DEFINITION AND PROPERTIES OF $\mathcal{M}_d$

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1. Goal

Let X be a (smooth, projective, geometrically connected) curve over a finite field k, and $\nu\colon X' \to X$ a degree 2 étale cover.

Let $T := (\mathrm{Res}_{X'/X}\,\mathbf{G}_m)/\mathbf{G}_m$. We can embed T into "PGL_2" $= \mathrm{Aut}(\nu_*\mathcal{O}_{X'})/\mathcal{O}_X^*$.

Remark 1.1. We can also view T as the norm-1 subgroup of the Weil restriction:

$$1 \to T \to \mathrm{Res}_{X'/X}\,\mathbf{G}_m \xrightarrow{\mathrm{Nm}} \mathbf{G}_m \to 1$$

We can put these two definitions together to get an exact sequence

$$1 \to \mathbf{G}_m \to \mathrm{Res}_{X'/X}\,\mathbf{G}_m \xrightarrow{t \mapsto t(\sigma^* t)^{-1}} \mathrm{Res}_{X'/X}\,\mathbf{G}_m \xrightarrow{\mathrm{Nm}} \mathbf{G}_m \to 1. \tag{1.1}$$

The goal is to compute the intersection number

$$\langle \mathrm{Sht}_T, h_D * \mathrm{Sht}_T \rangle_{\mathrm{Sht}_G}$$

where $G = \mathrm{PGL}_2$.

2. The moduli space $\mathcal{M}_d$

2.1. **Relation to intersection numbers.** Recall the shtuka space is

$$\begin{array}{ccc} \mathrm{Sht}_T & \longrightarrow & \mathrm{Hk}^\mu \\ \downarrow & & \downarrow \\ \mathrm{Bun}_T & \xrightarrow{\mathrm{Id},\mathrm{Frob}} & \mathrm{Bun}_T \times \mathrm{Bun}_T \end{array}$$

The idea is that the shtuka construction is complicated, so we should try to do the Bun_T intersection first. So we should try to compute "$\mathrm{Bun}_T \cap h_D * \mathrm{Bun}_T$".

Every time we want to do a PGL_2-computation we actually push it to a GL_2-computation. So as usual, set $\widetilde{T} = \mathrm{Res}_{X'/X}\,\mathbf{G}_m$ and $\widetilde{G} = \mathrm{GL}_2$. Note that $\mathrm{Bun}_{\widetilde{T}}$ can just be thought of as parametrizing line bundles on X, by the definition of Weil restriction.

So we want to compute the intersection

$$\begin{array}{ccc} ? & \longrightarrow & \mathrm{Hk}^d \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{\widetilde{T}} \times \mathrm{Bun}_{\widetilde{T}} & \longrightarrow & \mathrm{Bun}_2 \times \mathrm{Bun}_2 \end{array}$$

$$(\mathcal{L}, \mathcal{L}') \longrightarrow (\nu_*\mathcal{L}, \nu_*\mathcal{L}')$$

In terms of the previous talks, $d = \deg D$. Recall that Hk^d parametrizes maps of vector bundles $\mathcal{E} \hookrightarrow \mathcal{E}'$ with quotient a torsion sheaf of degree d.

Definition 2.1. We define $\widetilde{\mathcal{M}}_d$ to be the fibered product

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_d & \longrightarrow & \mathrm{Hk}^d \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{\widetilde{T}} \times \mathrm{Bun}_{\widetilde{T}} & \longrightarrow & \mathrm{Bun}_2 \times \mathrm{Bun}_2 \end{array}$$

Remark 2.2. What we are calling $\widetilde{\mathcal{M}}_d$ is called $\widetilde{\mathcal{M}}_d^{\heartsuit}$ in the paper, but we're going to omit it because we'll be working with it most often.

2.2. **The functor of points.** Let's try to compute the functor of points of $\widetilde{\mathcal{M}}_d$. View $\mathrm{Bun}_{\widetilde{T}}$ as $\mathrm{Pic}_{X'}$. The bottom horizontal map sends

$$(\mathcal{L}, \mathcal{L}') \mapsto (\nu_*\mathcal{L}, \nu_*\mathcal{L}').$$

The space $\widetilde{\mathcal{M}}_d$ (which is analogous to a "Hitchin space") parametrizes

$$\{(\mathcal{L}, \mathcal{L}', \psi \colon \nu_*\mathcal{L} \to \nu_*\mathcal{L}') \mid \deg \operatorname{coker} \psi = d\}.$$

