LTF FOR COHOMOLOGICAL CORRESPONDENCES

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1. Motivation

We want to compute an intersection number

$$\mathbb{I}_r(h_D) = \langle \mathrm{Sht}'_T, \mathrm{Sht}'_T, \rangle_{\mathrm{Sht}'_G}.$$

The shtuka involves some sort of Frobenius.

$$\begin{array}{ccc} \mathrm{Sht} & \longrightarrow & H \\ \downarrow & & \downarrow \\ M & \xrightarrow{\Gamma := \mathrm{Id} \times \mathrm{Frob}} & M \times M \end{array}$$

We'll rewrite this intersection in another order, so that at the end the answer will be presented as a refined Gysin pullback via Frobenius, which we can then compute in terms of a cohomological trace.

Recall the "usual" Grothendieck-Lefschetz trace formula.

Theorem 1.1. *Let X_0 be a variety over $\mathbf{F}_q$, and $X = X_0 \times_{\mathbf{F}_q} \overline{\mathbf{F}_q}$.*

$$\sum_i (-1)^i \operatorname{Tr}(\mathrm{Frob} \mid H^i_c(X, \mathcal{E})) = \sum_{x \in X_0(\mathbf{F}_q)} \operatorname{Tr}(\mathrm{Frob}_x \mid \mathcal{E}_{\overline{x}}).$$

Outline of

1. Cohomological correspondences.
2. Trace formula.
3. Application.

2. Cohomological correspondences

2.1. **Setup.** To convey the idea, we're just going to work with schemes. Let $k = \overline{k}$ be an algebraically closed field, and X a scheme of finite type over k. Let $D(X) := D^b_c(X, \mathbf{Q}_\ell)$. If $f \colon X \to Y$ is a map then we have functors

$$f_*, f_! \colon D^b(X) \to D^b(Y)$$

and

$$f^*, f^! \colon D^b(Y) \to D^b(X)$$

and adjunctions

$$\mathrm{Id} \to f_* f^*$$

$$f_! f^! \to \mathrm{Id}\,.$$

2.2. **Borel-Moore homology.** Let $\pi\colon X \to \operatorname{Spec} k$, then $K_X = \pi^! \mathbf{Q}_\ell$ is the dualizing sheaf.

Example 2.1. If X is smooth of dimension n, then $K_X = \mathbf{Q}_\ell[2n](n)$.

Definition 2.2. We define the *Borel-Moore homology*

$$H_d^{BM}(X) := H^{-d}(K_X).$$

If $f\colon X \to Y$ is proper, then we have a trace map

$$\operatorname{Tr}\colon H_0^{BM}(X) \to H_0^{BM}(Y)$$

via

$$f_! K_X = f_! f^! K_Y \to K_Y.$$

using that $K_X = f^! K_Y$.

Think of H_0^{BM} as being a receptacle for 0-cycles (it is the target of cycle class map from Ch_0) , and this as being pushforward of cycles. In particular, if X is proper over k then the pushforward for the structure map $X \to \operatorname{Spec} k$ is the degree map

$$H_0^{BM}(X) \xrightarrow{\deg} \mathbf{Q}_\ell.$$

2.3. **Cohomological correspondences.**

Definition 2.3. Given X_1, X_2 a *correspondence* between X_1, X_2 is a diagram

$$\begin{array}{ccc} & C & \\ {}^{c_1}\swarrow & & \searrow^{c_2} \\ X_1 & & X_2 \end{array}$$

Given $\mathcal{F}_i \in D(X_i)$, a *cohomological correspondence* is an element

$$u \in \operatorname{Hom}_C(c_1^* \mathcal{F}_1, c_2^! \mathcal{F}_2) = \operatorname{Hom}_{X_2}(c_{2!} c_1^* \mathcal{F}_1, \mathcal{F}_2).$$

Example 2.4. For a morphism $f\colon X \to Y$, we have $f^* \mathbf{Q}_\ell = \mathbf{Q}_\ell = \operatorname{Id}^! \mathbf{Q}_\ell$. This gives a cohomological correspondence, which is admittedly trivial.

