Proof of Geometric Langlands for GL(2), I

Notes by Tony Feng
for a talk by Stefan Patrikis

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1 Some recollections

1.1 Notation

Let

- $X/\mathbb{F}_q =: k$ be a smooth projective geometrically connected curve.
- $F = k(X)$,
- for all $x \in |X|$ we denote $\mathcal{O}_x, F_x$ to be the completed local ring and its fraction field, respectively.
- $G = \mathrm{GL}_n$, $G_x := G(F_x)$,
- $K_x := G(\mathcal{O}_X)$,
- $\mathcal{H}_x$ the spherical Hecke algebra at x,
- $\mathcal{O}_F = \prod \mathcal{O}_x$.

1.2 Goal

Given an everywhere unramified $\sigma \colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, i.e. a local system E on X, we want to

1. Construct an unramified automorphic form

   $$f_\sigma \colon G(F)\backslash G(\mathbb{A})/G(\mathcal{O}) \to \overline{\mathbb{Q}}_\ell$$

   such that for all $x \in |X|$, the action $\mathcal{H}_x$ on $\overline{\mathbb{Q}}_\ell \cdot f_\sigma$, the eigencharacter for the action of is

   $$\mathcal{H}_x \overset{\mathrm{Sat}}{\cong} R(G^\vee) \xrightarrow{\chi_{\gamma_x}} \overline{\mathbb{Q}}_\ell$$

   given by

   $$V \mapsto \mathrm{Tr}(\gamma_x \mid_V)$$

   where $\gamma_x = [\sigma(\mathrm{Frob}_x)^{ss}]$.

2. Upgrade f_σ to a (perverse Hecke eigen-)sheaf Aut_E on Bun_n, recalling that

$$|\mathrm{Bun}_n(\mathbb{F}_q)| = G(F)\backslash G(\mathbb{A})/G(\mathcal{O}_F).$$

In this talk we'll get as far as we can along the construction of a sheaf (not quite the Aut_E) on a Bun'_n which maps to Bun_n. Roughly speaking, Bun'_n is the moduli space of pairs $(L, \Omega^{\otimes(n-1)} \hookrightarrow L)$ where L is a GL_n-bundle, so the map to Bun_n is the forgetful map. In Heinloth's talk, this sheaf will be shown to descend.

2 Classical motivation

This section will be about how, given a Galois representation σ, we could make a guess of Aut_E. By analogy, suppose you had an elliptic curve over $\mathbb{Q}$ and you wanted to show that it was modular. A naïve strategy might be to write down the Fourier expansion of the modular form from the local data. Then you have to check some invariance properties. This is hard to carry out in that setting, but it's basically what we'll try to do here.

2.1 Fourier expansion of cusp forms on GL_n

For $n = 2$, let

$$\varphi\colon \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}) \to \overline{\mathbb{Q}}_\ell$$

be a cusp form. Fix $g \in \mathrm{GL}_2(\mathbb{A})$. Then the function

$$x \mapsto \varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right)$$

is periodic, i.e. descends to a function on $F\backslash\mathbb{A}_F$. This gives a Fourier expansion in the characters $\widehat{F\backslash\mathbb{A}}$:

$$\varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = \sum_{\widehat{F\backslash\mathbb{A}}} \cdots$$

Fixing a nontrivial character Ψ of F, we can identify $\widehat{F\backslash\mathbb{A}} \cong F$ by

$$\gamma \in F \mapsto (x \mapsto \Psi(\gamma x)).$$

Then the Fourier expansion of Ψ is

$$\varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = \sum_{\gamma\in F}\left(\int_{F\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} g\right)\Psi_\gamma(y)^{-1} dy\right) \cdot \Psi_\gamma(x).$$

For $\gamma = 0$ the integral vanishes by cuspidality. For $\gamma \neq 0$, a change of variables g

$$\int_{F\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & y \\ & 1 \end{pmatrix}\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right)\Psi(y)^{-1}\, dy.$$

The conclusion (taking $x = 0$) is that

$$\varphi(g) = \sum_{\gamma \in F^*} W_{\varphi,\psi}\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} g\right)$$

where

$$\begin{aligned} W_{\varphi,\psi}(g) &= \int_{F\backslash\mathbb{A}} \varphi\left(\begin{pmatrix}1 & y \\ & 1\end{pmatrix} g\right) \Psi(y)^{-1}\, dy \\ &= \int_{N(F)\backslash N(\mathbb{A})} \varphi(ng)\Psi(n)^{-1}\, dn. \end{aligned}$$

