INTERSECTION THEORY ON STACKS

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The aim is to introduce intersection theory on stacks which are only locally of finite type, like the moduli stack of shtukas. Fortunately, we only need the **Q**-theory, which makes things easier.

1. Definition of $\mathrm{Ch}(X)_{\mathbf{Q}}$

1.1. Chow groups for finite type.

Definition 1.1. Let X/k be a DM stack, finite type over k. Then we define

$$\mathrm{Ch}_*(X)_{\mathbf{Q}} = Z_*(X)_{\mathbf{Q}}/\partial W_*(X)_{\mathbf{Q}}$$

where

- $Z_*(X)_{\mathbf{Q}} = \bigoplus_V \mathbf{Q}$ with V running over irreducible reduced closed substacks of dimension $*$, and
- $W_*(X)_{\mathbf{Q}} = \bigoplus_W k(W)^* \otimes_{\mathbf{Z}} \mathbf{Q}$ with the same index set, and $k(W)$ viewed as a rational function to $\mathbf{A}^1_k$; the inclusion into $Z_*(X)$ is by the "boundary" as in the usual case for schemes.

1.2. Generalization to locally finite type.

When X is *locally* finite type over k, we replace $Z_*(X)_{\mathbf{Q}}$ with $Z_{c,*}(X)_{\mathbf{Q}}$ and $W_{c,*}(X)_{\mathbf{Q}}$, where the subscript c indicates that we only take substacks *proper* over Spec k. We have

$$\mathrm{Ch}_c(X) = \varinjlim_{Y \text{ f.t. } \subset X} \mathrm{Ch}_*(Y)_{\mathbf{Q}} = \varinjlim_{U \text{ open } \subset X} \mathrm{Ch}_{*,c}(U)_{\mathbf{Q}}.$$

1.3. Degree map.

We want to define a map

$$\deg\colon \mathrm{Ch}_{c,0}(X)_{\mathbf{Q}} \to \mathbf{Q}.$$

Since we are working with stacks, we need to account for stabilizers.

Definition 1.2. Let $x \in X$ be represented by a geometric point $\overline{x}\colon \mathrm{Spec}\, k^s \to X$. We define

$$\deg x := [(k^{\mathrm{sep}})^{\Gamma_x} \colon k] \cdot \frac{1}{|\,\mathrm{Aut}(x^s)|}.$$

1.4. Intersection pairing.

Now let X be smooth, locally of finite type, and pure dimension n. Then we have an intersection product

$$\mathrm{Ch}_{c,i}(X)_{\mathbf{Q}} \times \mathrm{Ch}_{c,j}(X)_{\mathbf{Q}} \to \mathrm{Ch}_{c,i+j-n}(X)_{\mathbf{Q}} \tag{1.1}$$

defined as follows. Let Y_1, Y_2 be closed substacks of X, which are proper over k. Then (1.1) is the colimit of the finite-type intersection products

$$\mathrm{Ch}_i(Y_1)_{\mathbf{Q}} \times \mathrm{Ch}_j(Y_2)_{\mathbf{Q}} \to \mathrm{Ch}_{i+j-n}(Y_1 \cap Y_2) \to \mathrm{Ch}_{c,i+j-n}(X)_{\mathbf{Q}}.$$

The first map is subtle to define: it is the *refined intersection product*

$$(\zeta_1, \zeta_2) \mapsto X \times_{(X,X)} (\zeta_1 \times \zeta_2).$$

What does this mean? It is a special case of the *refined Gysin morphism*. Start with the fibered product diagram

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{i} & Y \end{array}$$

where i is a regular embedding of codimension e. Then we get a *refined Gysin morphism*

$$i^! \colon \operatorname{Ch}_i(V)_{\mathbf{Q}} \to \operatorname{Ch}_{i+e}(W)_{\mathbf{Q}}$$

and we define

$$X \times_{(X,X)} (\zeta_1 \times \zeta_2) := \Delta^!(\zeta_1 \times \zeta_2).$$

Thus we have finally constructed the product

$$\operatorname{Ch}_{c,i}(X)_{\mathbf{Q}} \times \operatorname{Ch}_{c,j}(X)_{\mathbf{Q}} \to \operatorname{Ch}_{c,i+j-n}(X)_{\mathbf{Q}}$$

Then composing with the degree map, we get an intersection pairing

$$\langle , \rangle_X \colon \operatorname{Ch}_{c,j}(X)_{\mathbf{Q}} \times \operatorname{Ch}_{c,n-j}(X)_{\mathbf{Q}} \to \mathbf{Q}.$$

Remark 1.3. (i) We have a cycle class map

$$\operatorname{cl}_X \colon \operatorname{Ch}_{c,j}(X)_{\mathbf{Q}} \to H_c^{2n-2j}(X \otimes_k \overline{k}, \mathbf{Q}_\ell(n-j))$$

and the intersection product is compatible with cup product.