Example 2.5. Let X_2 be smooth of dimension n. Then $K_{X_2} = \mathbf{Q}_\ell[2n](n)$, so $c_2^! \mathbf{Q}_\ell = K_C[-2n](-n)$. So the cohomological correspondences between $\mathbf{Q}_\ell$ and $\mathbf{Q}_\ell$ are maps

$$\mathbf{Q}_\ell \to K_C[-2n](-n) = H_{2n}^{BM}(c)(-n).$$

We get a Borel-Moore homology class from any cycle, which gives a map

$$\operatorname{Ch}(C) \to \operatorname{Corr}_C(\mathbf{Q}_\ell, \mathbf{Q}_\ell).$$

2.4. **Maps on cohomology.** If c_1 is proper, then from a cohomological correspondence u we can define a map

$$R\Gamma_c(u)\colon R\Gamma_c(X_1, \mathcal{F}_1) \to R\Gamma_c(X_2, \mathcal{F}_2).$$

Indeed, we have a map of sheaves

$$\mathcal{F}_1 \to c_{1*} c_1^* \mathcal{F}_1 = c_{1!} c_1^* \mathcal{F}_1$$

(using that c_1 is proper in the second equality) which induces on cohomology

$$R\Gamma_c(X_1, \mathcal{F}_1) \to R\Gamma_c(C, c_1^* \mathcal{F}_1) \xrightarrow{u} R\Gamma_c(C, c_2^! \mathcal{F}_2) = R\Gamma_c(X_2, c_{2!} c_2^! \mathcal{F}_2) \to R\Gamma_c(X_2, \mathcal{F}_2).$$

More generally, given a diagram of correspondences

$$\begin{array}{ccccc} X_1 & \xleftarrow{c_1} & C & \xrightarrow{c_2} & X_2 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_2} \\ Y_1 & \xleftarrow{d_1} & D & \xrightarrow{d_2} & Y_2 \end{array}$$

if (a) f and f_1 are proper, and (b) c_1 and d_1 are proper then we can define a pushforward

$$[f]_! \colon \mathrm{Corr}_C(\mathcal{F}_1, \mathcal{F}_2) \to \mathrm{Corr}_D(f_{1!}\mathcal{F}_1, f_{2!}\mathcal{F}_2).$$

This generalizes the previous construction, which is the special case with $Y_1 = D = Y_2 = \mathrm{Spec}\ k$ sending a correspondence $u \mapsto R\Gamma_C(u) \in \mathrm{Corr}_{\mathrm{pt}}(R\Gamma_c(\mathcal{F}_1), R\Gamma_c(\mathcal{F}_2))$.

3. Trace formula

3.1. **Self correspondences.** Suppose we have a correspondence between X and itself:

$$\begin{array}{ccc} & C & \\ {\scriptstyle c_1}\swarrow & & \searrow{\scriptstyle c_2} \\ X & & X \end{array}$$

If c_1 is proper, then we have an endomorphism of $R\Gamma_c(u)$ on $R\Gamma_c(X, F)$. The fundamental question is: *what is its trace*?

In a relative situation, if we have a map of correspondences

$$\begin{array}{ccccc} & & C & & \\ & {\scriptstyle c_1}\swarrow & \downarrow & \searrow{\scriptstyle c_2} & \\ X & & \downarrow & & X \\ \downarrow{\scriptstyle f} & & S & & \downarrow{\scriptstyle f} \\ & \swarrow & & \searrow & \\ S & & & & S \end{array}$$

then $[f]_!(u)$ is an endomorpism of $f_!\mathcal{F}$.

3.2. **The trace.** Consider the cartesian square

$$\begin{array}{ccc} \mathrm{Fix}(c) & \xrightarrow{\Delta'} & C \\ \downarrow{\scriptstyle c'} & & \downarrow{\scriptstyle c = c_1 \times c_2} \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

Definition 3.1. We define a *trace map*

$$\mathrm{R}\mathscr{H}om_C(c_1^*\mathcal{F}, c_2^!\mathcal{F}) \to \Delta'_* K_{\mathrm{Fix}(c)}. \tag{3.1}$$

as follows. We have

$$\mathrm{R}\mathscr{H}om_C(c_1^*\mathcal{F}, c_2^!\mathcal{F}) \cong c^!(\mathbf{D}(\mathcal{F}) \boxtimes \mathcal{F}) \to c^!(\Delta_* K_X)$$

where $\mathbf{D}(-) = \mathrm{R}\mathscr{H}om(-, K_C)$ is Verdier duality, and then we apply base change.

