Data assimilation with stochastic reduced models

Fei Lu
Johns Hopkins University

Joint with

Xuemin Tu
Department Mathematics, University of Kansas

Alexandre J. Chorin
UC Berkeley, LBNL

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Data Assimilation

A dynamical system + Noisy partial observations

$$x_{n+1} = F(x_n, y_n),$$
$$y_{n+1} = G(x_n, y_n).$$

$$z_n = H(x_n) + V_n$$

- Estimate the states $x_{1:n}, y_{1:n}$ from $z_{1:n}$
- Predict the future evolution

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Bayesian approach: approximate the posteriors

- Sequential Monte Carlo: particle filters
- Ensemble Kalman filters
  - ✦ Run a forward model from an ensemble of IC's
    —> approximate Gaussian prior (mean + Cov)
  - ✦ Update the ensemble similar to Kalman Filter

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Computational cost: many simulations of the full system

- Costly (or even unknown): e.g. weather forecasting

DA with reduced models

Reduced order models (ROM): (suppose that x is of main interest)

$$x_{n+1} = F(x_n, y_n),$$
$$y_{n+1} = G(x_n, y_n).$$

$$\Rightarrow \quad x_{n+1} \approx f_0(x_n)$$

(by grid / truncation / PCA etc. **y** viewed as "sub-grid scales".)

Model error not negligible: $U(x_n, y_n) = F(x_n, y_n) - f_0(x_n)$

- Chaotic geoscience models
- Missing statistical-dynamical properties

Accounting for the model error:

$$x_{n+1} = F(x_n, y_n),$$
$$y_{n+1} = G(x_n, y_n).$$

$$\Rightarrow \quad x_{n+1} = f_0(x_n) + U(x_n, y_n)$$

Stochastic reduced model accounting for the model error

$$x_{n+1} = F(x_n, y_n), \quad y_{n+1} = G(x_n, y_n). \quad \longrightarrow \quad x_{n+1} = f_0(x_n) + U(x_n, y_n)$$

Stochastic parametrization:

- Approximate the model error U
- Memory/non-local effects (due to model reduction)

——> Represent U as a function of paths of x and noise

——> A non-Markovian reduced model

Stochastic reduced model accounting for the model error

$$x_{n+1} = F(x_n, y_n),$$
$$y_{n+1} = G(x_n, y_n).$$
$\rightarrow$ $x_{n+1} = f_0(x_n) + U(x_n, y_n)$

Stochastic parametrization:

NARMA (Nonlinear AutoRegression Moving Average)

$$\mathbf{x}_n = \mathbf{f}_0(\mathbf{x}_{n-1}) + \Phi(\mathbf{x}_{n-p:n-1}, \xi_{n-q:n-1}) + \xi_n,$$

$$\Phi_n := \Phi(\mathbf{x}_{n-p:n-1}, \xi_{n-q:n-1})$$
$$= \sum_{j=1}^{p} a_j \mathbf{x}_{n-j} + \sum_{i=0}^{r} \sum_{j=1}^{p} b_{i,j} \mathbf{f}_i(\mathbf{x}_{n-j}) + \sum_{j=1}^{q} c_j \xi_{n-j},$$

- ξ_n is i.i.d. Gaussian $N(0,\sigma^2)$
- Derive the terms $\{\mathbf{f}_i, i = 1, \ldots, r\}$
  - from physical laws, numerical schemes
- Estimate parameters $\{a_j, b_{i,j}, c_j, \sigma_\xi^2\}$
  - by conditional maximum likelihood methods

Other efforts to accounting for the model error in DA

$$x_{n+1} = F(x_n, y_n), \quad y_{n+1} = G(x_n, y_n). \quad \longrightarrow \quad x_{n+1} = f_0(x_n) + U(x_n, y_n)$$

Covariance inflation and localization (IL)

Idea: Quantify the effect of U to the prior in EnKF

- use the rough reduced model $x_{n+1} \approx f_0(x_n)$
- correct the ensemble covariance C_n^f (of the prior)

Additive / Multiplicative inflation:

$$\widehat{C}_n^f = C_n^f + \lambda I \qquad \widehat{C}_n^f = (1+\lambda) C_n^f$$

Localization:

$$\widehat{C}_n^f = C_n^f \circ C_r^{loc}$$

- Tune the inflation / localization parameters

Accounting for the model error in DA

$$x_{n+1} = F(x_n, y_n),$$
$$y_{n+1} = G(x_n, y_n).$$
$$\Rightarrow \quad x_{n+1} = f_0(x_n) + U(x_n, y_n)$$

