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Non-degeneracy of some Sobolev Pseudo-norms of fractional Brownian motion

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Abstract

Applying an upper bound estimate for L^2 small ball probability for fractional Brownian motion (fBm), we prove the non-degeneracy of some Sobolev pseudo-norms of fBm.

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1 Introduction

Let $B^H = \left\{B_t^H : t \in [0,1]\right\}$ be a fractional Brownian motion (fBm) on $(\Omega, \mathcal{F}, P)$. That is, $\left\{B_t^H : t \geq 0\right\}$ is a centered Gaussian process with covariance

$$R_H(t,s) = E(B_t^H B_s^H) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \tag{1.1}$$

where $H \in (0,1)$ is the Hurst parameter. Consider the random variable F given by a functional of B^H:

$$F = \int_0^1 \int_0^1 \frac{\left|B_t^H - B_{t'}^H\right|^{2p}}{|t-t'|^q} dt dt', \tag{1.2}$$

where $p, q \geq 0$ satisfy $(2p-2)H > q-1$.

In the case of $H = \frac{1}{2}$, B^H is a Brownian motion, and the random variable F is the Sobolev norm on the Wiener space considered by Airault and Malliavin in [1]. This norm plays a central role in the construction of surface measures on the Wiener space. Fang [4] showed that F is non-degenerate in the sense of Malliavin calculus (see the definition below). Then it follows from the well-known criteria on regularity of densities that the law of F has a smooth density.

The purpose of this note is to extend this result to the case $H \neq \frac{1}{2}$ and to show that F is non-degenerate.

In order to state our result precisely, we need some notations from Malliavin calculus (for which we refer to Nualart [9, Section 1.2]). Denote by $\mathcal{E}$ the set of all step functions

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on $[0,1]$. Let $\mathfrak{H}$ be the Hilbert space defined as closure of $\mathcal{E}$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathfrak{H}} = R_H(t,s), \text{ for } s,t \in [0,1].$$

Then the mapping $\mathbf{1}_{[0,t]} \mapsto B_t^H$ extends to a linear isometry between $\mathfrak{H}$ and the Gaussian space spanned by B^H. We denote this isometry by B^H. Then, for any $h, g \in \mathfrak{H}$, $B^H(f)$ and $B^H(g)$ are two centered Gaussian random variables with $E[B^H(h)B^H(g)] = \langle h, g \rangle_{\mathfrak{H}}$. We define the space $\mathbb{D}^{1,2}$ as the closure of the set of smooth and cylindrical random variable of the form

$$G = f(B^H(h_1), \dots, B^H(h_n))$$

with $h_i \in \mathfrak{H}$, $f \in C_p^\infty(\mathbb{R}^n)$ (f and all its partial derivatives has polynomial growth) under the norm

$$\|G\|_{1,2} = \sqrt{E[G^2] + E[\|DG\|_{\mathfrak{H}}^2]},$$

where the DF is the Malliavin derivative of F defined as

$$DG = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B^H(h_1), \dots, B^H(h_n))h_i.$$

We say that a random vector $\mathbf{V} = (V_1, \dots, V_d)$ whose components are in $\mathbb{D}^{1,2}$ is *non-degenerate* if its Malliavin matrix $\gamma_{\mathbf{V}} = \left(\langle DV_i, DV_j \rangle_{\mathfrak{H}}\right)$ is invertible a.s. and $(\det \gamma_{\mathbf{V}})^{-1} \in L^p(\Omega)$, for all $p \geq 1$ (see for instance [9, Definition 2.1.1]). Our main result is the following theorem.

Theorem 1.1. *For all $H \in (0,1)$, the functional F of a fBm B^H given in (1.2) is non-degenerate. That is,*

$$\|DF\|_{\mathfrak{H}}^{-1} \in L^k(\Omega), \text{ for all } k \geq 1. \tag{1.3}$$

We shall follow the same scheme introduced in [4] to prove Theorem 1.1. That is, it suffices to prove that for any integer n, there exists a constant C_n such that

$$P(\|DF\|_{\mathfrak{H}} \leq \varepsilon) \leq C_n \varepsilon^n \tag{1.4}$$

for all ε small. This kind of inequality is called upper bound estimate in small deviation theory (also called small ball probability theory, for which we refer to [6] and the reference therein). To prove (1.4), we will need an upper bound estimate of the small deviation for the path variance of the fBm, which is introduced in the following section.

