $\ker \theta = (\xi)$.

Definition 2.1. We define $B^+_{\text{dR}}(R)$ to be the ξ-adic completion of $W(R^{\flat+})[\frac{1}{[\varpi^\flat]}]$. We think of this as "the completion of $\operatorname{Spa} R \times_? \operatorname{Spa} \mathbb{Z}_p$ along the graph map

$$\Gamma_{\operatorname{Spa} R \to \operatorname{Spa} \mathbb{Z}_p} \colon \operatorname{Spa} R \hookrightarrow \operatorname{Spa} R \times \operatorname{Spa} \mathbb{Z}_p."$$

We also define $\operatorname{Fil}^n B^+_{\text{dR}}(R) = \xi^n B^+_{\text{dR}}(R)$ and $B_{\text{dR}}(R) = B^+_{\text{dR}}(R)[\xi^{-1}]$.

Proposition 2.2. *The ring $B^+_{\text{dR}}(R)$ enjoys the following properties:*

1. *$B^+_{\text{dR}}(R)$ is ξ-adically complete, ξ-torsion free, and $B^+_{\text{dR}}(R)/\xi = R$. (It looks like the completion along something of codimension one.)*
2. *Assume $p = 0$ in R. Then one can take $\xi = p$, obtaining $B^+_{\text{dR}}(R) = W(R)$. (Thus the characteristic 0 version can be thought of as a deformation of $W(R)$.)*

Remark 2.3. If $R = \mathbb{C}_p$ then we get Fontaine's ring $B^+_{\text{dR}} = B^+_{\text{dR}}(\mathbb{C}_p)$ of p-adic periods. This B^+_{dR} is a complete DVR with uniformizer ξ and residue field $\mathbb{C}_p$. That means that it is abstractly isomorphic to $\mathbb{C}_p[[\xi]]$. However, the topology and Galois structures are not compatible.

We would like to play the game of affine Grassmannians in this situation. (Whenever you have a DVR you can take think of the affine Grassmannian as the space of lattices in its fraction field.)

2.2 The B^+_{dR}-affine Grassmannian

Let $G/\mathbb{Q}_p$ be a reductive group.

Definition 2.4. We define $\operatorname{Gr}_G^{B^+_{\text{dR}}}$ to be the (pre)sheaf (which will be a sheaf for all our topologies) on $\operatorname{Perf}_{\mathbb{F}_p}$ with the following functor of points: if $\operatorname{Spa}(R, R^+) = S$ then

$$\operatorname{Gr}_G^{B^+_{\text{dR}}}(S) = \left\{ \begin{matrix} R^\# = \text{ untilt of } R/\mathbb{Q}_p \\ \mathcal{F} = G\text{-bundle / Spec } B^+_{\text{dR}}(R^\#) \\ \iota \colon \mathcal{F}|_{\text{Spec } B_{\text{dR}}(R^\#)} \cong G \times \text{Spec } B_{\text{dR}}(R^\#) \end{matrix} \right\}$$

Remark 2.5. There is a map

$$\operatorname{Gr}_G^{B^+_{\text{dR}}} \to \operatorname{Spa} \mathbb{Q}_p^\diamond$$

which in terms of the functor of points is

$$(R^\#, \mathcal{F}) \mapsto R^\#.$$

Therefore, we can consider $\operatorname{Gr}_G^{B^+_{\text{dR}}}$ as a (pre)sheaf on $\operatorname{Perf}_{\mathbb{F}_p} / \operatorname{Spa} \mathbb{Q}_p^\diamond$. But we have seen that this slice category is precisely $\operatorname{Perf}_{\mathbb{Q}_p}$. Under this identification $\operatorname{Gr}_G^{B^+_{\text{dR}}}$ has the functor of points

$$B \in \operatorname{Perf}_{\mathbb{Q}_p} \mapsto \left\{ \begin{matrix} \mathcal{F} = G\text{-bundle / Spec } B^+_{\text{dR}}(B) \\ + \text{ trivialiation on Spec } B_{\text{dR}}(B) \end{matrix} \right\} \tag{1}$$

This is maybe the more natural definition, but we have chosen to give a definition that already lives in the worlds of diamonds.

Example 2.6. If $G = GL_n$, then the right side of (1) is simply the set of finite projective $B_{\mathrm{dR}}^+(B)$-modules M plus an isomorphism $M[1/\xi] \cong B_{\mathrm{dR}}(R)^n$. In general, we can think in these terms using the Tannakian philosophy.