Generate training data using the full model

1. Stochastic parametrization: (**Improve the forecast model)**
   - Use NARMA as forecast mode in EnKF
   - The NARMA model from training data

2. Covariance Inflation/Localization (IL): **(improve the prior cov.)**
   - Use the reduced model with IL in EnKF
   - Tune the IL parameters using training data

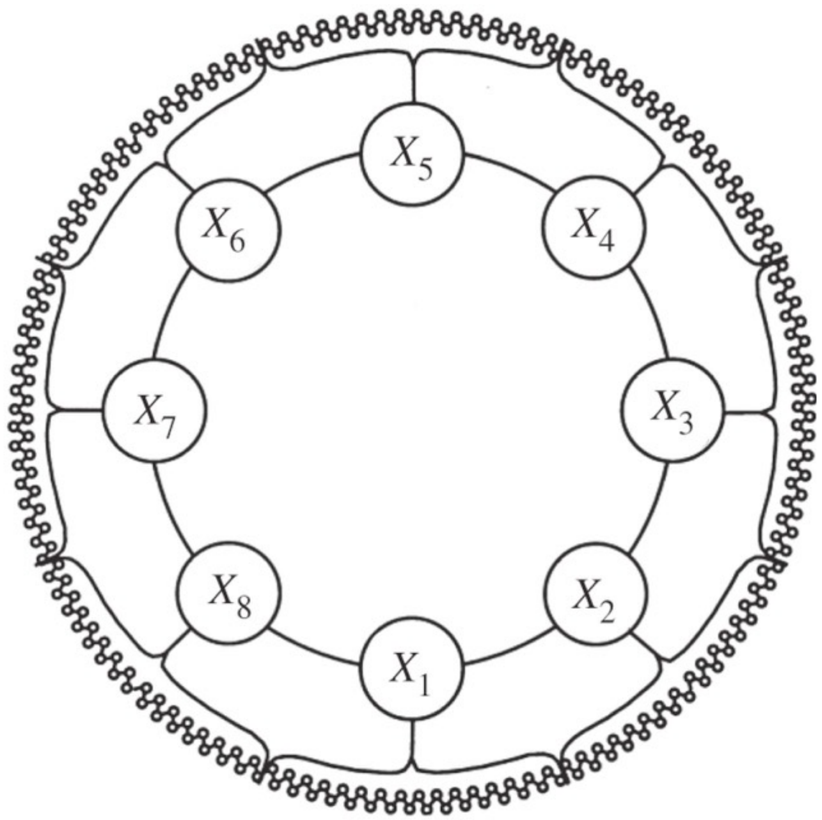
Numerical experiments:

The Lorenz 96 system

$$\frac{d}{dt}x_k = x_{k-1}\left(x_{k+1} - x_{k-2}\right) - x_k + 10 - \frac{1}{J}\sum_j y_{k,j},$$

$$\frac{d}{dt}y_{k,j} = \frac{1}{\varepsilon}[y_{k,j+1}(y_{k,j-1} - y_{k,j+2}) - y_{k,j} + x_k],$$

$$x \in \mathbb{R}^{18},\ y \in \mathbb{R}^{360}$$



Wilks 05

- Suppose x is of main interest
- Noisy observations of x
- To predict future evolution of x

A model without the “sub-grid” y:

$$\frac{d}{dt}x_k = x_{k-1}\left(x_{k+1} - x_{k-2}\right) - x_k + 10$$

$\Rightarrow$ **L96x** $x_{k,n} = x_{k,n-1} + f_k^h(x_{\cdot,n-1})$

Numerical experiments:

L96x $x_{k,n} = x_{k,n-1} + f_k^h(x_{\cdot,n-1})$

NARMA $$x_{k,n} = \sum_{j=1}^{2}(a_j x_{k,n-j} + b_j f_k^h(x_{\cdot,n-j})) + c_0 + c_1 x_{k,n-1}^2 + c_2 x_{k,n-1}^3 + \xi_{k,n},$$