We comment that Li and Shao [5, Theorem 4] proved that

$$P\left(\int_0^1 \int_0^1 \frac{\left|B_t^H - B_s^H\right|^{2p}}{|t-s|^q} dtds \leq \varepsilon\right) \leq \exp\{-\frac{C}{\varepsilon^\beta}\} \tag{1.5}$$

for $p > 0$, $0 \leq q < 1 + 2pH$, $q \neq 1$ and $\beta = 1/(pH - \max\{0, q-1\}$. But (1.5) gives the small ball probability of F, not of $\|DF\|_{\mathfrak{H}}$.

2 An estimate on the path variance of fBm

In this section we show the following useful lemma.

Lemma 2.1 (Estimate of the path variance of the fBm)**.** *Let $B^H = \left\{B_t^H : t \in [0,1]\right\}$ be a fBm. For $0 \leq a < b \leq 1$, consider the path variance $V_{[a,b]}(B^H)$ defined by*

$$V_{[a,b]}(B^H) = \int_a^b \left|B_t^H\right|^2 \frac{dt}{b-a} - (\int_a^b B_t^H \frac{dt}{b-a})^2.$$

Then for $c_H = H\left((2H+1)\sin\frac{\pi}{2H+1}\right)^{-\frac{2H+1}{2H}}\left(\Gamma(2H+1)\sin(\pi H)\right)^{\frac{1}{2H}}$*,*

$$\lim_{\varepsilon\to 0}\varepsilon^{\frac{1}{H}}\log P(V_{[a,b]}(B^H)\le\varepsilon^2) = -(b-a)c_H. \tag{2.1}$$

Actually, we will only need

$$\limsup_{\varepsilon\to 0}\varepsilon^{\frac{1}{H}}\log P(V_{[a,b]}(B^H)\le\varepsilon^2) < \infty. \tag{2.2}$$

In the case of $H=\frac{1}{2}$, this estimate of the path variance for Brownian motion was introduced by Malliavin [7, Lemma 3.3.2], using the following Payley–Wiener expansion of Brownian motion:

$$B_t = tG + \sqrt{2}\sum_{k=1}^{\infty}\frac{1}{2\pi k}(X_k\cos 2\pi kt + Y_k\sin 2\pi kt),\ \text{a.s. for all } t\in[0,1], \tag{2.3}$$

where G, X_k, Y_k, $k\in\mathbb{N}$, are i.i.d. standard Gaussian random variables. Then the estimate (2.2) follows by observing that $V_{[0,1]}(B)=\frac{1}{2\pi^2}\sum_{k=1}^{\infty}\frac{1}{2\pi k}(X_k^2+Y_k^2)$, a sum of $\chi^2(1)$ random variables. The above expansion of Brownian motion can be obtained by integrating an expansion of white noise on the orthonormal basis $\left\{1,\sqrt{2}\cos 2\pi kt,\sqrt{2}\sin 2\pi kt\right\}$ of $L^2[0,1]$. Payley–Wiener expansion of fBm has been established recently by Dzhaparidze and van Zanten [3]:

$$B_t^H = tX + \sum_{k=1}^{\infty}\frac{1}{\omega_k}\left[X_k(\cos 2\omega_k t - 1) + Y_k\sin 2\omega_k t\right], \tag{2.4}$$

where $0<\omega_1<\omega_2<\dots$ are the real zeros of J_{-H} (the Bessel function of the first kind of order $-H$), and X, X_k, Y_k, $k\in\mathbb{N}$, are independent centered Gaussian random variables with variance

$$EX^2=\sigma_H^2, EX_k^2=EY_k^2=\sigma_k^2,$$

with $\sigma_H^2=\frac{\Gamma(\frac{3}{2}-H)}{2H\Gamma(H+\frac{1}{2})\Gamma(3-2H)}$ and $\sigma_k^2=\sigma_H^2(2-2H)\Gamma^2(1-H)\left(\frac{\omega_k}{2}\right)^{2H}J_{-H}(\omega_k)$. Because the path variance $V_{[0,1]}(B^H)$ is difficult to evaluate in the case $H\neq\frac{1}{2}$, the techniques of [7, Lemma 3.3.2] to prove (2.2) no longer work.

Fortunately, recent developments in small deviation theory allow us to derive a simple proof of (2.1).