2.3 Schubert cells

Let μ be a conjugacy class of cocharacters $\mathbb{G}_m \to G$. This may not be defined until an extension of $\mathbb{Q}_p$, but let's assume it's defined over $\mathbb{Q}_p$ for simplicity. Then we have a closed Schubert cell

$$\mathrm{Gr}_{G,\mu}^{B_{\mathrm{dR}}^+} \subset \mathrm{Gr}_G^{B_{\mathrm{dR}}^+}$$

parametrizing bundles such that at all geometric points, the relative position is bounded by μ.

If $R = C$ is algebraically closed and complete, and we choose $T \subset G_C$, then we have a Cartan decomposition

$$G(B_{\mathrm{dR}}^+(C))\backslash G(B_{\mathrm{dR}}(C))/G(B_{\mathrm{dR}}^+(C)) = X_*(T)_+.$$

For a proof, choose an isomorphism with $\mathbb{C}_p[[\xi]]$ (see Remark 2.3).

Remark 2.7. We can think of $\mathrm{Gr}_G^{B_{\mathrm{dR}}^+}$ as the sheafification of

$$R \mapsto G(B_{\mathrm{dR}}(R))/G(B_{\mathrm{dR}}(R^+)).$$

Theorem 2.8 (Scholze). *$\mathrm{Gr}_{G,\mu}^{B_{\mathrm{dR}}^+}$ is a diamond.*

Example 2.9. (Caraiani-Scholze) If μ is miniscule and $P_\mu \subset G$ is the parabolic subgroup corresponding to μ, then

$$\mathrm{Gr}_{G,\mu}^{B_{\mathrm{dR}}^+} \cong (\underbrace{G/P_\mu}_{\text{rigid space}/\mathbb{Q}_p})^\diamond$$

This is an analogue of the result that for the classical affine Grassmannian, the Schubert cells are the usual flag varieties.

Remark 2.10. There is a fully faithful embedding

$$\{\text{seminormal rigid spaces}/\mathbb{Q}_p\} \hookrightarrow \{\text{diamonds}/\operatorname{Spa}\mathbb{Q}_p^\diamond\}$$

sending $X \mapsto X^\diamond$. Seminormality has to do with the topological difference between the curve and its normalization. (A node is seminormal; a cusp is not.) The point is that if $X \to Y$ is a universal homeomorphism, then $X^\diamond \cong Y^\diamond$. I like to think of diamonds as only remembering topological information. So this fully faithful embedding is saying that up to this defect, diamonds remember everything.

Example 2.11. For $G = \mathrm{GL}_2$ and $\mu = (n, 0)$ for $n \geq 2$,

$$\mathrm{Gr}_\mu := \mathrm{Gr}_{G,\mu}^{B_{\mathrm{dR}}^+} = \left\{ \begin{matrix} M = B_{\mathrm{dR}}^+ - \text{lattice} \subset B_{\mathrm{dR}}^2 : \\ \xi^n (B_{\mathrm{dR}}^+)^2 \overset{n}{\subseteq} M \overset{n}{\subseteq} (B_{\mathrm{dR}}^+)^2 \end{matrix} \right\}$$

There is a Bott-Samuelson resolution

$$\widetilde{\mathrm{Gr}_\mu} = \left\{ \begin{matrix} M \in \mathrm{Gr}_\mu + \text{flag} \\ M = M_n \overset{1}{\subseteq} M_{n-1} \overset{1}{\subseteq} \ldots \overset{1}{\subseteq} M_0 = (B_{\mathrm{dR}}^+)^2 \\ \text{each } M_i / M_{i-1} \text{ is a line bundle over } R \end{matrix} \right\}$$

Then $\widetilde{\mathrm{Gr}_\mu}$ is a succesion of $\mathbb{P}^1$-fibrations over $\mathbb{P}^1$. You might think that because it is inductively built from classical rigid spaces that it is itself a classical rigid space, but actually it is *not* a rigid space. (However, we would still like to think of it as being "smooth", whatever that means.) Why?

Locally (say $n = 2$) it looks like an extension

$$0 \to \mathbb{A}^1 \to B_{\mathrm{dR}}^+ / \mathrm{Fil}^2 \to B_{\mathrm{dR}}^+ / \mathrm{Fil}^1 = \mathbb{A}^1 \to 0$$

(the left $\mathbb{A}^1$ may be twisted over $\mathbb{Q}_p$, but the twist goes away over $\mathbb{Q}_p^{\mathrm{cyc}}$). The middle space $B_{\mathrm{dR}}^+ / \mathrm{Fil}^2$ is an example of a Banach-Colmez space. This is *not split* étale locally, so it cannot be a rigid space. (To split it we need to make a pro-étale extension adjoining all p-power roots of something.)

3 Bun_G

3.1 Construction of Bun_G

Recall that the Fargues-Fontaine curve lives over $\mathbb{Q}_p$. We know what its vector bundles are, but it is not clear what parametrizes families of vector bundles over X. The naïve guess is rigid spaces over $\mathbb{Q}_p$, but that's wrong. Instead, *we need to use the relative Fargues-Fontaine curve over* $S \in \mathrm{Perf}_{\mathbb{F}_p}$.