(ii) Consider

$${}_c\operatorname{Ch}_n(X \times X)_{\mathbf{Q}} = \varinjlim_{Z \subset X \times X} \operatorname{Ch}_*(Z)_{\mathbf{Q}}$$

such that $\operatorname{pr}_1|_Z$ is proper. This is a $\mathbf{Q}$-algebra. It acts on each $\operatorname{Ch}_{c,j}(X)_{\mathbf{Q}}$ via

$$(\xi, \zeta) = \operatorname{pr}_{2*}(\xi \cdot_{(X \times X)} \operatorname{pr}_1^* \zeta).$$

Now that we have a definition, the problem is that we can't really calculate. So instead we pass to K groups.

2. Relation to K-theory

For technical reasons, we need to relate the Chow groups to K-theory. First we recall K-theory of schemes of finite type over k. Let $K_0'(X)$ be the Grothendieck group of the abelian category of coherent $\mathcal{O}_X$-modules. Let $K_0'(X)_{\mathbf{Q}}$ be the rationalization.

2.1. **The naïve filtration.** We have a filtration

$$K_0'(X)^{\mathrm{naive}}_{\mathbf{Q},\le m} = \mathrm{Im}\left(K_0(\mathrm{Coh}(X)_{\le m})_{\mathbf{Q}} \to K_0'(X)_{\mathbf{Q}}\right)$$

where $\mathrm{Coh}(X)_{\le m}$ is the subcategory of coherent sheaves with support of dimension at most m.

We have a natural graded map

$$\phi_X\colon \mathrm{Ch}_*(X)_{\mathbf{Q}} \to \mathrm{Gr}^{\mathrm{naive}}_*(K_0'(X))_{\mathbf{Q}}$$

sending

$$[V]\colon \mapsto \text{class of } \mathcal{O}_V.$$

This is an isomorphism: we have a commutative diagram

$$\begin{array}{ccc} K_0(\mathrm{Coh}(X)_{\le m})_{\mathbf{Q}} & \longrightarrow & \mathrm{Gr}^{\mathrm{naive}}_m(X)_{\mathbf{Q}} \\ \big\downarrow{\scriptstyle \mathrm{supp}} & & \big\downarrow{\scriptstyle \psi_X} \\ Z_m(X)_{\mathbf{Q}} & \longrightarrow & \mathrm{Ch}_m(X)_{\mathbf{Q}} \end{array}$$

where the map supp sends $\mathcal{F} \mapsto \sum_{\dim V = m} \mu_V(\mathcal{F}) \cdot [V]$.

This discussion was for schemes. For stacks, all definitions extend but it's not clear if the map

$$\begin{array}{ccc} K_0(\mathrm{Coh}(X)_{\le m})_{\mathbf{Q}} & & \\ \big\downarrow & & \\ Z_m(X)_{\mathbf{Q}} & \longrightarrow & \mathrm{Ch}_m(X)_{\mathbf{Q}} \end{array}$$

factors through $K_0'(X)^{\mathrm{naive}}_{\le m}$.

2.2. **The not-so-naïve filtration.** This problem is solved in the paper under the assumption

> (*) there exists a finite flat presentation $U \to X$ where U is an algebraic space of finite type over k.

Define $K_0'(X)_{\mathbf{Q},\le m}$ to be the set of $\alpha \in K_0'(X)_{\mathbf{Q}}$ such that there exists a finite presentation $\pi\colon U \to X$ with $\pi^*(\alpha) \in K_0'(U)^{\mathrm{naive}}_{\mathbf{Q},\le m}$.

Example 2.1. It may happen that $K_0'(X)_{\mathbf{Q},\le m}$ is non-zero for $m < 0$. (Of course, this doesn't happen for the naïve filtration.) Let $X = [*/G]$. Then $K_0'(X)_{\mathbf{Q}} = \mathrm{Rep}_{\mathbf{Q}}(G)$, and $K_0'(X)_{\mathbf{Q},\le -1}$ is the augmentation ideal (in particular, non-zero). Indeed, when we pull back via the cover $* \to [*/G]$, anything in the augmentation ideal becomes 0 in $K_0(*)$.

