## DEFINITION AND PROPERTIES OF $\mathcal{M}_d$

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### 1. Goal

Let X be a (smooth, projective, geometrically connected) curve over a finite field k, and  $\nu: X' \to X$  a degree 2 étale cover.

Let  $T := (\operatorname{Res}_{X'/X} \mathbf{G}_m)/\mathbf{G}_m$ . We can embed T into "PGL<sub>2</sub>" = Aut $(\nu_* \mathcal{O}_{X'})/\mathcal{O}_X^*$ . **Remark 1.1.** We can also view T as the norm-1 subgroup of the Weil restriction:

$$1 \to T \to \operatorname{Res}_{X'/X} \mathbf{G}_m \xrightarrow{\operatorname{Nm}} \mathbf{G}_m \to 1$$

We can put these two definitions together to get an exact sequence

$$1 \to \mathbf{G}_m \to \operatorname{Res}_{X'/X} \mathbf{G}_m \xrightarrow{t \mapsto t(\sigma^* t)^{-1}} \operatorname{Res}_{X'/X} \mathbf{G}_m \xrightarrow{\operatorname{Nm}} \mathbf{G}_m \to 1.$$
(1.1)

The goal is to compute the intersection number

$$\langle Sht_T, h_D * Sht_T \rangle_{Sht_G}$$

where  $G = PGL_2$ .

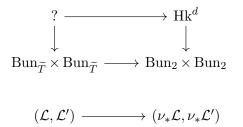
# 2. The moduli space $\mathcal{M}_d$

### 2.1. Relation to intersection numbers. Recall the shtuka space is

$$\begin{array}{ccc} \operatorname{Sht}_T & & \longrightarrow & \operatorname{Hk}^{\mu} \\ & & & \downarrow \\ & & & \downarrow \\ \operatorname{Bun}_T & \xrightarrow{\operatorname{Id},\operatorname{Frob}} & \operatorname{Bun}_T \times \operatorname{Bun}_T \end{array}$$

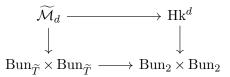
The idea is that the shtuka construction is complicated, so we should try to do the  $\operatorname{Bun}_T$  intersection first. So we should try to compute " $\operatorname{Bun}_T \cap h_D * \operatorname{Bun}_T$ ".

Every time we want to do a PGL<sub>2</sub>-computation we actually push it to a GL<sub>2</sub>computation. So as usual, set  $\tilde{T} = \operatorname{Res}_{X'/X} \mathbf{G}_m$  and  $\tilde{G} = \operatorname{GL}_2$ . Note that  $\operatorname{Bun}_{\tilde{T}}$ can just be thought of as parametrizing line bundles on X, by the definition of Weil restriction. So we want to compute the intersection



In terms of the previous talks,  $d = \deg D$ . Recall that  $\operatorname{Hk}^d$  parametrizes maps of vector bundles  $\mathcal{E} \hookrightarrow \mathcal{E}'$  with quotient a torsion sheaf of degree d.

**Definition 2.1.** We define  $\widetilde{\mathcal{M}}_d$  to be the fibered product



**Remark 2.2.** What we are calling  $\widetilde{\mathcal{M}}_d$  is called  $\widetilde{\mathcal{M}}_d^{\heartsuit}$  in the paper, but we're going to omit it because we'll be working with it most often.

2.2. The functor of points. Let's try to compute the functor of points of  $\widetilde{\mathcal{M}}_d$ . View  $\operatorname{Bun}_{\widetilde{T}}$  as  $\operatorname{Pic}_{X'}$ . The bottom horizontal map sends

$$(\mathcal{L}, \mathcal{L}') \mapsto (\nu_* \mathcal{L}, \nu_* \mathcal{L}').$$

The space  $\widetilde{\mathcal{M}}_d$  (which is analogous to a "Hitchin space") parametrizes

$$\{(\mathcal{L}, \mathcal{L}', \psi \colon \nu_* \mathcal{L} \to \nu_* \mathcal{L}') \mid \deg \operatorname{coker} \psi = d\}.$$

Let's digest this. We need  $\mathcal{L}, \mathcal{L}' \in \operatorname{Pic}_X^* \times \operatorname{Pic}_X^{*+d}$ , and  $\psi$  is equivalent to (by adjunction)

$$\varphi \colon \nu^* \nu_* \mathcal{L} \to \mathcal{L}$$

We have  $\nu^* \nu_* \mathcal{L} = \mathcal{L} \oplus \sigma^* \mathcal{L}$ . So  $\varphi$  is equivalent to

$$\nu^*\nu_*\mathcal{L} = \mathcal{L} \oplus \sigma^*\mathcal{L} \xrightarrow{\alpha,\beta} \mathcal{L}'$$

which amounts to the data of two maps

$$\mathcal{L} \xrightarrow{\alpha} \mathcal{L}'$$
$$\sigma^* \mathcal{L} \xrightarrow{\beta} \mathcal{L}'$$