More generally, for GL_n we get the Fourier expansion

$$\varphi(g) = \sum_{N_{n-1}(F)\backslash \mathrm{GL}_{n-1}(F)} W_{\varphi,\psi}\left(\begin{pmatrix}\boxed{\gamma} & \\ & 1\end{pmatrix} g\right). \tag{1}$$

The Whittaker property is

$$W_{\varphi,\psi}(ng) = \Psi(n) W_{\varphi,\psi}(g)$$

for all $n \in N(\mathbb{A})$. Here $\Psi(n)$ is defined by

$$n = \begin{pmatrix} 1 & u_{12} & & & \\ & \ddots & \ddots & & \\ & & 1 & u_{n-1,n} \\ & & & 1 \end{pmatrix} \mapsto \Psi\left(\sum u_{i,i+1}\right).$$

More precisely, this expansion yields a $G(\mathbb{A})$-equivariant isomorphism

$$C^\infty(G(\mathbb{A}))^{(N(\mathbb{A}),\psi)} \overset{\mathbb{G}(A)}{\cong} C^\infty(P_1(F)\backslash G(\mathbb{A}))_{\mathrm{cusp}}$$

where P_1 is the mirabolic subgroup

$$P_1 = \begin{pmatrix}\boxed{*} & \boxed{*} \\ & 1\end{pmatrix} \subset P = \begin{pmatrix}\boxed{*} & \boxed{*} \\ & *\end{pmatrix}$$

the isomorphism being given by $\varphi \mapsto W_{\varphi,\psi}$, and the inverse being the Whittaker expansion (1).

The strategy for producing f_σ is to use the local theory write down an element W_σ on the left side, and then take the Fourier expansion to get an element of the right side. The hard work is the descent on the right hand side.

2.2 Building W_σ

Let γ_x be the semisimpole conjugacy class of $\sigma(\mathrm{Frob}_x)$. It is a fact that for all G and all $x \in |X|$, there exists a unique $W_{\gamma_x} : G(F_x) \to \overline{\mathbb{Q}}_\ell$ satisfying the conditions:

1. (NORMALIZATION) $W_{\gamma_x}(1) = 1$,
2. (SPHERICAL WHITTAKER CONDITION) for all $n \in N(F_x), g \in G(F_x), k \in G(O_x)$
$$W_{\gamma_x}(ngk) = \psi_x(n) W_{\gamma_x}(g)$$
3. (HECKE EIGENVALUES) for all $h \in \mathcal{H}_x$,
$$h \cdot W_{\gamma_x} = \chi_{\gamma_x}(h) W_{\gamma_x}.$$

This builds local Whittaker functions. To build the global ones, we take their product.

Definition 2.1. Define $W_\sigma : G(\mathbb{A}) \to \overline{\mathbb{Q}}_\ell$ by
$$W_\sigma((g_x)) = \prod W_{\gamma_x}(g_x).$$

We then define f'_σ to be the Fourier expansion of W_σ, as in (1). This is a priori only left invariant under the mirabolic, so
$$f'_\sigma \in C^\infty(P_1(F)\backslash G(\mathbb{A})/G(O))$$
and f'_σ has the correct Hecke eigenvalues.

Remark 2.2. For general G the local Whittaker functions exist, but not global (what is a generalization of the mirabolic?).

Conjecture 2.3. *The function f'_σ is (left) $\mathrm{GL}_n(F)$-invariant.*

3 Geometrization

The aim of the rest of the talk is to geometrize f'_σ on a subset of its domain, corresponding to $|\,\mathrm{Bun}'_n(\mathbb{F}_q)| \subset P_1(F)\backslash G(\mathbb{A})/G(O)$. We won't elaborate on this yet, but we emphasize that this applies to a particular *subset* of the domain.

3.1 Setup

We now replace GL_n by a group *scheme* over X, denoted GL_n^J, whose functor points is
$$\mathrm{GL}_n^J(R) = \{\text{invertible } n \times n \text{ matrices } (A) \in \Gamma(\mathrm{Spec}\ R, \Omega^{j-i})\}.$$

Example 3.1. For GL_2,
$$\mathrm{GL}_2^J = \begin{pmatrix} O & \Omega^1 \\ \Omega^{-1} & O \end{pmatrix}.$$

Likewise, we define $N^J, P_1^J, B^J, \ldots$

We can then construct a (more) canonical character

$$\Psi\colon N(F)\backslash N(\mathbb{A})/N(\mathcal{O}) \to \overline{\mathbb{Q}}_\ell^*$$

which depends only on the choice of character of the residue field $\psi\colon k \to \overline{\mathbb{Q}}_\ell^*$, setting

$$\Psi = \prod \Psi_x$$

where

$$\Psi_x\colon \begin{pmatrix} 1 & u_{12} & & \\ & \ddots & \ddots & \\ & & 1 & u_{n-1,n} \\ & & & 1 \end{pmatrix} \mapsto \prod_{i=1}^{n-1} \psi(\mathrm{Tr}_{k(x)/k} \mathrm{Res}_x(u_{i,i+1} \in \Omega^1)).$$

This is invariant by $N(\mathcal{O})$ and $N(F)$ by the residue theorem.