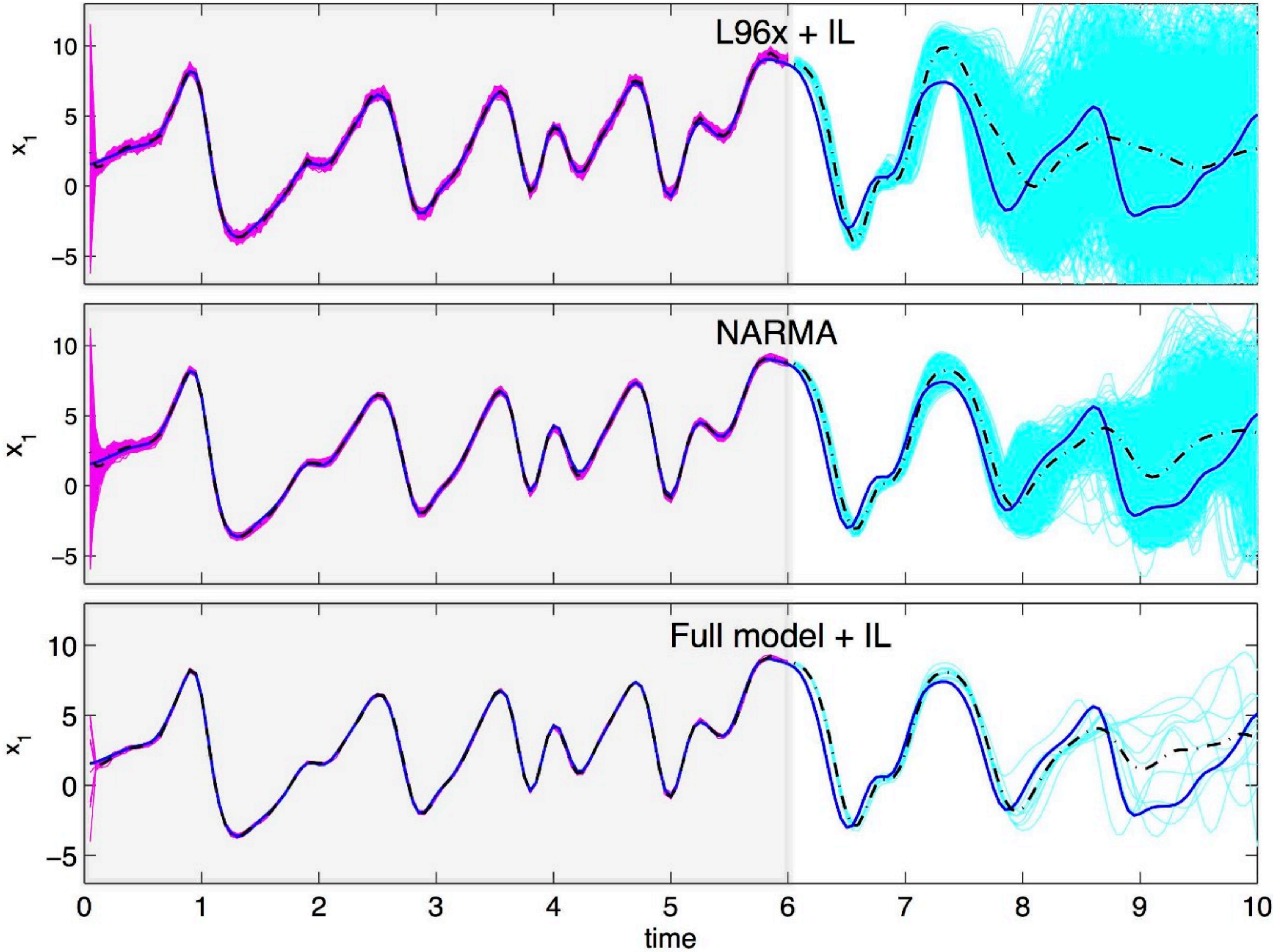
A typical EnKF assimilation:

Ensemble size

L96x/NARMA: 1000

Full model: 10

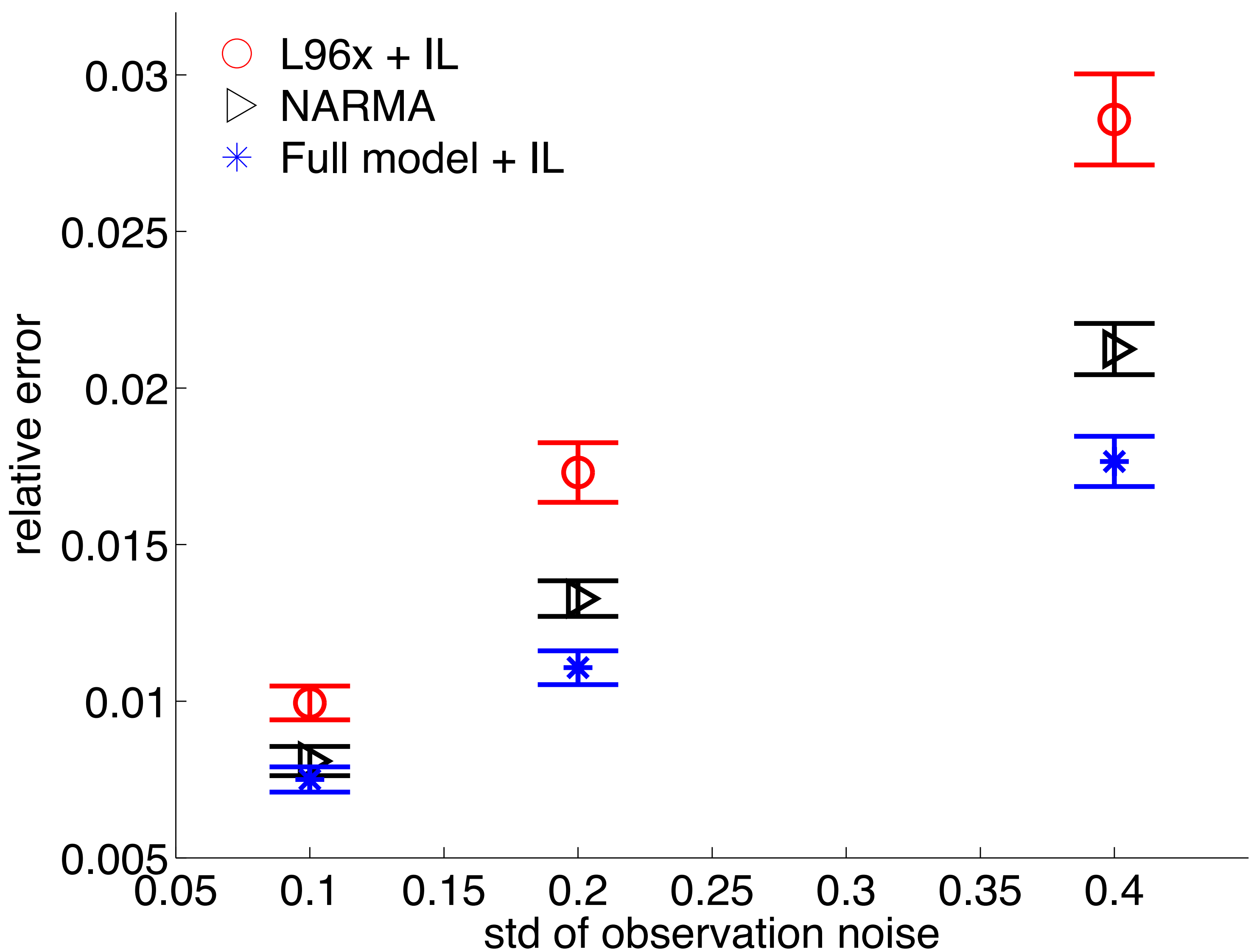
State estimation Forecast

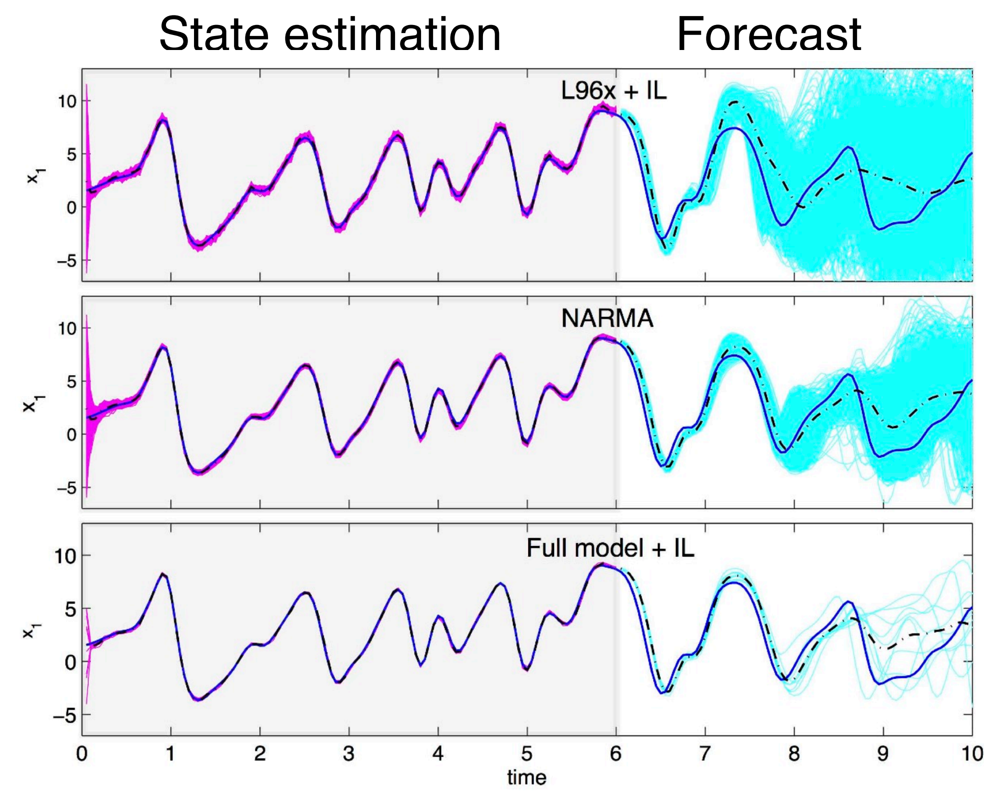


Numerical experiments:

Many simulations:

Mean relative error of state estimation



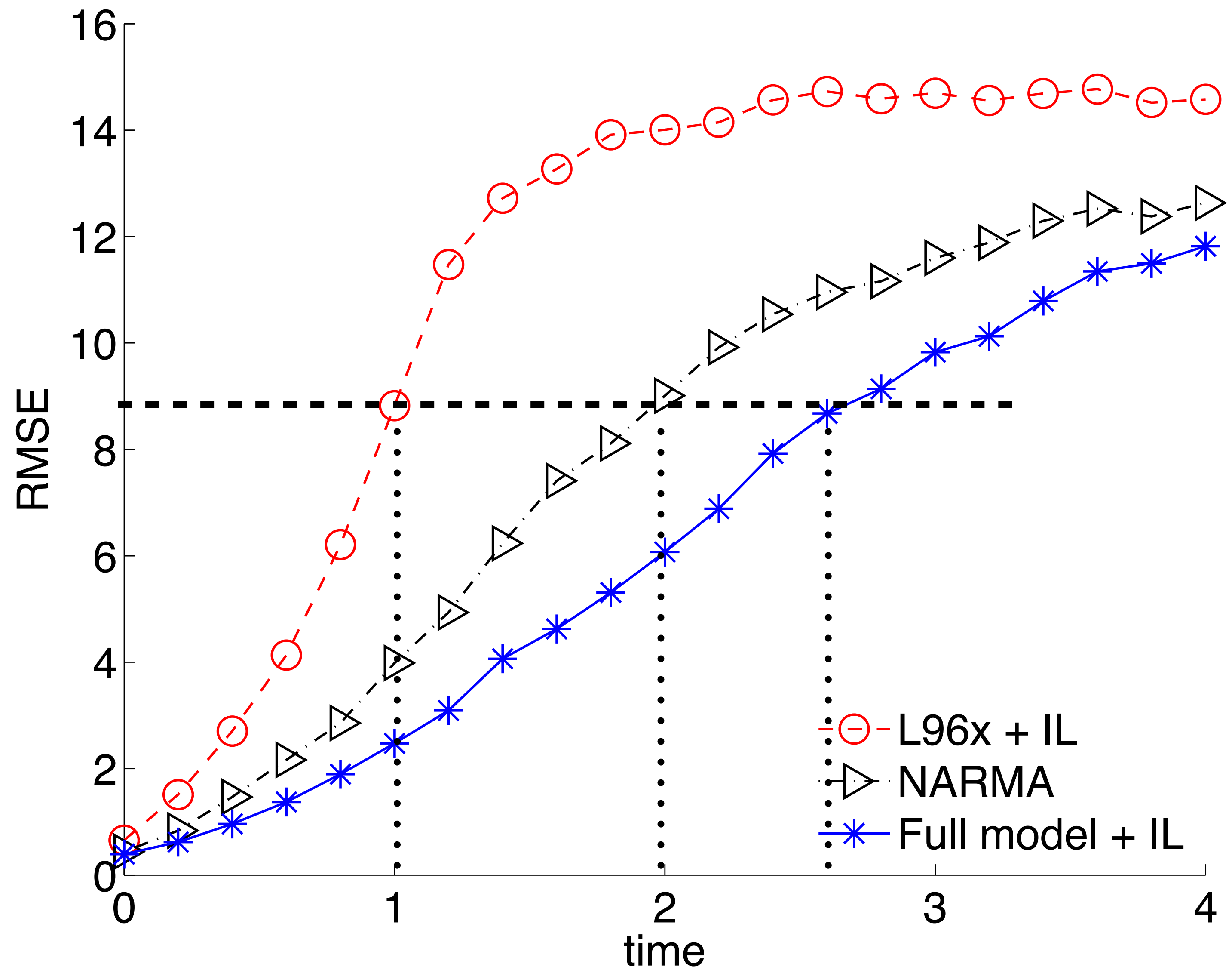


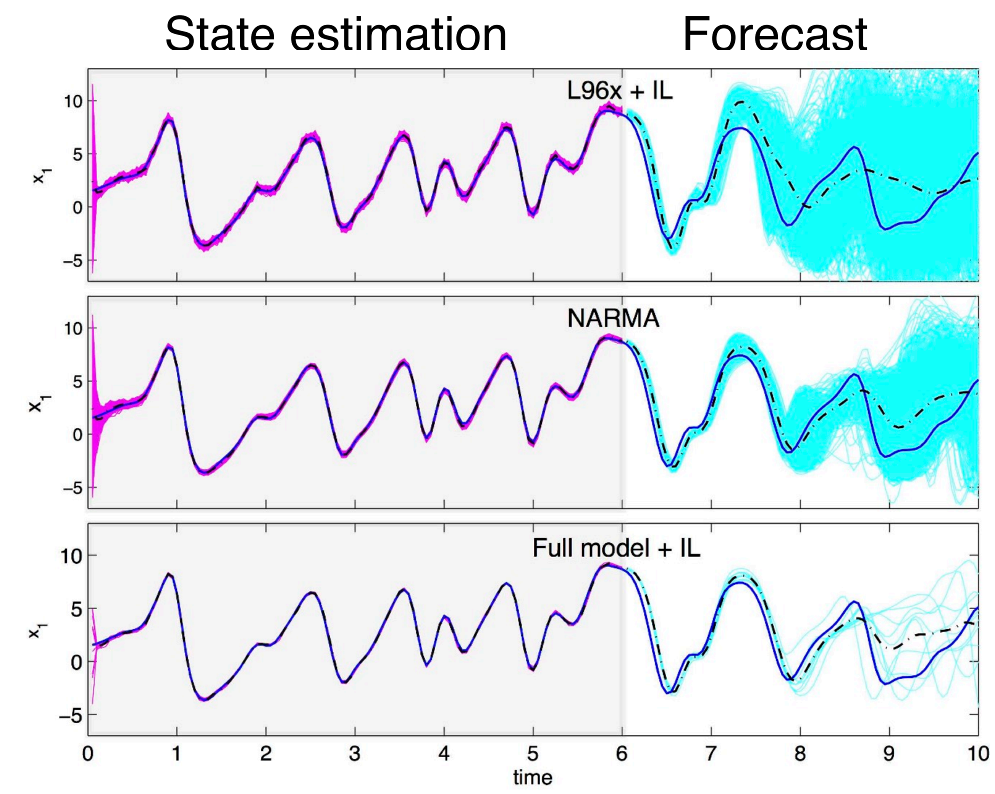
>>> NARMA yields better state estimation

Numerical experiments:

Many simulations:

Root mean square error of prediction





Forecast time

L96x+IL: ≈ 1.0

NARMA: ≈ 2.0

Full model: ≈ 2.6

>>> NARMA yields better forecast ensembles

Summary:

DA with reduced model: accounting for model errors

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$$\Rightarrow \quad x_{n+1} = f_0(x_n) + U(x_n, y_n)$$

DA with stochastic model reduction:

Quantify the model error due to subgrid scales

—> **improve the forecast model**

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DA with stochastic model reduction:

Quantify the model error due to subgrid scales

—> improve the forecast model

Open 1: Estimate/represent of U

- parametric /nonparametric regression and beyond
  - A. when simulated data are available
  - B. when only noisy observations of x

Open 2: DA with non-Markovian models

- particle filter/EnKF for non-Markovian SSM

References

- Chorin-Lu. Discrete approach to stochastic parametrization and dimension reduction in nonlinear dynamics. PNAS, 2015.
- Lu-Tu-Chorin. Accounting for model error from unresolved scales in ensemble Kalman filters. Mon. Wea. Rev. 2017.

Thank you!