2.3. Compactification. We now introduce a compactification of  $\mathcal{M}_d$ .

**Definition 2.3.** We define  $\widetilde{\mathcal{M}}_d$  to be the moduli space classifying

- $\mathcal{L}, \mathcal{L}' \in \operatorname{Pic}_{X'}^* \times \operatorname{Pic}_{X'}^{*+d}$ ,
- Maps

$$\alpha \colon \mathcal{L} \to \mathcal{L}'$$
$$\beta \colon \mathcal{L} \to \sigma^* \mathcal{L}'$$

such that  $\alpha, \beta$  are not both 0.

**Remark 2.4.** The bar on  $\widetilde{\mathcal{M}}_d$  is because we haven't imposed an injectivity condition on  $\psi$ . This space is just called  $\widetilde{\mathcal{M}}_d$  in the paper.

There is an action of  $\operatorname{Pic}_X$  on  $\widetilde{\mathcal{M}}_d$ , and we finally define  $\overline{\mathcal{M}}_d := \widetilde{\mathcal{M}}_d / \operatorname{Pic}_X$ .

**Remark 2.5.** Obviously  $\widetilde{\mathcal{M}}_d$  isn't of finite type, since it has infinitely many components. Since  $\nu^* \colon \operatorname{Pic}_X^* \to \operatorname{Pic}_{X'}^{2*}$  hits "half" the components,  $\overline{\mathcal{M}}_d$  is of finite type. In fact it has exactly 2 components.

The map

$$\psi \colon \nu_* \mathcal{L} \to \nu_* \mathcal{L}'$$

when pulled back to X' becomes

$$\nu^*\psi\colon\nu^*\nu_*\mathcal{L}\to\nu^*\nu_*\mathcal{L}'$$

and is given by

$$\nu^*\psi = \begin{pmatrix} \alpha & \sigma^*\beta\\ \beta & \sigma^*\alpha \end{pmatrix}$$

so det  $\nu^* \psi = \operatorname{Nm} \alpha - \operatorname{Nm} \beta$ . We have

$$\widetilde{\mathcal{M}}_d = \overline{\widetilde{\mathcal{M}}_d} \setminus \{\operatorname{Nm} \alpha = \operatorname{Nm} \beta\},\$$

and

$$M_d = [\widetilde{\mathcal{M}}_d / \operatorname{Pic}_X].$$

2.4. The moduli space  $\mathcal{A}_d$ .

**Definition 2.6.** We define the moduli space  $\overline{\mathcal{A}_d}$  parametrizing

- $\Delta \in \operatorname{Pic}_X$ ,
- $a, b \in H^0(X, \Delta)$  where a and b never simultaneously vanish.

Thus

$$\overline{\mathcal{A}}_d = \widehat{X}_d \times_{\operatorname{Pic}_X} \widehat{X}_d - Z(\operatorname{Pic}_X^d)$$

where  $Z(\operatorname{Pic}_X^d) = (\operatorname{Pic}_X \times_{\operatorname{Pic}_X} \widehat{X}_d \cup \widehat{X}_d \times_{\operatorname{Pic}_X} \operatorname{Pic}_X)$ , embedding as the locus where a or b vanish.

**Remark 2.7.** Again we point out that the notation has changed from the paper and previous talks. What is being called  $\overline{\mathcal{A}}_d$  used to be called  $\mathcal{A}_d$ , and what is being called  $\mathcal{A}_d$  is called  $\mathcal{A}_d^{\heartsuit}$  in the paper.

2.5. The map f. There is a map

$$f \colon \overline{\mathcal{M}}_d \to \overline{\mathcal{A}}_d = \widehat{X}_d \times_{\operatorname{Pic}_X} \widehat{X}_d - Z(\operatorname{Pic}_X^d)$$

sending

$$(\mathcal{L}, \mathcal{L}', \alpha, \beta) \mapsto (\operatorname{Nm}(\mathcal{L}') \otimes \operatorname{Nm}(\mathcal{L})^{-1}, a := \operatorname{Nm}(\alpha), b := \operatorname{Nm}(\beta))$$

So  $\overline{\mathcal{M}}_d$  is the pre-image of  $\mathcal{A}_d := \langle (\mathcal{L}, a, b) \colon a = b \rangle$ .

