

The Classification of G -bundles

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We now want to generalize the preceding classification of vector bundles to G -bundles (recovering the old results when $G = \mathrm{GL}_n$).

1 Background

1.1 Notation

We fix the notation for this talk: let

- E be a local field (of characteristic 0 or p),
- ϖ_E the uniformizer,
- \mathbb{F}_q the residue field,
- \check{E} the completion of the maximal unramified extension, and
- F an algebraically closed perfectoid field of characteristic p .

1.2 Classical G -bundles

Definition 1.1. Let G be a connected linear algebraic group over E . A *connected G -bundle* on X can be defined in either of the following two ways:

1. (“INTERNAL”) A principal homogeneous space \mathcal{T} under G on X which is locally trivial for the (étale or fppf) topology.
2. (“EXTERNAL”) An exact faithful E -linear \otimes -functor $\mathrm{Rep}_E G \rightarrow \mathrm{Vect}_X$.

Example 1.2. Why are the two definitions equivalent? We sketch one direction. Given a G -torsor \mathcal{T} , we can define the functor

$$V_{\mathcal{T}}((V, \rho)) = \mathcal{T} \times^{G, \rho} V.$$

Definition 1.3. We denote by $|\mathrm{Bun}_G|$ the set of isomorphism classes of connected G -bundles on X .

Example 1.4. If $G = \mathrm{GL}_n$ then $|\mathrm{Bun}_G| = \mathrm{Vect}_{X,n}$.

1.3 The classification of Vect_X

We have a functor

$$\mathcal{E}: \varphi - \text{Mod}_{\check{E}} \rightarrow \text{Vect}_X$$

sending

$$(V, \varphi) \mapsto \bigoplus_{d \geq 0} (B_{E,F}^+ \otimes_{\check{E}} V)^{\varphi = \varpi_E^d}.$$

Theorem 1.5. *This \mathcal{E} is a faithful exact E -linear \otimes -functor, which is essentially surjective (but not fully faithful, see Warning 1.6).*

It also induces an equivalence of categories

$$(\text{isoclinic } \varphi\text{-isocrystals}) \leftrightarrow (\text{semi-stable vector bundles})$$

and a bijection of objects

$$|\varphi - \text{Mod}_{\check{E}}| = |\text{Vect}_X|.$$

Warning 1.6. The functor is *not* fully faithful because $\text{End}(\text{Triv} \oplus \text{Triv}(1))$ is $E \oplus E$ in the category of isocrystals but a ‘‘Banach-Colmez-like object’’ $\begin{pmatrix} E & BC \\ & E \end{pmatrix}$ in the category of vector bundles.

This theorem is what we want to generalize, from vector bundles to G -bundles.

2 G -isocrystals (following Kottwitz)

2.1 The definition

Definition 2.1. Let G be a connected linear algebraic group over E . A G -isocrystal can be defined in either of the following two ways:

1. (EXTERNAL) An exact faithful E -linear \otimes -functor

$$N: \text{Rep}_E G \rightarrow \varphi - \text{Mod}_{\check{E}}.$$

2. (INTERNAL) An element $b \in G(\check{E})$. These form a category via

$$\text{Hom}(b, b') = \{g \in G(\check{E}) \mid gb\sigma(g)^{-1} = b'\}.$$

We denote by $B(G)$ the set of G -isocrystals up to isomorphism.

Example 2.2. Why are the internal and external versions equivalent? Given $b \in G(\check{E})$, we can associate the functor N_b defined by

$$N_b(V, \rho) = (V \otimes_E \check{E}, \rho(b) \circ (\text{Id} \otimes \sigma))$$

Example 2.3. For $G = \text{GL}_n$, the classical isocrystal description of an element $b \in G(\check{E})$ is $(\check{E}^{\oplus n}, b \circ \sigma)$.

2.2 The Newton and Kottwitz invariants

Let G be reductive. We construct two invariants associated to G -bundles.

The Newton Invariant. Let $b \in G(\check{E})$. Then we can associate a homomorphism

$$\nu_b : \mathbb{D}_{\check{E}} \rightarrow G_{\check{E}}$$

where \mathbb{D} is the split torus over E with $X^*(\mathbb{D}) = \mathbb{Q}$. This homomorphism ν_b is characterized by the property that for all (V, ρ) , the morphism

$$\rho \circ \nu_b : \mathbb{D}_{\check{E}} \rightarrow \mathrm{GL}(V_{\check{E}})$$

has induced \mathbb{Q} -grading on $V_{\check{E}}$ equal to the slope filtration of $(V_{\check{E}}, b\sigma)$.

