Beauville-Laszlo Uniformization

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1 Statement of the results

1.1 Setup

Let

- G be a split reductive group over $k = \overline{k}$ an algebraically closed field of characteristic p.
- X be a smooth projective geometrically connected curve over k.
- $x \in |X|$ and $X^0 = X - x$.
- S be a scheme over k.
- $\mathcal{F}$ be a G-bundle over $X \times S$.

1.2 Statement of Theorems

Theorem 1.1. *There is a surjective étale map $S' \to S$ such that the G-bundle $\mathcal{F} \times_S S' \to X \times_S S'$ has a B-structure.*

Definition 1.2. Let $\mathcal{F} \to Y$ be a G-bundle and $B \subset G$ a fixed Borel subgroup. By a *B-structure* of $\mathcal{F}$ we mean a pair $(\mathcal{E}, \eta)$ such that E is a B-bundle and

$$\begin{array}{ccc} \eta\colon G \times^B \mathcal{E} & \xrightarrow{\sim} & \mathcal{F} \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

Remark 1.3. There is a natural bijection

$$(B\text{-structures of } \mathcal{F} \to Y) \leftrightarrow (\text{sections } s\colon Y \to B\backslash\mathcal{F}).$$

Here $B\backslash\mathcal{F} = G\backslash B \times^G \mathcal{F}$.

Theorem 1.4. *If G is semisimple, then there exists a faithfully flat morphism $S' \to S$ of finite presentation such that $\mathcal{F} \times_S S'|_{X^0 \times_S S'}$ is trivial. In general, if $p \nmid \#\pi_1(G)$ then $S' \to S$ can be chosen to be étale (if $p = 0$, then there are no restrictions).*

Remark 1.5. The statements and proofs generalize immediately to a relative curve $X \xrightarrow{\pi} S$, e.g. in Theorem 2 and $D \subset X$ a divisor such that $\pi|_D \colon D \cong S$ then $\mathcal{F}|_{X-D}$ is trivial after base change.

1.3 The affine Grassmannian

Recall that $\mathrm{Gr}_G = LG/L^+G$ is an ind-scheme classifying

$$\left\{ (\mathcal{F}, i) \colon \begin{array}{c} \mathcal{F} = G\text{-bundle}/D = \mathrm{Spec}\ k[[t]] \\ i = \text{trivialization of } \mathcal{F}|_{D^\times} \end{array} \right\}$$

Definition 1.6. Define $\mathrm{Gr}_{G,x}$ to be the moduli space defined by

$$\mathrm{Gr}_{G,x}(S) = \left\{ (\mathcal{F}, i) \colon \begin{array}{c} \mathcal{F} = G\text{-bundle}/X \times S \\ i \colon \mathcal{F}|_{X^0 \times S} \xrightarrow{\sim} \mathcal{F}^0|_{X^0 \times S} \end{array} \right\}$$

where F^0 is the trivial bundle.

It is easy to see by a "gluing Lemma" that

$$\mathrm{Gr}_{G,x} \cong \mathrm{Gr}_G .$$

This isomorphism is almost canonical (up to a choice of uniformizer at x).

Why is this relevant? There is a natural map

$$\pi \colon \mathrm{Gr}_{G,x} \to \mathrm{Bun}_n(X)$$

sending $(\mathcal{F}, i) \mapsto \mathcal{F}$.

Theorem 1.7. *Theorem 1.4 says that the map π is surjective in the the faithfully flat topology.*

This statement of the theorem will be generalized to the Fargues-Fontaine curve.

2 Proof of Theorem 1.1

2.1 A simple case

Suppose $S = k = \overline{\mathbb{F}}_p$. For the function field of a curve, Steinberg's Theorem in characteristic 0 or Springer's Theorem in characteristic p tells us that $H^1(k(X), G) = 0$. From this it follows that any G-bundle over the generic point $\eta = \mathrm{Spec}\ k(X)$ is trivial. Of course the trivial bundle has a B-structure, which by Remark 1.3 is equivalent to a section of $B\backslash\mathcal{F}|_\eta$ at the generic point. Such a section spreads out to some open subset $U \subset X$. By the valuative criterion for properness applied to $B\backslash\mathcal{F} \to X$, the section extends (uniquely) to all of X.

2.2 Moduli space of B-structures

Remark 2.1. We can replace G by G/Z^0, and so assume that G is semisimple.

The idea is to consider the moduli space of all B-structures. We want to show that this has a section after an étale cover; for this it suffices to show that the map from the moduli space to S is smooth and surjective.

Definition 2.2. (1) Let $T \subset B$ be the maximal torus and $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ the set of simple roots. For all i and all B-bundles $\mathcal{E} \to X$ we can form a line bundle $\alpha_i(\mathcal{E}) \to X$ via $\alpha_i \colon B \to T \to \mathbb{G}_m$, and we define

$$\deg_i(\mathcal{E}) := \deg \alpha_i(\mathcal{E}).$$

(2) Let $M_{\mathcal{F}}$ be the moduli space of B-structures of F, so

$$M_{\mathcal{F}}(T) = \{B\text{-structures of } \mathcal{F} \times_S T\}.$$

By the way, there are no automorphisms because a B-structure is a section, and sections have no automorphisms.

