Data Assimilation with Stochastic Model Reduction of Chaotic Systems

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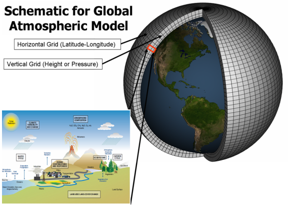
Snowbird, SIAM-DS, May 23, 2019

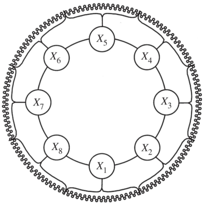
Model error from sub-grid scales



ECMWF: 16 km horizontal

grid $\rightarrow 10^9$ freedoms

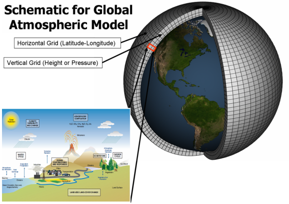
From: Wikipedia, ECWMF



The Lorenz 96 system

Wilks 2005

Model error from sub-grid scales



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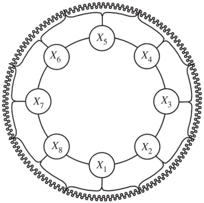


The Lorenz 96 system

Wilks 2005

High-dimensional Full system + **Discrete partial data** $\dashrightarrow$ **Prediction**

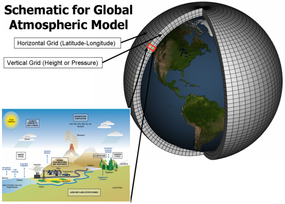
$$x' = f(x) + U(x, y),$$
$$y' = g(x, y).$$

Observe only $\{x(nh)\}_{n=1}^{N}$.

Forecast $x(t), t \geq Nh$.

- HighD multiscale full systems:
  - can only afford to resolve $x' = f(x)$
  - y: unresolved variables (subgrid-scales)
- Discrete noisy observations: missing i.c.
- Ensemble prediction: need many simulations

Model error from sub-grid scales



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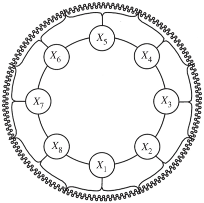

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$\rightarrow$ How to account for the model error $U(x, y)$?

Given a highD multiscale full system:

$$x' = f(x) + \mathbf{U}(\mathbf{x}, \mathbf{y}), y' = g(x, y).$$

Ensemble prediction: can afford to resolve $x' = f(x)$ online.

Accounting model error $U(x, y)$ from subgrid scales

- Indirect approaches: correct the forecast ensemble
  - e.g. Inflation and Localization;[1] relaxation/bias correction[2],
  - in Assimilation step; deficiency in forecast model remains
- Direct approach: improve the forecast model
  - parametrization methods [3], non-Markovian [4],
  - random perturbation, averaging+homogenization [5]

[1] Mitchell-Houtekamer(00), Hamill-Whitaker(05), Anderson(07)

[2] Zhang-etc (04), Dee-Da Silva(98)

[3] Palmer, Arnold+(01,13), Wilks(05), Meng-Zhang(07), Danforth-Kalnay-Li+(08,09), Berry-Harlim(14), Mitchell-Carrissi(15), Van Leeuwen etc(18)

[4] Chorin+(00-15), Marjda-Timofeyev-Harlim+(03-13), Chekroun-Kondrashov-Gil+(11,15), Cromellin+Vanden-Eijinden(08), Gottwald+(15)

[5] Hamill-Whitaker(05),Houtekamer+(09) Pavliotis-Stuart(08), Gottwald+(12-13)

Outline

1. Stochastic model reduction
   (reduction from simulated data)

   - Discrete-time stochastic parametrization (NARMA)

2. Data assimilation with the reduced model
   (Noisy data + reduced model $\rightarrow$ state estimation and prediction)

Stochastic model reduction

$x' = f(x) + U(x,y)$, $y' = g(x,y)$.
Data $\{x(nh)\}_{n=1}^{N}$

Why stochastic reduced models?

- The system is "ergodic": $\frac{1}{N}\sum_{n=1}^{N} F(x(nh)) \xrightarrow{N\to\infty} \int F(x)\mu(dx)$
- $U(x,y)$ acts like a stochastic force

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Memory effects (Mori, Zwanzig, Chorin, Kubo, Majda, Wilks, Ghil, . . .)