Let's digest this. We need $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}_X^* \times \mathrm{Pic}_X^{*+d}$, and ψ is equivalent to (by adjunction)

$$\varphi \colon \nu^*\nu_*\mathcal{L} \to \mathcal{L}'$$

We have $\nu^*\nu_*\mathcal{L} = \mathcal{L} \oplus \sigma^*\mathcal{L}$. So φ is equivalent to

$$\nu^*\nu_*\mathcal{L} = \mathcal{L} \oplus \sigma^*\mathcal{L} \xrightarrow{\alpha,\beta} \mathcal{L}'$$

which amounts to the data of two maps

$$\mathcal{L} \xrightarrow{\alpha} \mathcal{L}'$$

$$\sigma^*\mathcal{L} \xrightarrow{\beta} \mathcal{L}'$$

2.3. **Compactification.** We now introduce a compactification of $\mathcal{M}_d$.

Definition 2.3. We define $\overline{\widetilde{\mathcal{M}}_d}$ to be the moduli space classifying

- $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}^*_{X'} \times \mathrm{Pic}^{*+d}_{X'}$,
- Maps

$$\alpha\colon \mathcal{L} \to \mathcal{L}'$$
$$\beta\colon \mathcal{L} \to \sigma^*\mathcal{L}'$$

 such that α, β are not both 0.

Remark 2.4. The bar on $\overline{\widetilde{\mathcal{M}}_d}$ is because we haven't imposed an injectivity condition on ψ. This space is just called $\widetilde{\mathcal{M}}_d$ in the paper.

There is an action of Pic_X on $\overline{\widetilde{\mathcal{M}}_d}$, and we finally define $\overline{\mathcal{M}_d} := \overline{\widetilde{\mathcal{M}}_d}/\mathrm{Pic}_X$.

Remark 2.5. Obviously $\overline{\widetilde{\mathcal{M}}_d}$ isn't of finite type, since it has infinitely many components. Since $\nu^*\colon \mathrm{Pic}^*_X \to \mathrm{Pic}^{2*}_{X'}$ hits "half" the components, $\overline{\mathcal{M}_d}$ is of finite type. In fact it has exactly 2 components.

The map

$$\psi\colon \nu_*\mathcal{L} \to \nu_*\mathcal{L}'$$

when pulled back to X' becomes

$$\nu^*\psi\colon \nu^*\nu_*\mathcal{L} \to \nu^*\nu_*\mathcal{L}'$$

and is given by

$$\nu^*\psi = \begin{pmatrix} \alpha & \sigma^*\beta \\ \beta & \sigma^*\alpha \end{pmatrix}$$

so $\det \nu^*\psi = \mathrm{Nm}\,\alpha - \mathrm{Nm}\,\beta$. We have

$$\widetilde{\mathcal{M}}_d = \overline{\widetilde{\mathcal{M}}_d} \setminus \{\mathrm{Nm}\,\alpha = \mathrm{Nm}\,\beta\},$$

and

$$M_d = [\widetilde{\mathcal{M}}_d/\mathrm{Pic}_X].$$

2.4. **The moduli space $\mathcal{A}_d$.**

Definition 2.6. We define the moduli space $\overline{\mathcal{A}_d}$ parametrizing

- $\Delta \in \mathrm{Pic}_X$,
- $a, b \in H^0(X, \Delta)$ where a and b never simultaneously vanish.

Thus

$$\overline{\mathcal{A}}_d = \widehat{X}_d \times_{\mathrm{Pic}_X} \widehat{X}_d - Z(\mathrm{Pic}^d_X)$$

where $Z(\mathrm{Pic}^d_X) = (\mathrm{Pic}_X \times_{\mathrm{Pic}_X} \widehat{X}_d \cup \widehat{X}_d \times_{\mathrm{Pic}_X} \mathrm{Pic}_X)$, embedding as the locus where a or b vanish.

Remark 2.7. Again we point out that the notation has changed from the paper and previous talks. What is being called $\overline{\mathcal{A}}_d$ used to be called $\mathcal{A}_d$, and what is being called $\mathcal{A}_d$ is called $\mathcal{A}^{\heartsuit}_d$ in the paper.

2.5. **The map f.** There is a map

$$f\colon \overline{\mathcal{M}}_d \to \overline{\mathcal{A}}_d = \widehat{X}_d \times_{\mathrm{Pic}_X} \widehat{X}_d - Z(\mathrm{Pic}^d_X)$$

sending

$$(\mathcal{L}, \mathcal{L}', \alpha, \beta) \mapsto (\mathrm{Nm}(\mathcal{L}') \otimes \mathrm{Nm}(\mathcal{L})^{-1}, a := \mathrm{Nm}(\alpha), b := \mathrm{Nm}(\beta))$$

So $\overline{\mathcal{M}}_d$ is the pre-image of $\mathcal{A}_d := \langle(\mathcal{L}, a, b)\colon a = b\rangle$.