Proof of Lemma 2.1. In [8, Theorem 3.1 and Remark 3.1] Nazarov and Nikitin proved that for any square integrable random variable G and any nonnegative function $\psi\in L^1[0,1]$,

$$\lim_{\varepsilon\to 0}\varepsilon^{\frac{1}{H}}\log P(\int_0^1(B_t^H-G)^2\psi(t)dt\le\varepsilon^2) = -c_H\left(\int_0^1\psi(t)^{\frac{1}{2H+1}}dt\right)^{\frac{2H+1}{2H}}. \tag{2.5}$$

Notice that by the self-similarity property of fBm,

$$V_{[a,b]}(B^H)=\int_a^b\left(B_t^H-\overline{B^H}\right)^2\frac{dt}{b-a} = b\int_{a/b}^1\left(B_{bu}^H-\overline{B^H}\right)^2\frac{du}{b-a}$$

has the same distribution as $b^{2H+1}\int_{a/b}^1\left(B_u^H-b^{-H}\overline{B^H}\right)^2\frac{du}{b-a}$. Then, Lemma 2.1 follows from (2.5) by taking $G=b^{-H}\overline{B^H}$ and $\psi(t)=\mathbf{1}_{[a/b,1]}(t)$. $\square$

We comment that Bronski [2] proved (2.5) for the case $G=0$ and $\psi\equiv 1$ by estimating the asymptotics of the Karhunen–Loeve eigenvalues of fBm. Actually, the assumption $G=0$ is not necessary, because a random variable G here doesn't contribute to the asymptotics of the Karhunen–Loeve eigenvalues.

3 Proof of the main theorem

In this section we prove (1.3) by estimating $P(\|DF\|_{\mathfrak{H}} \leq \varepsilon)$ for ε small.

For simplicity, we denote

$$\begin{aligned} I &= \left\{(t,t') \in [0,1]^2, t' \leq t\right\}, \\ \vec{t} &= (t,t'),\ d\vec{t} = dtdt'. \end{aligned}$$

Lemma 3.1. *Let $Q(\vec{t},\vec{s}) = \langle \mathbf{1}_{[t',t]}, \mathbf{1}_{[s',s]}\rangle_{\mathfrak{H}}$. Then the operator Q on $L^2(I)$ defined by*

$$Qf(\vec{t}) = \int_I Q(\vec{t},\vec{s})f(\vec{s})d\vec{s},\ f \in L^2(I)$$

is symmetric positive and compact.

Proof. Compactness follows from $Q(\vec{t},\vec{s}) \in L^2(I \times I)$. The function $Q(\vec{t},\vec{s})$ is symmetric, so is the operator Q. Finally, Q is positive because for any $f \in L^2(I)$,

$$\langle Qf, f\rangle_{L^2(I)} = \int_I\int_I Q(\vec{t},\vec{s})f(\vec{s})d\vec{s}f(\vec{t})d\vec{t} = \left\|\int_I \mathbf{1}_{[t',t]}f(\vec{t})d\vec{t}, \right\|_{\mathfrak{H}}^2.$$

□

Then, it follows that Q has a sequence of decreasing eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$, i.e. $\lambda_1 \geq \cdots \geq \lambda_n > 0$, and $\lambda_n \to 0$. The corresponding normalized eigen-functions $\{\varphi_n\}_{n\in\mathbb{N}}$ form an orthonormal basis of $L^2(I)$. Each of them is continuous because $\phi_n(\vec{t}) = \lambda_n^{-1}\int_I Q(\vec{t},\vec{s})\phi_n(\vec{s})d\vec{s}$ and $Q(\vec{t},\vec{s})$ is continuous. We can write

$$Q(\vec{t},\vec{s}) = \sum_{n\geq 1}\lambda_n\varphi_n(\vec{t})\varphi_n(\vec{s}). \tag{3.1}$$

From the definition of Malliavin derivative we have

$$D_rF = 4p\int_I \frac{\left|B_t^H - B_{t'}^H\right|^{2p-1}}{|t-t'|^q}\mathrm{sign}(B_t^H - B_{t'}^H)\mathbf{1}_{[t',t]}(r)dtdt'.$$