In general, we have an inclusion $K_0'(X)^{\mathrm{naive}}_{\mathbf{Q},\le m} \subset K_0'(X)_{\mathbf{Q},\le m}$, which is an equality if X is an algebraic space.

The filtration just defined enjoys expected functoriality properties: compatibility with flat pullback and under proper pushforward.

Let X be a DM stack satisfying (*). Then there is a homomorphism

$$\psi_X\colon \mathrm{Gr}_m(K_0'(X)_{\mathbf{Q}}) \to \mathrm{Ch}_*(X)_{\mathbf{Q}}$$

induced by a commutative diagram

$$\begin{array}{ccccc} K_0(\mathrm{Coh}(X)_{\leq m})_{\mathbf{Q}} & \longrightarrow & K_0'(X)^{\mathrm{naive}}_{\mathbf{Q},\leq m} & \longrightarrow & K_0'(X)_{\mathbf{Q},\leq m} \\ \downarrow & & & & \downarrow \psi_X \\ Z_m(X)_{\mathbf{Q}} & & \longrightarrow & & \mathrm{Ch}_m(X)_{\mathbf{Q}} \end{array}$$

We now come to a key technical point, which the compatibility of K-theory with the refined Gysin homomorphism. We will describe two situations in which we can deduce a good compatibility relationship.

2.3. **(A): Compatibility with the refined Gysin homomorphism.** Consider the cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow g & & \downarrow h \\ X & \xrightarrow{f} & Y \end{array}$$

Assumptions (A).

- Assume that X' satisfies (*).
- Assume that f is the composition of a regular embedding of codimension e and smooth morphism of relative dimension $e-d$. (Note that this is automatic if X and Y are smooth.)

We have two maps: the refined Gysin morphism

$$f^!\colon \mathrm{Ch}_*(Y')_{\mathbf{Q}} \to \mathrm{Ch}_{*-d}(X')_{\mathbf{Q}}$$

and the pullback on K-theory

$$f^*\colon K_0'(Y')_{\mathbf{Q}} \to K_0'(X')_{\mathbf{Q}}$$

sending $\mathcal{F} \mapsto (f')^{-1}(\mathcal{F}) \overset{\mathbf{L}}{\otimes}_{(f\circ g)^{-1}\mathcal{O}_Y} (f')^{-1}(\mathcal{O}_Y)$.

Proposition 2.2. *Under the assumptions (A):*

(1) The pullback f^* sends $K_0'(Y')^{\mathrm{naive}}_{\mathbf{Q},\leq m}$ to $K_0'(X')_{\mathbf{Q},\leq m}$ and hence induces a map

$$\mathrm{Gr}^{\mathrm{naive}}_m f^*\colon \mathrm{Gr}^{\mathrm{naive}}_m K_0'(Y')_{\mathbf{Q}} \to \mathrm{Gr}_{m-d} K_0'(X')_{\mathbf{Q}}.$$

(2) We have a commutative diagram

$$\begin{array}{ccccc} & \nearrow & \mathrm{Gr}^{\mathrm{naive}}_m K_0'(Y')_{\mathbf{Q}} & \xrightarrow{\mathrm{Gr}^{\mathrm{naive}}(f')^*} & \searrow \\ K_0(\mathrm{Coh}(X)_{\leq m})_{\mathbf{Q}} & & & & \mathrm{Gr}_{m-d}(X')_{\mathbf{Q}} \\ \downarrow \mathrm{supp} & & & & \downarrow \\ Z_m(Y')_{\mathbf{Q}} & & \longrightarrow & & \mathrm{Ch}_{m-d}(X')_{\mathbf{Q}} \end{array}$$