The cocharacter group $X_*(G)$ has an action of G , and we set

$$X_*(G)_{\mathbb{Q}}/G = \mathrm{Hom}_{\check{E}}(\mathbb{D}_{\check{E}}, G_{\check{E}})/G(\check{E}).$$

There is an action of σ on $X_*(G)_{\mathbb{Q}}$, and one can show that $\nu_b \in (X_*(G)_{\mathbb{Q}}/G)^{\sigma}$ only depends on $[b]$, thus inducing a well-defined map

$$\nu : B(G) \rightarrow (X_*(G)_{\mathbb{Q}}/G)^{\sigma}. \quad (1)$$

This is the *Newton invariant*.

Example 2.4. If G is quasi-split, say with Borel B , maximal torus $T \subset B$, and maximal split torus $A \subset T \subset B$ then the right side of (1) can be identified with $X_*(A)_{\mathbb{Q}}^+$.

Remark 2.5. There is also an internal definition of the Newton invariant. Given b , there exists b' with $b \sim b'$ such that $s \gg 0$ such that

$$(b'\sigma)^s = s \cdot \nu_{b'}(\varpi_E) \cdot \sigma^s$$

with the equality taking place in $G(\check{E}) \rtimes \langle \sigma \rangle$. This characterizes $\nu_{[b]} = \nu_{[b']}$ (since ν is supposed to be defined on isomorphism classes).

The Kottwitz invariant. Consider

$$\pi_1(G) = X_*(T)/X_*(T_{sc}).$$

This is canonically and functorially associated to G , and admits an action of Γ . The Kottwitz invariant is described in terms of this fundamental group, as a map

$$\kappa : B(G) \rightarrow \pi_1(G)_{\Gamma}.$$

This is not so easy to define, but we will try to give some feeling for it. Roughly $B(G)$ is similar to $\pi_0(LG)$ (but not quite on the nose) and $\pi_0(LG) = \pi_1(G)_{\Gamma}$.

Theorem 2.6. *The map $B(G) \rightarrow (X_*(G)_{\mathbb{Q}}/G)^{\sigma} \times \pi_1(G)_{\Gamma}$ is injective.*

The description of the image is not easy in general, but in the quasi-split case it is fairly easy to describe it.

Example 2.7. Let $G = \mathrm{GL}_n$. Then $X_*(A)_{\mathbb{Q}}^+ = (\mathbb{Q}^n)_+$ and $\pi_1(G)_{\Gamma} = \mathbb{Z}$. In this case the first component of the map gives the slopes of the Newton polygon, and the second component gives the endpoint of the Newton polygon. So in this case the 1st component determines the second, since the endpoint can be determined from the slopes via the formula

$$(\lambda_i) \in (\mathbb{Q}^n)_+ \mapsto \sum \lambda_i.$$

Therefore, the image can be characterized as the tuples whose break points are integers.

Example 2.8. Let $G = T$. Then $X(A)_{\mathbb{Q}}^+ = X_*(T)_{\Gamma} \otimes \mathbb{Q}$. (There are no positivity conditions because there are no roots.) The second component is $\pi_1(T)_{\Gamma} = X_*(T)_{\Gamma}$. In this case the second component determines the first, via

$$\gamma \in X_*(T)_{\Gamma} \rightarrow X_*(T)_{\Gamma} \otimes \mathbb{Q}.$$

(And the first component determines the second up to torsion.)

2.3 More structure to $B(G)$

First, there is an analogue of the semistable/isoclinic set.

Definition 2.9. Let $B(G)_{\mathrm{basic}} = \{[b] \mid \nu_b = \text{central homomorphism}\}$.

Example 2.10. For GL_n , this means isoclinic.

Inside $B(G)_{\mathrm{basic}}$ there is the subset $B(G)_{\mathrm{basic}}^0 = \{[b] \mid \nu_b = \text{trivial}\}$. This is the analogue of the unit root isocrystals.

These form a section to the Kottwitz invariant. In other words, κ induces *bijections*

$$B(G)_{\mathrm{basic}} \rightarrow \pi_1(G)_{\Gamma}$$

and

$$B(G)_{\mathrm{basic}}^0 \rightarrow \pi_1(G)_{\Gamma, \mathrm{tors}} \cong H^1(E, G).$$

In this sense $B(G)$ is a generalization of Galois cohomology.

2.4 The automorphism group

Another piece of structure is the automorphism group. For $b \in G(\check{E})$, we can associate a a group

$$J_b(R) := \{g \in G(\check{E} \otimes R) \mid gb\sigma(g)^{-1} = b\}.$$

Then $J_b(E) = \mathrm{Aut}(b)$. This turns out to always be a reductive group over E .

Remark 2.11. The $J_b(E)$ are Levis if G is quasiplit.

Some facts.

- An element $b \in G(\check{E})$ is basic if and only if J_b is an inner form of G .
- If $Z(G)$ is connected then *every* inner form comes from some basic b .
- If G is quasisplit, then $B(G)$ can be described in terms $B(M)_{\text{basic}}$ for standard Levi subgroups $M \subset G$.
- $B(G)$ is a partially ordered set and its basic elements are the minimal ones.