Definition 3.1. Let $S \in \mathrm{Perf}_{\mathbb{F}_p}$. Then we have a relative curve $X_S = Y_S / \varphi^{\mathbb{Z}}$, which is an adic space over $\mathrm{Spa}\,\mathbb{Q}_p$. A *$G$-bundle on X_S* is an exact faithful $\mathbb{Q}_p$-linear $\otimes$-functor

$$\mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Bun}_{X_S}.$$

Let Bun_G be the (pre)stack (which again will turn out to be a stack for all possible topologies) on $\mathrm{Perf}_{\mathbb{F}_p}$ which sends

$$S \mapsto \{G\text{-bundles}/X_S\}.$$

Remark 3.2. A theorem of Kedlaya-Liu implies that Bun_{X_S} is well-defined. Basically it says that for any analytic adic space, the category of bundles behaves as one would expect (with respect to gluing, etc.).

Remark 3.3. Say $S/\overline{\mathbb{F}}_p$ and $b \in G(\breve{\mathbb{Q}}_p)$. Then we can form $\mathcal{E}_b$ over X_S. If we wrote the internal definition we would say that this is "the trivial G-torsor on Y_S, descended via b to X_S".

Proposition 3.4. Bun_G *is a stack for the v-topology.*

This uses that vector bundles form a stack for the v-topology, which was discussed in Eugen Hellman's talk.

Remark 3.5. In the algebraic case one only gets a stack for the fppf topology. Thus, the proposition is stronger than you might have expected from reasoning by analogy with schemes. But for *perfect* schemes one also gets it for the v-topology, so it's the perfection that makes this possible.

3.2 "Smooth Artin stacks" in the category of diamonds

One of the main ideas is that Bun_G is a "smooth Artin stack" (i.e. admits a "smooth" cover by a "smooth" perfectoid space). Unfortunately, we haven't yet figured out what "smooth" should mean. We have some basic examples of things that should be smooth.

Example 3.6. If $X \to Y$ is a smooth rigid space over $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, then $X^\diamond \to Y^\diamond$ should be "smooth". (In these cases taking the diamond is like taking the perfection.) Why? We are in the process of developing a six-functor sheaf formalism. Smooth maps should imply that $f^! = f^*$ up to shift. Because all étale information is preserved by taking the diamond, if this is satisfied for $X \to Y$ then it should also be satisfied at the level of diamonds.

Example 3.7. If you believe this then you run into funny phenomena. For instance, considering the classifying stack $B\underline{\mathbb{Q}_p}$ for $\underline{\mathbb{Q}_p}$-torsors. Then we claim that $\mathrm{Spa}\,\mathbb{Q}_p^{\mathrm{cyc},\diamond} \times B\underline{\mathbb{Q}_p}$ is smooth.

Under the equivalence of categories of

$$\mathrm{Perf}_{\mathbb{F}_p} / \mathrm{Spa}\,\mathbb{Q}_p^{\mathrm{cyc}\,\diamond} \cong \mathrm{Perf}_{\mathbb{Q}_p^{\mathrm{cyc}}}$$

the two stacks correspond:

$$\mathrm{Spa}\,\mathbb{Q}_p^{\mathrm{cyc}\,\diamond} \times B\underline{\mathbb{Q}_p} \leftrightarrow B\underline{\mathbb{Q}_p}$$

There is an exact sequence (in the category of pro-étale sheaves on $\mathrm{Perf}_{\mathbb{Q}_p^{\mathrm{cyc}}}$):

$$0 \to \underline{\mathbb{Q}_p} \to \widetilde{\mu_{p^\infty}}^{\mathrm{an}} \to \mathbb{G}_a \to 0$$

which induces a map $\mathbb{G}_a \twoheadrightarrow B\underline{\mathbb{Q}_p}$ with fiber $\widetilde{\mu_{p^\infty}}^{\mathrm{an}}$ (the surjectivity is because the map from a point to $B\underline{\mathbb{Q}_p}$ is always surjective; this just says that every torsor is locally trivial). We've declared $\mathbb{G}_a$ to be smooth, since it comes from a smooth rigid analytic space, but also $\widetilde{\mu_{p^\infty}}^{\mathrm{an}}$ is smooth because it is the perfection of the open unit disk. Therefore we are forced to believe that $B\underline{\mathbb{Q}_p}$ is smooth.