Now construct W_σ and f'_σ on $\mathrm{GL}_n^J(\mathbb{A})$ in the same way as before.

3.2 Difficulties with geometrization

Now we've arrived at the proper work, which has to do with trying to geometrize this. There are difficulties in doing this. Perhaps the two main ones are:

1. Local: If we were to try to geometrize the local Whittaker functions, then we would run into the problem that the orbits of $N(F_x)$ on the affine Grassmanin $G(F_x)/G(\mathcal{O}_X)$ on ∞-dimensional over k.

2. Global: $P_1(F)\backslash G(\mathbb{A})/G(\mathcal{O})$ are n-dimensional bundles L plus "generic embeddings" $\Omega^{n-1} \hookrightarrow L$. There can be poles with no control. Thus this is not an object of classical algebraic geometry.

A lot of the work in geometric Langlands in recent years is about developing a sensible theory of such things. However, in this case there is a hack to get around the obstacles: there are two things which come together to save us.

1. Local: we have the *Shintani (for* GL(n)*) and Casselman-Shalika (for general G)* formula, which tells us about the support of the spherical Whittaker function.

2. Global: $G(\mathbb{A})^+/G(\mathcal{O})$ is the set of $\mathbb{F}_q$-points of a scheme, where (using $G = \mathrm{GL}_n$) $G(F_x)^+ := \mathrm{GL}_n(F_x) \cap M_n(\mathcal{O}_X)$.

It turns out that modulo center, the interesting part of the Whittaker function is supported on the scheme underlying $G(\mathbb{A})^+/G(\mathcal{O})$.

3.3 Casselman-Shalika formula

We have the NAK decomposition. We know how the Whittaker function transforms under left translation by N and right translation by K, so we need to figure out what happens on A. The answer is that for all $x \in |X|$ and γ_x a semisimple conjugacy class (which gives rise to the Whittaker function W_{γ_x}):

1. $W_{\gamma_x}(\lambda(\varpi_x)) = 0$ for $\lambda \in X_*(T) - X_*(T)_+$ (*non*-dominant cocharacters).
2. For $\lambda \in X_*(T)_+$,
$$W_{\gamma_x}(\lambda(\varpi_x)) = q_x^{\text{scalar}(\lambda)} \operatorname{Tr}(\gamma_x \mid_{V(\lambda)}).$$
the right hand side being "trace of γ on the highest weight representation with weight λ of $G^\vee$".

Remark 3.2. The first part is an easy exercise.

3.4 Consequence

We're going to draw a picture of where various things live.

Define
$$\widetilde{Q} := N(F)\backslash N(\mathbb{A}) \times_{N(O)} G(\mathbb{A})^+/G(O).$$
The Whittaker sheaf W_σ is supported mod center in $G(\mathbb{A})^+/G(O)$. The map $(u, g) \mapsto ug$ presents $\widetilde{Q}$ over a space
$$B(F_x)^+ = N(F_x)(T(F_x)^+ := T(F_x) \cap \mathrm{Mat}_n(O_x)).$$
This is where the Whittaker function lives. It in admits a map down to
$$|\operatorname{Bun}'_n(\mathbb{F}_q)| \subset P_1(F)\backslash G(\mathbb{A})/G(O)$$
where our fake automorphic form lives, which then maps down to where the actual automorphic form lives
$$\begin{array}{ccc} \widetilde{Q} := N(F)\backslash N(\mathbb{A}) \times_{N(O)} G(\mathbb{A})^+/G(O) & & \\ \downarrow & & \\ Q := N(F)\backslash B(\mathbb{A})^+/B(O) & \hookrightarrow & N(F)\backslash G(\mathbb{A})/G(O) \\ \downarrow & & \\ |\operatorname{Bun}_n(\mathbb{F}_q)| := \mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A})/\mathrm{GL}_n(O). & & \end{array}$$

We start with W_σ on the top and descend it to f'_σ on the second layer. To descend further down the ladder, we want to geometrize, but we can only do so on the subsets $\widetilde{Q}$ and Q.