- Mori-Zwanzig formalism $\to$ generalized Langevin equ.

$$\frac{dx}{dt} = \underbrace{\mathbb{E}[RHS|x]}_{\text{Markov term}} + \underbrace{\int_0^t K(x(s), t-s)ds}_{\text{memory}} + \underbrace{W_t}_{\text{noise}},$$

- Fluctuation-dissipation theory $\to$ Hypoelliptic SDEs

$$dX = a(X, Y)dt + Y; \quad dY = b(X, Y)dt + c(X, Y)dW,$$

- Parametrization: multi-layer stochastic models

Goal: develop a non-Markovian stochastic reduced system for x

Discrete-time stochastic parametrization

NARMA(p, q)

$$
\begin{aligned}
X_n &= X_{n-1} + R_h(X_{n-1}) + Z_n,\\
Z_n &= \Phi_n + \xi_n,\\
\Phi_n &= \underbrace{\sum_{j=1}^{p} a_j X_{n-j} + \sum_{j=1}^{r}\sum_{i=1}^{s} b_{i,j} P_i(X_{n-j})}_{\text{Auto-Regression}} + \underbrace{\sum_{j=1}^{q} c_j \xi_{n-j}}_{\text{Moving Average}}
\end{aligned}
$$

- $R_h(X_{n-1})$ from a numerical scheme for $x' \approx f(x)$
- Φ_n depends on the past

Tasks:

Structure derivation: terms and orders (p, r, s, q) in Φ_n;

Parameter estimation: $a_j, b_{i,j}, c_j$, and σ.

Overview:

$x' = f(x) + U(x,y), \; y' = g(x,y).$
Data $\{x(nh)\}_{n=1}^{N}$

Discrete-time stochastic parametrization

NARMA

$$\begin{aligned} X_n &= X_{n-1} + R_h(X_{n-1}) + Z_n, \\ Z_n &= \Phi_n + \xi_n, \\ \Phi_n &= \sum_{j=1}^{p} a_j X_{n-j} + \sum_{j=1}^{q} c_j \xi_{n-j} \\ &\quad + \sum_{j=1}^{r} \sum_{i=1}^{s} b_{i,j} P_i(X_{n-j}). \end{aligned}$$

1. compute $R_h(x)$
2. derive structure
3. estimate parameters

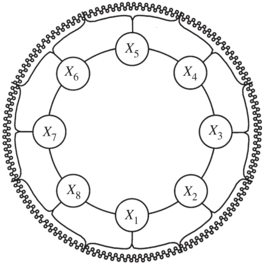
Application to the chaotic Lorenz 96 system

A chaotic dynamical system (a simplified atmospheric model)

$$\frac{d}{dt}x_k = x_{k-1}\left(x_{k+1} - x_{k-2}\right) - x_k + 10 - \frac{1}{J}\sum_j y_{k,j},$$

$$\frac{d}{dt}y_{k,j} = \frac{1}{\varepsilon}[y_{k,j+1}(y_{k,j-1} - y_{k,j+2}) - y_{k,j} + x_k],$$

where $x \in \mathbb{R}^{18}$, $y \in \mathbb{R}^{360}$.



Wilks 2005

Find a reduced system for $x \in \mathbb{R}^{18}$ based on

- Data $\{x(nh)\}_{n=1}^{N}$
- $\frac{d}{dt}x_k \approx x_{k-1}\left(x_{k+1} - x_{k-2}\right) - x_k + 10$.

- NARMA:

$$\begin{aligned} x^n &= x^{n-1} + R_h(x^{n-1}) + z^n;\ z^n = \Phi^n + \xi^n, \\ \Phi^n &= a + \sum_{j=1}^{p} \sum_{l=1}^{d_x} b_{j,l} (x^{n-j})^l + \sum_{j=1}^{p} c_j R_h(x^{n-j}) + \sum_{j=1}^{q} d_j \xi^{n-j}. \end{aligned}$$

$$p = 2, d_x = 3;\ q = \begin{cases} 1, & h = 0.01; \\ 0, & h = 0.05. \end{cases}$$

[6] Wilks 05: an MLR model in atmosphere science

- NARMA:

$$
\begin{aligned}
x^n &= x^{n-1} + R_h(x^{n-1}) + z^n;\; z^n = \Phi^n + \xi^n, \\
\Phi^n &= a + \sum_{j=1}^{p} \sum_{l=1}^{d_x} b_{j,l} (x^{n-j})^l + \sum_{j=1}^{p} c_j R_h(x^{n-j}) + \sum_{j=1}^{q} d_j \xi^{n-j}.
\end{aligned}
$$

$$
p = 2, d_x = 3;\; q = \begin{cases} 1, & h = 0.01; \\ 0, & h = 0.05. \end{cases}
$$