3 G -bundles on the Fargues-Fontaine Curve

3.1 Semistable G -bundles

We want to define a functor

$$\mathcal{E}_G : G\text{-isocrystals} \rightarrow \text{Bun}_G.$$

There are again two definitions.

1. (EXTERNAL) Given a G -isocrystal $\text{Rep}_E \rightarrow \varphi\text{-Mod}_{\check{E}}$ in the external sense, composing with \mathcal{E} gives

$$\text{Rep}_E \rightarrow \varphi\text{-Mod}_{\check{E}} \xrightarrow{\mathcal{E}} \text{Vect}_X.$$

This is a G -bundle in the external sense.

2. (INTERNAL) Given $b \in G(\check{E})$, form $G_{\check{E}} \times_{\check{E}} Y_E / \varphi^{\mathbb{Z}}$ with φ acting diagonally by φ on Y_E and by $g \mapsto b\sigma(g)$ on $G_{\check{E}}$.

Theorem 3.1. *Assume that $\text{ch } E = 0$. Then this functor \mathcal{E}_G is faithful and induces a bijection*

$$B(G) \rightarrow |\text{Bun}_G|.$$

Furthermore, \mathcal{E}_G induces an equivalence of categories between $B(G)_{\text{basic}}$ and the category of semi-stable G -bundles.

Definition 3.2. A G -bundle \mathcal{T} is *semi-stable* if

1. (HALF-EXTERNAL) $\mathcal{T}(\text{Lie } G, \text{Ad})$ is a semi-stable vector bundle.
2. (EXTERNAL) $\mathcal{T}(V, \rho)$ is a semi-stable vector bundle if ρ is homogeneous. (Remark: we are using here that tensor of semistable is semistable, which follows from the classification of vector bundles.)

3. (INTERNAL) Let $P \subset G$ be a power-bounded subgroup. Let A_P be the split part of the center of P . We have dually A'_P the split part of the cocenter of P . Then the map

$$A_P \rightarrow A'_P$$

is an isogeny, identifying the rational cocharacter groups. Let \mathcal{T} be a G -bundle and suppose \mathcal{T}_P is a P -structure on \mathcal{T} . Then we define the *slope* cocharacter $\mu(\mathcal{T}_P) \in X_*(A_P)_{\mathbb{Q}}$ which is characterized by the property

$$\langle \mu(\mathcal{T}_P), \lambda \rangle = \deg \lambda_*(\mathcal{T}_P) \quad \text{for all } \lambda \in X^*(A'_P).$$

Finally, we define \mathcal{T} to be semi-stable if and only if

$$\langle \mu(\mathcal{T}_P), \alpha \rangle \leq 0 \forall \alpha \in \text{Lie } N_P.$$

3.2 The two invariants

How are the two invariants expressed in terms of the corresponding G -bundles?

Newton invariant. First assume that G is quasi-split, with $A \subset T \subset B$ as before. Let \mathcal{T} be a G -bundle. The *Harder-Narasimhan reduction theorem* says that there exists a unique pair (P, \mathcal{T}_P) with P a standard parabolic subgroup and \mathcal{T}_P a P -bundle such that

1. $\mathcal{T}_P \times^P M_P$ is a semistable M_P -bundle, and
2. $\mu(\mathcal{T}_P) \in X_*(A_P)_{\mathbb{Q}}^{++}$.

Now the maximal split subtorus $A_P \subset A$ gives a map from $X_*(A_P)_{\mathbb{Q}}^{++} \rightarrow X_*(A)_{\mathbb{Q}}^+$, sending $\mu(\mathcal{T}_P) \mapsto \nu_{\mathcal{T}} \in X_*(A)_{\mathbb{Q}}^+$.

Proposition 3.3. *We have $[\nu_b] = -\nu_{\mathcal{T}(b)}$.*

Why the minus sign? It came up already in Dospinescu's talk: a minus sign was taken to get compatibility of endomorphisms.

Kottwitz invariant. We know that

$$|\text{Bun}_G| = H_{\text{ét}}^1(X, G).$$

Fargues defines a G -equivariant Chern class

$$c_1^G : H_{\text{ét}}^1(X, G) \rightarrow \pi_1(G)_{\Gamma}.$$

Proposition 3.4. *We have*

$$\kappa(b) = c_1^G(\mathcal{E}_G(b)).$$

3.3 What's wrong in characteristic p ?

In the book of Fargues and Fontaine, they construct various categories of φ -isocrystals which give vector bundles. One of these functors is not exact. When you want to apply the external definition of G -bundles you need an exact functor; this uses the fact that in characteristic 0 the representation theory is semisimple.