3. Properties of $\mathcal{M}_d$

We begin with an important alternate description of $\overline{\mathcal{M}}_d$. There is a map

$$\iota\colon \overline{\mathcal{M}}_d \to \widehat{X'}_d \times_{\mathrm{Pic}_X} \widehat{X'}_d.$$

Recall that $\widehat{X'}_d \times_{\mathrm{Pic}_X} \widehat{X'}_d$ parametrizes

- $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}_{X'}$,
- $\alpha \in H^0(X', \mathcal{L}), \beta \in H^0(X', \mathcal{L}')$ not both 0,
- $c\colon\ \mathrm{Nm}(\mathcal{L}) \cong \mathrm{Nm}(\mathcal{L}')\}$.

In these terms, ι sends

$$(\mathcal{L}, \mathcal{L}', \alpha, \beta) \mapsto (\mathcal{L}' \otimes \mathcal{L}^{-1}, \mathcal{L}' \otimes \sigma^*\mathcal{L}^{-1}, \alpha, \beta, \text{canonical}).$$

Proposition 3.1. *Keeping the notation above, the map ι is an isomorphism onto the open subset where a, b don't both vanish.*

Proof. We can ignore the sections; the interesting part is to keep track of the map on bundles, which looks like

$$(\mathrm{Pic}^*_{X'} \times \mathrm{Pic}^{*+d}_{X'})/\mathrm{Pic}_X \to (\mathrm{Pic}^d_{X'} \times_{\mathrm{Pic}_X} \mathrm{Pic}^d_{X'})$$

sending

$$(\mathcal{L}, \mathcal{L}') \mapsto (\mathcal{L}' \otimes \mathcal{L}^{-1}, \mathcal{L}' \otimes \sigma^*\mathcal{L}^{-1}). \tag{3.1}$$

We'll show that this is an isomorphism by describing the inverse.

By "looping" the sequences

$$1 \to T \to \mathrm{Res}_{F'/F}\mathbf{G}_m \xrightarrow{\mathrm{Nm}} \mathbf{G}_m \to 1$$

and

$$1 \to \mathbf{G}_m \to \mathrm{Res}_{F'/F}\mathbf{G}_m \to T \to 1$$

we obtain exact sequences of groups stacks

$$1 \to \mathrm{Bun}_T \to \mathrm{Pic}_{X'} \xrightarrow{\mathcal{L}\otimes\sigma^*\mathcal{L}} \mathrm{Pic}_X \to 1 \tag{3.2}$$

and

$$1 \to \mathrm{Pic}_X \to \mathrm{Pic}_{X'} \xrightarrow{\mathcal{L}\otimes\sigma^*\mathcal{L}^{-1}} \mathrm{Bun}_T \to 1. \tag{3.3}$$

Suppose we have a point $(\mathcal{M}, \mathcal{M}', c\colon\ \mathrm{Nm}(\mathcal{M}) \cong \mathrm{Nm}(\mathcal{M}'))$ on the right hand side of (3.1). Then (3.2) tells us that since $\mathcal{M}$ and $\mathcal{M}'$ have the same norm, they differ by a T-bundle. By (3.3), there exist $\mathcal{L}, \mathcal{L}'$ such that $\mathcal{M} = \mathcal{L}' \otimes \mathcal{L}^{-1}$ and $\mathcal{M}' = \mathcal{L}' \otimes \sigma^*\mathcal{L}^{-1}$, and this choice is unique up to multiplication by an element of Pic_X.

□

Proposition 3.2. *If* char$k \neq 2$ *then* $\overline{\mathcal{M}}_d$ *is a Deligne-Mumford stack.*

Proof. $\overline{\mathcal{M}}_d$ is covered by the open stacks $X'_d \times_{\mathrm{Pic}_X} \widehat{X'}_d$ and $\widehat{X'}_d \times_{\mathrm{Pic}_X} X'_d$, describing when the sections α and β don't vanish, respectively. By symmetry, it suffices to show that one of these is Deligne-Mumford. Consider the cartesian diagram

$$\begin{array}{ccc} X'_d \times_{\mathrm{Pic}_X} \widehat{X'}_d & \longrightarrow & \widehat{X'}_d \\ & & \downarrow \pi \\ \downarrow & & \mathrm{Pic}_{X'} \\ & & \downarrow \mathrm{Nm} \\ X'_d & \longrightarrow \mathrm{Pic}_{X'} \xrightarrow{\mathrm{Nm}} & \mathrm{Pic}_X \end{array}$$

The map π is representable, since the fiber over $\mathcal{L}$ is $H^0(X', \mathcal{L})$.