# 3. Properties of $\mathcal{M}_d$

We begin with an important alternate description of  $\overline{\mathcal{M}}_d$ . There is a map

$$\iota \colon \overline{\mathcal{M}}_d \to \widehat{X'}_d \times_{\operatorname{Pic}_X} \widehat{X'}_d.$$

Recall that  $\widehat{X'}_d \times_{\operatorname{Pic}_X} \widehat{X'}_d$  parametrizes

- $\mathcal{L}, \mathcal{L}' \in \operatorname{Pic}_{X'},$
- $\alpha \in H^0(X', \mathcal{L}), \beta \in H^0(X', \mathcal{L}')$  not both 0,
- $c: \operatorname{Nm}(\mathcal{L}) \cong \operatorname{Nm}(\mathcal{L}') \}.$

In these terms,  $\iota$  sends

$$(\mathcal{L}, \mathcal{L}', \alpha, \beta) \mapsto (\mathcal{L}' \otimes \mathcal{L}^{-1}, \mathcal{L}' \otimes \sigma^* \mathcal{L}^{-1}, \alpha, \beta, \text{canonical})$$

**Proposition 3.1.** Keeping the notation above, the map  $\iota$  is an isomorphism onto the open subset where a, b don't both vanish.

*Proof.* We can ignore the sections; the interesting part is to keep track of the map on bundles, which looks like

$$(\operatorname{Pic}_{X'}^* \times \operatorname{Pic}_{X'}^{*+d}) / \operatorname{Pic}_X \to (\operatorname{Pic}_{X'}^d \times_{\operatorname{Pic}_X} \operatorname{Pic}_{X'}^d)$$

sending

$$(\mathcal{L}, \mathcal{L}') \mapsto (\mathcal{L}' \otimes \mathcal{L}^{-1}, \mathcal{L}' \otimes \sigma^* \mathcal{L}^{-1}).$$
(3.1)

We'll show that this is an isomorphism by describing the inverse.

By "looping" the sequences

$$1 \to T \to \operatorname{Res}_{F'/F} \mathbf{G}_m \xrightarrow{\operatorname{Nm}} \mathbf{G}_m \to 1$$

and

$$1 \to \mathbf{G}_m \to \operatorname{Res}_{F'/F} \mathbf{G}_m \to T \to 1$$

we obtain exact sequences of groups stacks

$$1 \to \operatorname{Bun}_T \to \operatorname{Pic}_{X'} \xrightarrow{\mathcal{L} \otimes \sigma^* \mathcal{L}} \operatorname{Pic}_X \to 1$$
(3.2)

and

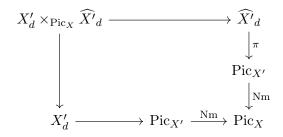
$$1 \to \operatorname{Pic}_X \to \operatorname{Pic}_{X'} \xrightarrow{\mathcal{L} \otimes \sigma^* \mathcal{L}^{-1}} \operatorname{Bun}_T \to 1.$$
(3.3)

Suppose we have a point  $(\mathcal{M}, \mathcal{M}', c: \operatorname{Nm}(\mathcal{M}) \cong \operatorname{Nm}(\mathcal{M}'))$  on the right hand side of (3.1). Then (3.2) tells us that since  $\mathcal{M}$  and  $\mathcal{M}'$  have the same norm, they differ by a T-bundle. By (3.3), there exist  $\mathcal{L}, \mathcal{L}'$  such that  $\mathcal{M} = \mathcal{L}' \otimes \mathcal{L}^{-1}$  and  $\mathcal{M}' = \mathcal{L}' \otimes \sigma^* \mathcal{L}^{-1}$ , and this choice is unique up to multiplication by an element of  $\operatorname{Pic}_X$ .

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**Proposition 3.2.** If chark  $\neq 2$  then  $\overline{\mathcal{M}}_d$  is a Deligne-Mumford stack.

*Proof.*  $\overline{\mathcal{M}}_d$  is covered by the open stacks  $X'_d \times_{\operatorname{Pic}_X} \widehat{X'}_d$  and  $\widehat{X'}_d \times_{\operatorname{Pic}_X} X'_d$ , describing when the sections  $\alpha$  and  $\beta$  don't vanish, respectively. By symmetry, it suffices to show that one of these is Deligne-Mumford. Consider the cartesian diagram



The map  $\pi$  is representable, since the fiber over  $\mathcal{L}$  is  $H^0(X', \mathcal{L})$ .