A section can be identified with a subscheme of the product. By the theory of Hilbert schemes, $M_{\mathcal{F}} \to S$ is a scheme locally of finite presentation (we do not say "locally of finite type" because S may not be Noetherian).

We said that we would like the map $M_{\mathcal{F}} \to S$ to be smooth and surjective. Actually it is surjective but not smooth. To rectify this, we look at a certain subspace of it.

(3) For every geometric point $y \in M_{\mathcal{F}}$ (corresponding to a B-bundle $\mathcal{E}_y \to X$) we can consider $d_i(y) := \deg \alpha_i(\mathcal{E}_y) \in \mathbb{Z}$. Then $d_i \colon M_{\mathcal{F}} \to \mathbb{Z}$ is locally constant. Define $M_{\mathcal{F}}^+ \subset M_{\mathcal{F}}$ to be the set of $y \in M_{\mathcal{F}}$ such that $d_i(Y) < \min\{1, 2-2g\}$ for all i. This is a union of connected components.

Then Theorem 1.1 follows from the two propositions.

Proposition 2.3. *The map $M_{\mathcal{F}}^+ \to S$ is smooth.*

Proposition 2.4. *The map $M_{\mathcal{F}}^+ \to S$ is surjective.*

2.3 Proof of Proposition 2.3

The first proposition is standard deformation theory. Indeed, a geometric point $y \in M_{\mathcal{F}}^+$ corresponds to a section $\sigma \colon X \to B\backslash\mathcal{F}$. A deformation of this B-structure is controlled by $H^0(X, \sigma^* T_{(B\backslash\mathcal{F})/X})$. One checks that $\sigma^* T_{(B\backslash\mathcal{F})/X} = (\operatorname{Lie} G / \operatorname{Lie} B) \times^B \mathcal{E}_y$ where $\mathcal{E}_y$ is the B-bundle corresponding to y. By deformation theory it is enough that the obstruction space

$$H^1(X, (\operatorname{Lie} G / \operatorname{Lie} B) \times^B \mathcal{E}_y) = 0 \text{ for all geometric points } y \in M_F^+.$$

The reason is that

$$\operatorname{Lie} G/\operatorname{Lie} B = \bigoplus_{\alpha<0} \mathfrak{g}_\alpha.$$

In particular, it is enough to show that $H^1(X, \mathfrak{g}_\alpha) = 0$ for all $\alpha < 0$. But we assumed that $\deg \mathfrak{g}_{\alpha_i} < 2 - 2g$ for each simple root α_i, so each simple negative weight space has degree at least $2g - 2$. By Riemann-Roch, $H^1(X, \mathfrak{g}_\alpha) = 0$.

Remark 2.5. There was some confusion about why we need the assumption $d_i(Y) < 1$. The answer is that otherwise if $g = 0$ then we could have $d_i(Y) = 1$. For $\alpha = \alpha_i + \alpha_j$ we would then have $\deg \alpha = 2$, so $H^1(X, \mathfrak{g}_{-\alpha})$ would have non-vanishing cohomology.

2.4 Proof of Proposition 2.4

We can check Proposition 2 at the level of geometric points. It follows from a more precise result:

Proposition 2.6. *Let $\mathcal{F} \to X$ be a G-bundle. Then for all N there exists a B-structure $\mathcal{E}$ of $\mathcal{F}$ such that* $\deg_i \mathcal{E} < -N$ *for all i.*

Example 2.7. Let $G = \mathrm{SL}_2$ and $\mathcal{F} \to X$ a rank 2 bundle. The proposition is saying that there is a line sub-bundle of degree as small as desired; this is an easy consequence of Riemann-Roch.

Proof. We proceed with several reductions.

Step 1: we may assume that $\mathcal{F}$ is the trivial bundle $\mathcal{F}_0$. The reason is that we know that $\mathcal{F}|_\eta$ is trivial (by Steinberg's or Springer's theorems), so there is an isomorphism

$$\mathcal{F}|_{X-D} \cong \mathcal{F}^0|_{X-D}$$

for some divisor $D \subset X$. But then every B-structure of $\mathcal{F}$ gives one for $\mathcal{F}^0$, by the valuative criterion. If the isomorphism $v|_{X-D} \cong \mathcal{F}^0|_{X-D}$ has "relative position h" then there exists $c(h)$ such that for every B-structure $\mathcal{E}$ of $\mathcal{F}$ the corresponding B-structure $\mathcal{E}^0$ of $\mathcal{F}^0$ satisfies

$$-c(h) < \deg_i \mathcal{E} - \deg_i(\mathcal{E}^0) < c(h).$$

Step 2: we may assume that $X = \mathbb{P}^1$ and G is simply connected. Indeed, take any map $X \to \mathbb{P}^1$. The pullback of a B-structure for the trivial bundle on $\mathbb{P}^1$ will be a B-structure for the trivial bundle on X.