Theorem 3.3. *There is a sheaf $\mathcal{F}_E$ on an algebraic stack $\widetilde{\mathcal{Q}} \xrightarrow{\widetilde{\nu}} \mathrm{Bun}'_n$ such that on $\mathbb{F}_q$-points*

$$|\widetilde{\mathcal{Q}}(\mathbb{F}_q)| = \widetilde{Q}$$

and admitting a map

$$\widetilde{\nu}\colon \widetilde{\mathcal{Q}} \to |\,\mathrm{Bun}'_n(\mathbb{F}_q)|$$

such that

$$\mathrm{Tr}(\widetilde{\nu}_! \mathcal{F}_E) = f'_\sigma|_{\mathrm{Bun}'_n(\mathbb{F}_q)}.$$

In the last few minutes we'll try to tell you as much as possible about the construction of $\mathcal{F}_E$.

4 Laumon's construction

The most significant part of $\mathcal{F}_E$ is that which geometrizes the Casselman-Shalika formula. This is a remarkable construction due to Laumon.

Laumon defined $\mathcal{L}_E$ on a stack $\mathrm{Coh}^n \leftarrow \widetilde{\mathcal{Q}}$. Here Coh is the algebraic stack/$\mathbb{F}_q$ parametrizing torsion coherent sheaves of finite length on X. That is, $\mathrm{Hom}(S, \mathrm{Coh})$ is the groupoid whose objects are coherent sheaves $\mathcal{T}$ on $X \times S$ that are finite flat over S.

We can then define a substack Coh^n of Coh which is the open substack of those $\mathcal{T}$ such that at each closed point of S, $\mathcal{T}|_{\mathrm{pt}}$ is a sum of at most n indecomposable summands:

$$\mathcal{T} = \bigoplus_i \mathcal{O}_X/\mathcal{O}_X(-D_i).$$

This breaks up into a union of components by degree:

$$\mathrm{Coh}^n = \coprod_{m>0} \mathrm{Coh}^{n,m}$$

where $\mathrm{Coh}^{n,m}$ is the "degree m part" of Coh^n. For a local system E on X, we get a local system $E^{\boxtimes m}$ on X^m, which then admits an obvious map π to $X^{(m)}$. We descend

$$E^{\boxtimes m} \rightsquigarrow (\pi_* E^{\boxtimes m})^{S_m} =: E^{(m)}.$$

Then $E^{(m)}|_{X^{(m),rss}}$ is a local system. We have a map

$$X^{(m),rss} \to \mathrm{Coh}^{n,m,rss}$$

and $E^{(m)}|_{X^{(m)},rss}$ descends to a local system $\mathcal{L}^{rss}_{E,m}$ on $\mathrm{Coh}^{n,m,rss}$. Finally, we define $\mathcal{L}_{E,m}$ as a middle extension sheaf for $j\colon \mathrm{Coh}^{n,m,rss} \hookrightarrow \mathrm{Coh}^n$:

$$\mathcal{L}_{E,m} := j_{!*}\mathcal{L}^{rss}_{E,m}.$$

Definition 4.1. *Laumon's sheaf* $\mathcal{L}_E$ is (up to shift) the perverse sheaf on Coh^n whose restriction to each $\mathrm{Coh}^{n,m}$ is $\mathcal{L}_{E,m}$.

The relation to Casselman-Shalika is described in (the second part of) the following theorem:

Theorem 4.2 (Laumon)**.** *The function* $\mathrm{Tr}(\mathcal{L}_E)\colon \mathrm{Coh}^n(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell$ *is given by*

1. $\mathrm{Tr}(\mathcal{L}_{E,m})(\mathcal{T}) = \prod_{x\in|X|} \mathrm{Tr}\,\mathcal{L}_{E,m_x,x}(\mathcal{T}_x)$ *where* $\mathrm{Coh}^{n,m}(x) \to \mathrm{Coh}^{n,m}$ *is defined in the same way but for torsion sheaves supported at x,* $\mathcal{L}_{E,m_x,x}$ *is the pullback of* $\mathcal{L}_{E,m}$*, and* $\mathcal{T}_x$ *is the restriction of* $\mathcal{T}$ *to a neighborhood of x where it is only supported at x.*

2. *We have*

$$\mathcal{L}_{E,m,x} \cong \bigoplus_{\lambda\in X_*(T)_{++,m}} IC(\mathrm{Coh}^{n,m,\lambda}(x)) \otimes E_x(\lambda)$$

   (where indexing is means $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n \geq 0$ *and* $m = \sum \lambda_i$*). Here* $Coh^{n,m,\lambda}$ *are st rata of* $\mathrm{Coh}^{n,m}(x)$ *and* $E_x(\lambda)$ *is obtained by composing* E_x *with the representation of heighest weight* λ*. Then* Frob_x *acts by* $\mathrm{Tr}(\sigma(\mathrm{Frob}_x)|_{V(\lambda)})$ *(i.e. the Casselman-Shalika formula).*