- Polynomial autoregression (POLYAR)[6]

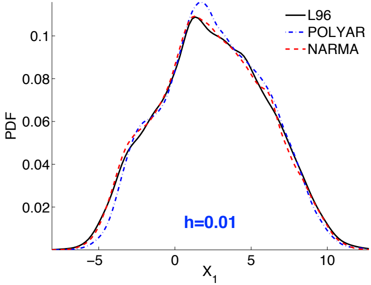
$$
\begin{aligned}
\frac{d}{dt} x_k &= x_{k-1}\,(x_{k+1} - x_{k-2}) - x_k + 10 + \boxed{U}, \\
U &= P(x_k) + \eta_k,\; with\ d\eta_k(t) = \phi \eta_k(t) + dB_k(t).
\end{aligned}
$$

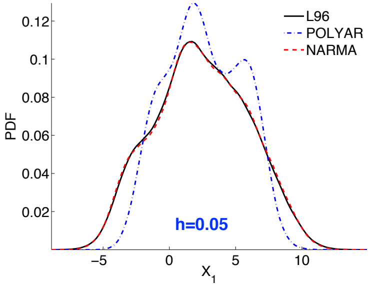
where $P(x) = \sum_{j=0}^{d_x} a_j x^j$. Optimal $d_x = 5$.

[6] Wilks 05: an MLR model in atmosphere science

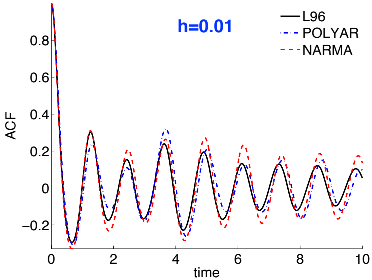
Long-term statistics

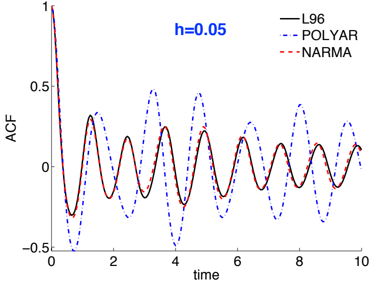
Empirical probability density function (PDF)





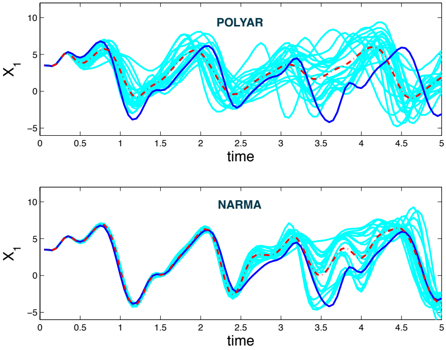
Empirical autocorrelation function (ACF)





Prediction ($h = 0.05$)

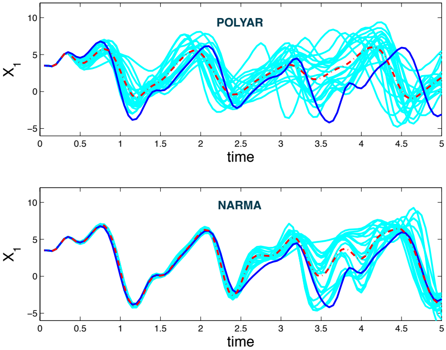
A typical ensemble forecast:



- forecast trajectories in cyan
- true trajectory in blue

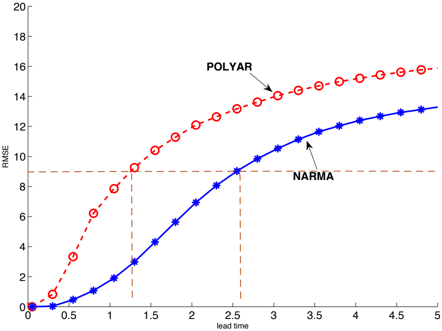
Prediction ($h = 0.05$)

A typical ensemble forecast:



- forecast trajectories in cyan
- true trajectory in blue

RMSE of many forecasts:



Forecast time:

POLYAR: $T \approx 1.25$

NARMA: $T \approx 2.5$

(Full model: $T \approx 2.5$

"Best" forecast time achieved!)