The map Nm is a torsor under ker( $\operatorname{Pic}_{X'} \xrightarrow{\operatorname{Nm}} \operatorname{Pic}_X$ ), which is the Prym variety  $\operatorname{Prym}(X'/X)/\mu_2$ , since  $\mu_2$  is precisely the group of automorphisms of the norm map on line bundles.

This implies that the fibered product is Deligne-Mumford.

**Remark 3.3.** Alternatively, we can establish the Deligne-Mumford property by showing that the automorphisms groups are étale, i.e. have vanishing tangent space. We can compute the tangent space to the map

$$\operatorname{Pic}_{X'} \xrightarrow{\operatorname{Nm}} \operatorname{Pic}_X$$

as follows. The map on tangent spaces is

$$T_{\mathrm{Nm}} = H^1(X', \mathcal{O}_{X'}) = H^1(X, \nu_*\mathcal{O}_X) \xrightarrow{\mathrm{trace}} H^1(X, \mathcal{O}_X)$$

and the infinitesimal deformations of this map is the kernel of  $H^0(X', \mathcal{O}_{X'}) \xrightarrow{\text{trace}} H^0(X, \mathcal{O}_X)$ , which is just multiplication by 2.

Corollary 3.4.  $\overline{\mathcal{M}_d}$  is smooth if d > 2g' - 1.

*Proof.* The map  $\widehat{X'}_d \to \operatorname{Pic}_{X'}^d$  is a vector bundle if d > 2g' - 1 by Riemann-Roch, and  $\widehat{X'}_d \times_{\operatorname{Pic}_X} X'_d = \overline{\mathcal{M}_d}$ .

**Proposition 3.5.** The morphism  $f: \overline{\mathcal{M}}_d \to \overline{\mathcal{A}}_d$  is proper. Therefore its restriction to  $f: \mathcal{M}_d \to \mathcal{A}_d$  is also proper.

*Proof.* Recall that f is the map

$$\widehat{X'}_d \times_{\operatorname{Pic}_X} \widehat{X'}_d - (\text{both } 0) \xrightarrow{\operatorname{Nm}} \widehat{X}_d \times_{\operatorname{Pic}_X} \widehat{X}_d - (\text{both } 0)$$

where (both 0) refers to the substack where both global sections vanish. So it suffices to show that the norm map  $\widehat{X'}_d \to \widehat{X}_d$  is proper. Note that this is obvious on fibers, since both  $X'_d \xrightarrow{\nu_d} X_d$  and  $\operatorname{Pic}_{X'} \xrightarrow{\operatorname{Nm}} \operatorname{Pic}_X$  are proper, the first map being even finite and the second map having the Prym variety as its kernel.

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To give a formal proof, we compactify. If we define

 $\overline{\widehat{X}_d} = \{ \mathcal{L} \in \operatorname{Pic}_X^d, s \in \mathbf{P}H^0(X, \mathcal{L} \oplus \mathcal{O}_X) \}$ 

then the natural map  $\overline{\hat{X}_d} \to \operatorname{Pic}_X$  is obviously proper, so  $\overline{\hat{X}_d}$  is proper. We have an open embedding  $\widehat{X}_d \hookrightarrow \overline{\hat{X}_d}$  sending  $(\mathcal{L}, s) \mapsto (\mathcal{L}, [s:1])$ . Note that

$$\widehat{\hat{X'}_d} = [(\widehat{X'}_d \times \mathbf{A}^1 - \text{both } 0) / \mathbf{G}_m]$$
$$\overline{\hat{X}_d} = [(\widehat{X}_d \times \mathbf{A}^1 - \text{both } 0) / \mathbf{G}_m]$$

where  $\widehat{X}_d \times \mathbf{A}^1$  parametrizes  $(\mathcal{L}, s \in H^0(\mathcal{L}), f \in H^0(\mathcal{O}_X))$ , and similarly for X'. The substack (both 0) refers to the locus where s = f = 0. Then we have a cartesian diagram

$$\widehat{X'}_d \longleftrightarrow \overline{\widehat{X'}_d} \\ \downarrow \qquad \qquad \downarrow \\ \widehat{X}_d \longleftrightarrow \overline{\widehat{X}_d}$$

and  $\overline{\widehat{X'}_d} \to \overline{\widehat{X}_d}$  is proper, so  $\widehat{X'}_d \to \widehat{X}_d$  is proper.