Theorem 3.8 (Kedlaya-Liu, Fargues)**.** *The semistable locus* $\mathrm{Bun}_G^{ss} \subset \mathrm{Bun}_G$ *is open, and*

$$\mathrm{Bun}_{G^{ss},\overline{\mathbb{F}}_p} \cong \coprod_{b \in B(G)_{\mathrm{basic}} \overset{\kappa}{\cong} \pi_1(G)_\Gamma} B\underline{J_b(\mathbb{Q}_p)}$$

(If G is a locally finite group then $\underline{G}$ *is the sheaf* $\underline{G}(S) = Map_{cont}(|S|, G)$*.)*

Remark 3.9. This is not what you get in the algebraic case (where the semistable locus is open). That may be surprising; it's because we took a different notion of family.

Note that the automorphisms of the trivial G-torsor are a locally profinite group $G(\mathbb{Q}_p)$, and *not* the algebraic group G. That's the reason p-adic groups appear. In the usual case we take the classifying space for a smooth group so it makes sense that we get an Artin stack, but here we are taking the classifying space for a p-adic group and we're not sure what we should get.

3.3 Uniformization of G-bundles

We have seen that if $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbb{Q}_p}$, then we get a relative Cartier divisor

$$S \hookrightarrow X_{S^\flat}.$$

As discussed in Definition 2.1, we can think of $B_{\mathrm{dR}}^+(R)$ as the completion of $X_{S^\flat}$ along S.

Lemma 3.10. *There is a functor*

$$\{B_{\mathrm{dR}}^+ - \textit{lattices in } B_{\mathrm{dR}}(R)^{\oplus n}\} \to \mathrm{Bun}(X_{S^\flat})$$

given by modifying the trivial vector bundle.

This is the Beauville-Laszlo Lemma in this setting. It was also proved by Kedlaya-Liu.

By the Tannakian formalism, for any G we get a map

$$\mathrm{Gr}_G^{B_{\mathrm{dR}}^+}(R, R^+) \to \mathrm{Bun}_G(R^\flat, R^{\flat+}).$$

Theorem 3.11 (Fargues)**.** *Assume G is quasisplit. Then the map*

$$\mathrm{Gr}_G^{B_{\mathrm{dR}}^+} \to \mathrm{Bun}_G$$

is surjective. More precisely, for $C/\mathbb{Q}_p$ *we get a point* $\infty \in X := X_{C^\flat}$*, and any G-bundle on* X *is trivial on* $X \setminus \{\infty\}$*.*

This follows easily from the classification of G-bundles

We claim that one can use this map and its surjectivity to give a smooth cover of G from a smooth space.

Example 3.12. Let $G = \mathrm{GL}_2$ and $\mu = (1,0)$. We have a Schubert cell $\mathrm{Gr}_{G,\mu}^{B_{\mathrm{dR}}^+} \subset \mathrm{Gr}_G^{B_{\mathrm{dR}}^+}$. What does it look like under the uniformization map?

$$\begin{array}{ccc} \mathbb{P}^1 = \mathrm{Gr}_{G,\mu}^{B_{\mathrm{dR}}^+} \subset & \mathrm{Gr}_G^{B_{\mathrm{dR}}^+} \\ & \downarrow \\ & \mathrm{Bun}_{\mathrm{GL}_2} \end{array}$$

Inside $\mathbb{P}^1$ we have Drinfeld's upper half space $\Omega^2 \subset \mathbb{P}^1$ and its complement $\mathbb{P}^1(\mathbb{Q}_p) \subset \mathbb{P}^1$. The former maps to $\mathcal{O}(1/2)$ and $\mathbb{P}^1(\mathbb{Q}_p)$ maps to $\mathcal{O} \oplus \mathcal{O}(1)$.

$$\begin{array}{ccc} \Omega^2 & & \mathbb{P}^1(\mathbb{Q}_p) \\ \downarrow & & \downarrow \\ \mathcal{O}(1/2) & & \mathcal{O} \oplus \mathcal{O}(1) \end{array}$$

So we see that the stratifications on flag varieties are highly non-algebraic!

What is the image of the Schubert cell $\mathrm{Gr}_{G,\mu}^{B_{\mathrm{dR}}^+}$? It is a subset of $B(G)$ called $B(G)(\mu)$, familiar from the theory of Shimura varieties.

The map is $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant. If you quotient by the $\mathrm{GL}_2(\mathbb{Q}_p)$-action then the map is surjective in some smooth topology.

The semistable locus has dimension 0, while its complement has negative dimension. For example, the bundle $[\mathcal{O} \oplus \mathcal{O}(1)] \in \mathrm{Bun}_{\mathrm{GL}_2}$ has automorphism scheme

$$B\begin{pmatrix} \underline{\mathbb{Q}_p^*} & \widetilde{\mu_{p^\infty}}^{\mathrm{an}} \\ & \underline{\mathbb{Q}_p^*} \end{pmatrix}.$$