Applying H^0 to (3.1), we get

$$\mathrm{Tr}\colon \mathrm{Corr}_C(\mathcal{F}, \mathcal{F}) \to H^0(\mathrm{Fix}, K_{\mathrm{Fix}(c)}) = H_0^{BM}(\mathrm{Fix}(c)).$$

Now suppose β is a connected component of $\mathrm{Fix}(c)$, so we have

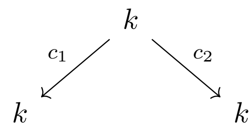
$$H_0^{BM}(\mathrm{Fix}) = \bigoplus_{\beta \in \pi_C(\mathrm{Fix})} H_0^{BM}(\mathrm{Fix}_\beta).$$

Assume further that β is proper over k. Then we can push forward to k and take the degree.

Definition 3.2. In the situation above, we define the *local terms*

$$LT_\beta(u) = \deg(\mathrm{Tr}(u)_\beta) \in \mathbf{Q}_\ell.$$

Example 3.3. For the correspondence



the cohomological correspondences are just $\mathrm{Hom}(\mathcal{F}, \mathcal{F})$ and the trace as defined above coincides with the usual trace.

3.3. **The local-global formula.**

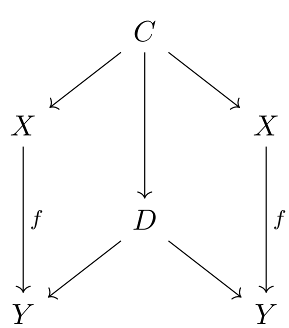
Example 3.4. For X smooth of dimension n and $\mathcal{F} = \mathbf{Q}_\ell$, we have $\mathrm{Corr}_C(\mathbf{Q}_\ell, \mathbf{Q}_\ell) = H_{2n}^{BM}(C)(-n)$. There is a cycle class map

$$\mathrm{Ch}_n(C) \to \mathrm{Corr}_C(\mathbf{Q}_\ell, \mathbf{Q}_\ell) = H_{2n}^{BM}(C)(-n) \xrightarrow{\mathrm{Tr}} H_0^{BM}(\mathrm{Fix})$$

The claim is that the diagram commutes:

$$\begin{array}{ccc} \mathrm{Ch}_n(C) & \longrightarrow & \mathrm{Corr}_C(\mathbf{Q}_\ell, \mathbf{Q}_\ell) = H_{2n}^{BM}(C)(-n) \\ \downarrow{\scriptstyle \Delta^!} & & \downarrow{\scriptstyle \mathrm{Tr}} \\ \mathrm{Ch}_0(\mathrm{Fix}) & \longrightarrow & H_0^{BM}(\mathrm{Fix}) \end{array}$$

Theorem 3.5. *The trace commutes with proper pushforward. In other words, if*



is a map of correspondences, with f proper, then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Corr}_C(F,F) & \xrightarrow{\mathrm{Tr}} & H_0^{BM}(\mathrm{Fix}(c)) \\ \downarrow{\scriptstyle [f]_!} & & \downarrow{\scriptstyle f_!} \\ \mathrm{Corr}_D(f_!F, f_!F) & \xrightarrow{\mathrm{Tr}} & H_0^{BM}(\mathrm{Fix}(d)) \end{array} \tag{3.2}$$

Corollary 3.6. *If C, X are proper over k, then*

$$\mathrm{Tr}(R\Gamma_c(u)) = \sum_\beta LT_\beta(u).$$

Proof. The left side corresponds to the left path of the commutative diagram in (3.2), and the right side corresponds to the right path in (3.2). ☐

This is what is usually called the Lefschetz-Verdier trace formula.

3.4. **The naïve local terms.** There are two issues with the trace formula. First, how do you actually compute the local terms? Consider a correspondence

$$\begin{array}{ccccc} & & C & & \\ & \swarrow^{c_1} & & \searrow^{c_2} & \\ X & & & & X \end{array}$$

with c_2 is quasifinite. Given $y \in \mathrm{Fix}(c)$, with $x = c_1(y) = c_2(y)$, we can define

$$u_y \colon F_x \to F_x$$

as follows. We have a cohomological correspondence

$$(c_{2!}, c_1^* F) = \bigoplus_{z \mapsto x} c_1^* F|_z \to F_x$$

by adjunction from

$$F_x \hookrightarrow \bigoplus_{z \mapsto x} F|_{c_1(z)}.$$

Definition 3.7. The $\mathrm{Tr}(u_y)$ defined above is called the *naïve local term.*

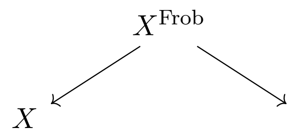
Example 3.8. The naïve local term does not necessarily coincide with the local terms computed above. Consider translation $x \mapsto x+1$ on $\mathbf{P}^1 \to \mathbf{P}^1$. Then

$$LT_\infty(u) = 2$$

whereas the naive local term is $\mathrm{Tr}(u_\infty) = 1$. The naïve local term doesn't know that the fixed point ∞ should have multiplicity 2; it only counts the physical fixed points. An example in the same spirit is the map $x \mapsto x+1$ on $\mathbf{A}^1$.