If we also assume that Y' satisfies $*$, then we can fill this in to

$$\begin{tikzcd}
 & \mathrm{Gr}_m^{\mathrm{naive}} K_0'(Y')_{\mathbf{Q}} \arrow[d, dashed] \arrow[dr, "\mathrm{Gr}^{\mathrm{naive}}(f')^*"] & \\
K_0(\mathrm{Coh}(X)_{\leq m})_{\mathbf{Q}} \arrow[ur] \arrow[r, dashed] \arrow[d, "\mathrm{supp}"] & \mathrm{Gr}_m K_0'(Y')_{\mathbf{Q}} s \arrow[r, dashed] \arrow[d, dashed] & \mathrm{Gr}_{m-d}(X')_{\mathbf{Q}} \arrow[d] \\
Z_m(Y')_{\mathbf{Q}} \arrow[r] & \mathrm{Ch}_m(Y')_{\mathbf{Q}} \arrow[r] & \mathrm{Ch}_{m-d}(X')_{\mathbf{Q}}
\end{tikzcd}$$

2.4. **(B): Compatibility with Gysin map.** Again consider a cartesian diagram

$$\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\
X & \xrightarrow{f} & Y
\end{array}$$

Assumptions (B).

- Assume h is representable.
- Assume that the normal cone of f is a vector bundle of constant virtual dimension. (We will apply this to $(\mathrm{Id}, \mathrm{Frob})\colon X \to X \times X$, where X is smooth, so this is certainly satisfied.)
- Assume that there exists a commutative diagram

$$\begin{array}{ccc}
U & \longrightarrow & V \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\
X & \xrightarrow{f} & Y
\end{array}$$

  where U and V are smooth surjective maps from schemes of finite type and i is a regular embedding.

Write $\dim Y' = n$ and $\dim X' = n - d$.

Proposition 2.3. *Under the assumptions (B), the following diagram is commutative:*

$$\begin{array}{ccc}
K_0'(Y')_{\mathbf{Q}} & \xrightarrow{\quad f^* \quad} & K_0'(X')_{\mathbf{Q}} \\
\downarrow{\scriptstyle supp} & & \downarrow \\
\mathrm{Ch}_n(Y')_{\mathbf{Q}} = Z_n(Y')_{\mathbf{Q}} & \longrightarrow & Z_{n-d}(X')_{\mathbf{Q}} = \mathrm{Ch}_{n-d}(X')_{\mathbf{Q}}.
\end{array}$$

3. The octahedron lemma

Consider a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & X & \longleftarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ U & \longrightarrow & S & \longleftarrow & V \\ \uparrow & & \uparrow & & \uparrow \\ C & \longrightarrow & Y & \longleftarrow & D \end{array}$$

Let N be the fiber product as in

$$\begin{array}{ccc} N & \longrightarrow & A \times B \times C \times D \\ \downarrow & & \downarrow \\ X \times_S Y \times_S U \times_S V & \longrightarrow & (X \times_S U) \times (X \times_S Y) \times (Y \times_S U) \times (X \times_S V) \end{array}$$

Lemma 3.1. *There are canonical isomorphisms*

$$(C \times_Y D) \times_{U \times_S V} (A \times_X B) \cong N \cong (C \times_U A) \times_{Y \times_S X} (D \times_V B).$$

Theorem 3.2. *Assume everybody is smooth, except B (the "bad" object) of dimension $d_A, d_B, \ldots$. Also assume that the fiber products (on the left) $C \times_Y D$, $U \times_S V$, $C \times_U A, Y \times_S X$ have the expected dimension. Further assume that each of the fiber diagrams*

$$\begin{array}{ccc} A \times_X B & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{array}$$

and

$$\begin{array}{ccc} D \times_V B & \longrightarrow & B \\ \downarrow & & \downarrow \\ D & \longrightarrow & V \end{array}$$

satisfy the compatibility conditions (A) or (B). Finally assume that both fiber diagrams

$$\begin{array}{ccc} N & \longrightarrow & A \times_X B \\ \downarrow & & \downarrow \\ C \times_Y D & \longrightarrow & U \times_S V \end{array}$$

and

$$\begin{array}{ccc} N & \longrightarrow & D \times_V B \\ \downarrow & & \downarrow \\ C \times_U A & \longrightarrow & Y \times_S X \end{array}$$

satisfies the compatibility condition (A). Let $n = \dim N$*. For the diagram*

$$\begin{array}{ccccc} N & \xrightarrow{\alpha} & D \times_V B & \xrightarrow{d} & B \\ \| & & & & \| \\ N & \xrightarrow{\delta} & A \times_X B & \xrightarrow{a} & B \end{array}$$

we have $\delta^! a^! [B] = d^! \alpha^! [B]$.

Roughly speaking, the proof proceeds by using the relation to K-theory, and lifting the statement to the level of derived stacks.