Step 3: Let $\mathrm{Bun}_B^{<-N}$ be the space of B-bundles $\mathcal{E}$ with $\deg_i \mathcal{E} < -N$ for all i. Then we claim that $\mathrm{Bun}_B^{-N} \neq \emptyset$ for all N. Indeed, a T-bundle induces a B-bundle, and a T-bundle is just a direct sum of line bundles, which we can arrange to have any degree we want.

Step 4. We claim that the map $\mathrm{Bun}_B^{-N} \to \mathrm{Bun}_G$ is smooth by Proposition 2.3 if $N \gg 0$. Indeed the fibers of this map over a G-bundle are the B-structures on it, and Proposition 2.3

shows that this is smooth for $N > 2g - 2$.

Step 5. Since $X = \mathbb{P}^1$, $\mathrm{Bun}_G^{\mathrm{triv}} \subset \mathrm{Bun}_G$ is an open substack. To see this it suffices to calculate that the map $B(G = \mathrm{Aut}(\mathrm{triv})) \to \mathrm{Bun}_G$ is étale, which can be done using tangent spaces since both are smooth. The dimension of BG is $-\dim G$. To calculate the dimension of Bun_G, we use that its tangent complex is $\mathrm{Lie}(G)[1]$, so the dimension of the tangent space is

$$h^0(X, \mathrm{Lie}(G)) - h^1(X, \mathrm{Lie}(G)).$$

This is a bundle of rank $\dim G$ and degree 0 (since it's self-dual by the Killing form). Then Riemann-Roch shows that

$$\chi(X, \mathrm{Lie}(G)) = 0 + (\dim G)(1 - g).$$

The dimension of Bun_G is $(g-1)\dim G$ in general.

Step 6. Finally, it is a general fact that if G is simply-connected then $\mathrm{Bun}_G(X)$ is irreducible. The trivial bundle is open in Bun_G, and the map $\mathrm{Bun}_B^{-N} \to G$ has open image because it is smooth, so its image intersects the trivial bundle. □

3 Proof of Theorem 1.4

Step 1. We may assume that F comes from a T-torsor $\mathcal{E}_T$.

Proof. By Proposition 2.3, we may assume that F has a B-structure $\mathcal{E} \to X \times S$. We have

$$B \twoheadrightarrow T \hookrightarrow B.$$

This gives a map

$$\mathrm{Bun}_B \to \mathrm{Bun}_T \to \mathrm{Bun}_B.$$

In particular from $\mathcal{E} \in \mathrm{Bun}_B$ we get $\mathcal{E}' \in \mathrm{Bun}_B$.

We may assume that S is affine since we are proving a local assertion. We want to show that

$$G \times^B \mathcal{E}|_{X^0 \times S} \cong G \times^B \mathcal{E}'|_{X^0 \times S}.$$

You'll see the idea if we just do the proof for GL_2. In that case a GL_2-bundle is a rank 2 bundle $\mathcal{F}/X$. A B-structure $\mathcal{E}$ corresponds to a line sub-bundle $\mathcal{F}_0 \hookrightarrow \mathcal{F}$. In terms of the notation above, the G-bundle obtained from $\mathcal{E}'$ is $\mathcal{F}_0 \oplus \mathcal{F}/\mathcal{F}_0$. The claim then boils down to the assertion that

$$\mathcal{F} \cong \mathcal{F}_0 \oplus \mathcal{F}/\mathcal{F}_0.$$

The result then follows from $X^0 \times S$ is affine, so all the extension groups $\mathrm{Ext}^1(\dots)$ vanish. □

Step 2: Reduce to G being simply-connected.

Step 3. Reduce to the GL_2 case. The point is that if G is simply-connected then all T-bundles are controlled by coroots. One can then reduce to showing that two T-bundles differing by a single coroot are isomorphic locally on S, which moves us into the rank 2 case.

Step 4. Doing the case of GL_2. This isn't semisimple, so one has to find an appropriate formulation. The statement becomes:

Let $\mathcal{F}, \mathcal{F}' \to S$ *be two rank* 2 *bundles such that* $\det \mathcal{F} \cong \det \mathcal{F}'$. *Then we have*

$$\mathcal{F}|_{X^0 \times S} \cong \mathcal{F}^0|_{X^0 \times S}$$

after Zariski localization on S.

The proof is that after localizing on S we have a filtration

$$0 \to \mathcal{O} \to \mathcal{F}|_{X^0 \times S} \to \det \mathcal{F}|_{X^0 \times S} \to 0$$

since any bundle has "enough" sections after localizing on S and puncturing X. Then the result follows form the fact that extension groups will vanish after localizing (e.g. so that S is affine).