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Data assimilation with the reduced model

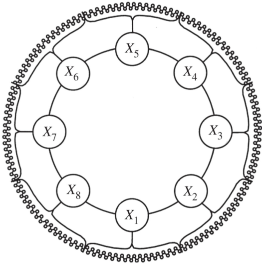
$x' = f(x) + U(x, y)$, $y' = g(x, y)$.
Noisy data: $x(nh) + W(n), \ n = 1, 2, \ldots$

Data assimilation:

- estimate the state of a forward model
- make prediction (by ensembles of solutions)

The widely used method: Ensemble Kalman filters (EnKF)

The Lorenz 96 system

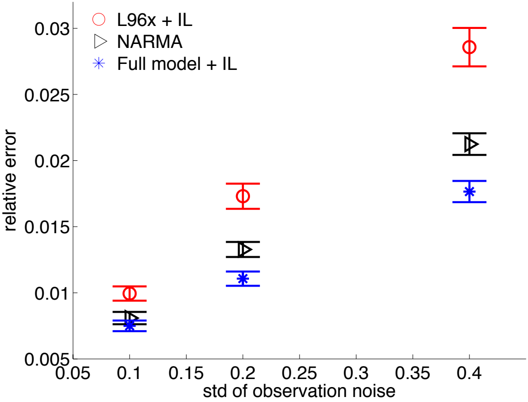


Wilks 2005

Estimate and predict x based on

- Noisy Data $z(n) = x(nh) + \mathbf{W(n)}$
- **Forward models**
  - **L96x**: the truncated model
    $\frac{d}{dt}x_k \approx x_{k-1}\left(x_{k+1} - x_{k-2}\right) - x_k + 10$
    (account for the model error by IL in EnKF)
  - **NARMA** (account for the model error by parametrization in the forward model)

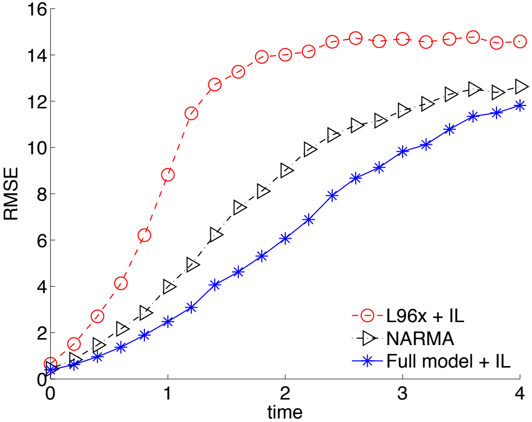
Relative error of state estimation



Relative error for different observation noises.

(ensemble size: =1000 for L96x and NARMA; =10 for the full model)

RMSE of state prediction



RMSE of 10^4 ensemble forecasts.

(ensemble size: =1000 for L96x and NARMA; =10 for the full model)

Summary: The stochastic model improves performance of DA.

Summary and ongoing work

Accounting model error $U(x, y)$ from subgrid scales

$x' = f(x) + U(x,y)$, $y' = g(x,y)$.
Data $\{x(nh)\}_{n=1}^{N}$

Inference

“$X' = f(X) + Z(t,\omega)$”

Discretization

Inference

“$X_{n+1} = X_n + R_h(X_n) + Z_n$”
for prediction

- Stochastic model reduction by
  Discrete-time stochastic parametrization
  - simplifies the inference from data
  - incorporates memory flexibly
  - effective reduced model (**NARMA**)
    - capture key statistical-dynamical features
    - make medium-range forecasting

- **Improve the forecast model**
  $\rightarrow$ Improve performance of DA

Ongoing work:

- noisy data: state estimation and model inference
  - data assimilation with non-Markovian models
  - inference for hidden non-Markovian models
- model reduction for (stochastic) PDEs
  - stochastic Burgers equation, N-S equation

References

- Data-driven stochastic model reduction
  - Chorin-Lu: Discrete approach to stochastic parametrization and dimension reduction in nonlinear dynamics. **PNAS 112 (2015)**, no. 32, 9804–9809.
  - Lu-Lin-Chorin: Comparison of continuous and discrete-time data-based modeling for hypoelliptic systems. **CAMCoS, 11 (2016)**, no. 8, 4227–4246. (With K. Lin and A. Chorin).
  - Lu-Lin-Chorin: Data-based stochastic model reduction for the Kuramoto – Sivashinsky equation. **Physica D, 340 (2017)**, 46–57. (With K. Lin and A. Chorin)
- Data assimilation
  - Lu-Tu-Chorin: Accounting for model error from unresolved scales in EnKFs: improving the forecast model. **MWR, 340 (2017)**.

Thank you!