The map Nm is a torsor under $\ker(\mathrm{Pic}_{X'} \xrightarrow{\mathrm{Nm}} \mathrm{Pic}_X)$, which is the Prym variety $\mathrm{Prym}(X'/X)/\mu_2$, since μ_2 is precisely the group of automorphisms of the norm map on line bundles.

This implies that the fibered product is Deligne-Mumford. □

Remark 3.3. Alternatively, we can establish the Deligne-Mumford property by showing that the automorphisms groups are étale, i.e. have vanishing tangent space. We can compute the tangent space to the map

$$\mathrm{Pic}_{X'} \xrightarrow{\mathrm{Nm}} \mathrm{Pic}_X$$

as follows. The map on tangent spaces is

$$T_{\mathrm{Nm}} = H^1(X', \mathcal{O}_{X'}) = H^1(X, \nu_* \mathcal{O}_X) \xrightarrow{\text{trace}} H^1(X, \mathcal{O}_X)$$

and the infinitesimal deformations of this map is the kernel of $H^0(X', \mathcal{O}_{X'}) \xrightarrow{\text{trace}} H^0(X, \mathcal{O}_X)$, which is just multiplication by 2.

Corollary 3.4. $\overline{\mathcal{M}_d}$ *is smooth if* $d > 2g' - 1$.

Proof. The map $\widehat{X'}_d \to \mathrm{Pic}^d_{X'}$ is a vector bundle if $d > 2g' - 1$ by Riemann-Roch, and $\widehat{X'}_d \times_{\mathrm{Pic}_X} X'_d = \overline{\mathcal{M}_d}$. □

Proposition 3.5. *The morphism* $f\colon \overline{\mathcal{M}}_d \to \overline{\mathcal{A}}_d$ *is proper. Therefore its restriction to* $f\colon \mathcal{M}_d \to \mathcal{A}_d$ *is also proper.*

Proof. Recall that f is the map

$$\widehat{X'}_d \times_{\mathrm{Pic}_X} \widehat{X'}_d - (\text{both } 0) \xrightarrow{\mathrm{Nm}} \widehat{X}_d \times_{\mathrm{Pic}_X} \widehat{X}_d - (\text{both } 0)$$

where (both 0) refers to the substack where both global sections vanish. So it suffices to show that the norm map $\widehat{X'}_d \to \widehat{X}_d$ is proper. Note that this is obvious on fibers, since both $X'_d \xrightarrow{\nu_d} X_d$ and $\mathrm{Pic}_{X'} \xrightarrow{\mathrm{Nm}} \mathrm{Pic}_X$ are proper, the first map being even finite and the second map having the Prym variety as its kernel.

To give a formal proof, we compactify. If we define

$$\overline{\widehat{X}_d} = \{\mathcal{L} \in \mathrm{Pic}^d_X, s \in \mathbf{P}H^0(X, \mathcal{L} \oplus \mathcal{O}_X)\}$$

then the natural map $\overline{\widehat{X}_d} \to \mathrm{Pic}_X$ is obviously proper, so $\overline{\widehat{X}_d}$ is proper. We have an open embedding $\widehat{X}_d \hookrightarrow \overline{\widehat{X}_d}$ sending $(\mathcal{L}, s) \mapsto (\mathcal{L}, [s : 1])$. Note that

$$\overline{\widehat{X'}_d} = [(\widehat{X'}_d \times \mathbf{A}^1 - \text{both } 0)/\mathbf{G}_m]$$
$$\overline{\widehat{X}_d} = [(\widehat{X}_d \times \mathbf{A}^1 - \text{both } 0)/\mathbf{G}_m]$$

where $\widehat{X}_d \times \mathbf{A}^1$ parametrizes $(\mathcal{L}, s \in H^0(\mathcal{L}), f \in H^0(\mathcal{O}_X))$, and similarly for X'. The substack (both 0) refers to the locus where $s = f = 0$. Then we have a cartesian diagram

$$\begin{array}{ccc} \widehat{X'}_d & \hookrightarrow & \overline{\widehat{X'}_d} \\ \downarrow & & \downarrow \\ \widehat{X}_d & \longrightarrow & \overline{\widehat{X}_d} \end{array}$$

and $\overline{\widehat{X'}_d} \to \overline{\widehat{X}_d}$ is proper, so $\widehat{X'}_d \to \widehat{X}_d$ is proper. □