Another issue is that we need properness. We can solve that by compactifying everything, but then you get local terms at infinity, which may be non-zero (as we saw in the preceding example).

3.5. **A special case.** Let X_0 be a variety over $k = \overline{\mathbf{F}}_q$ and $X = X_0 \times_{\mathbf{F}_q} \overline{\mathbf{F}}_q$. Consider the correspondence



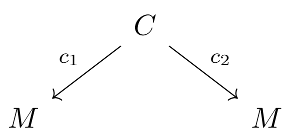
Let $u = \mathrm{Frob}^* \mathcal{E} \to \mathcal{E}$. Then the local terms coincide with the naïve local terms. In other words,

(1) For all $s \in X_0(\mathbf{F}_q)$, we have
$$LT_s(u) = \mathrm{Tr}(u_s).$$
(2) We have
$$\mathrm{Tr}(R\Gamma_c(u)) = \sum_s \mathrm{Tr}(u_s, E_s).$$

Why? The idea is that Frobenius is contracting near fixed points. For $s \in \mathrm{Fix}$,
$$\mathrm{Frob}^{-1}(\mathfrak{m}_x^n)\mathcal{O}_X \subset \mathfrak{m}_x^{n+1}\mathcal{O}_X$$
for some $n \geq 0$. Geometrically, this means that if we pass to the normal cone we get an endomorphism which contracts everything to the origin.

4. Applications to the appendix

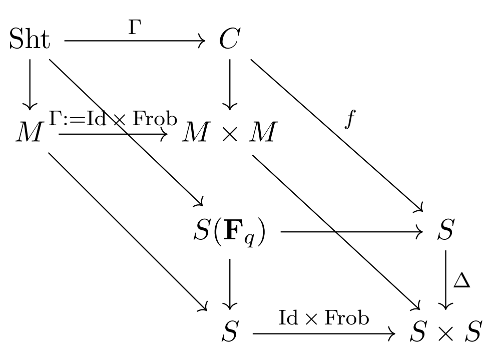
Consider a correspondence



Assume

- c_1 is proper, and
- M is smooth of dimension n, and
- we have a proper map $f\colon C \to S$.

Let $\gamma \in \mathrm{Ch}_n(C)_{\mathbf{Q}}$. Suppose we have a map of cartesian squares



Then we can write

$$\mathrm{Sht} = \coprod_{s \in S(\mathbf{F}_q)} \mathrm{Sht}_s \,.$$

We can pull back $(\Gamma^! \gamma)_s$ = contribution of Sht_s. This is in $\mathrm{Ch}_0(\mathrm{Sht}_S)_{\mathbf{Q}}$, which is proper, so we can apply the degree map to get something in $\mathbf{Q}$. We want a formula for it, so set

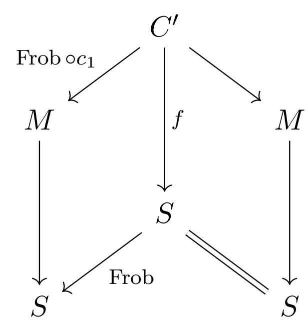
$$\langle \gamma, \Gamma_{\mathrm{Frob}} \rangle_s := \deg(\Gamma^! \gamma)_s.$$

Theorem 4.1. *We have*

$$\langle \gamma, \Gamma_{\mathrm{Frob}} \rangle_s = \mathrm{Tr}((f_! \mathrm{cl}(\gamma))_s \circ \mathrm{Frob}_s \mid (f_! \mathbf{Q}_\ell)_{\overline{s}}).$$

The argument has two steps: compatibility of trace with proper pushforward, and the special case discussed in §3.5.

The first idea is to replace the correspondence C with Frobenius, by composing $C \xrightarrow{c_1} M$ with $C \xrightarrow{c_1} M \xrightarrow{\mathrm{Frob}} M$. This gives a C' which lives over the Frobenius correspondence for S.



The second idea is to use the compatibility of trace with proper pushforward to express this as a trace on S, from which one gets the answer.