Then

$$\begin{aligned} \|DF\|_{\mathfrak{H}}^2 &= 16p^2\left\|\int_I \mathbf{1}_{[t',t]}(\cdot)\frac{\left|B_t^H - B_{t'}^H\right|^{2p-1}}{|t-t'|^q}\mathrm{sign}(B_t^H - B_{t'}^H)dtdt'\right\|_{\mathfrak{H}}^2 \qquad (3.2)\\ &= 16p^2\int_{I\times I}\langle \mathbf{1}_{[t',t]}, \mathbf{1}_{[s',s]}\rangle_{\mathfrak{H}}\frac{\left|B_t^H - B_{t'}^H\right|^{2p-1}}{|t-t'|^q}\mathrm{sign}(B_t^H - B_{t'}^H) \\ &\quad \times\frac{\left|B_s^H - B_{s'}^H\right|^{2p-1}}{|s-s'|^q}\mathrm{sign}(B_s^H - B_{s'}^H)d\vec{t}d\vec{s}. \end{aligned}$$

Using (3.1) to evaluate the inner product in (3.2) yields

$$\|DF\|_{\mathfrak{H}}^2 = 16p^2\sum_{i\geq 1}\lambda_i V_i^2, \tag{3.3}$$

where we denote

$$V_i = \int_I \varphi_i(t,t')\frac{\left|B_t^H - B_{t'}^H\right|^{2p-1}}{|t-t'|^q}\mathrm{sign}(B_t^H - B_{t'}^H)dtdt'. \tag{3.4}$$

For each $\beta = (\beta_1,\ldots,\beta_n) \in S^{n-1}$ (the unit sphere in $\mathbb{R}^n$), let $\Psi_\beta(\vec{t}) = \sum_{i=1}^n \beta_i\varphi_i(\vec{t})$. We denote

$$G_\beta = \int_I \Psi_\beta^2(\vec{t})\frac{\left|B_t^H - B_{t'}^H\right|^{2p-2}}{|t-t'|^q}d\vec{t}. \tag{3.5}$$

Lemma 3.2. *There exists a constant $C_{p,H} > 0$ such that for all $\beta \in S^{n-1}$ and $\varepsilon > 0$,*

$$P\left(G_\beta \le \varepsilon\right) \le \exp\left\{-C_{p,H}\varepsilon^{-\frac{1}{2H(p-1)}}\right\}. \tag{3.6}$$

Proof. Fix an arbitrary $\beta \in S^{n-1}$. Then $\Psi_\beta \not\equiv 0$ since $\varphi_i, \dots, \varphi_n$ are linearly independent. Since Ψ_β is continuous on I, there exists $\vec{t}_\beta = (t'_\beta, t_\beta) \in I$, δ_β and ρ_β such that for all $\vec{t} \in A_\beta := [t'_\beta - \delta_\beta, t'_\beta + \delta_\beta] \times [t_\beta - \delta_\beta, t_\beta + \delta_\beta] \subset I$,

$$\Psi^2_\beta(\vec{t}) \ge \rho_\beta > 0.$$

Let $C = 2\max_{i\in\{1,\dots,n\}} \sup_{\vec{t}\in I} \left|\varphi(\vec{t})\right| < \infty$. Then for any $\beta' \in S^{n-1}$,

$$\left|\Psi^2_\beta(\vec{t}) - \Psi^2_{\beta'}(\vec{t})\right| \le C\left\|\beta - \beta'\right\|.$$

Then for any $\beta' \in S^{n-1}$ satisfying $\|\beta' - \beta\| \le \rho_\beta/2C$, one has

$$\Psi^2_{\beta'}(\vec{t}) \ge \Psi^2_\beta(\vec{t}) - \left|\Psi^2_\beta(\vec{t}) - \Psi^2_{\beta'}(\vec{t})\right| \ge \rho_\beta/2, \tag{3.7}$$

for any $\vec{t} \in A_\beta$.

Note that S^{n-1} has a finite cover $S^{n-1} \subset \cup_{i=1}^m B(\beta^i, \frac{\rho_{\beta^i}}{2C})$. Denote $\rho_i = \rho_{\beta^i}$, $\delta_i = \delta_{\beta^i}$, $\vec{t}_i = \vec{t}_{\beta^i}$ and $A_i = A_{\beta^i}$. Then it follows from (3.7) that for any $\beta \in S^{n-1}$, there exists a $\beta^i \in S^{n-1}$ such that

$$\Psi^2_\beta(\vec{t}) \ge \rho_i/2, \text{ for all } \vec{t} \in A_i.$$

Then noticing that $|t - t'| \le 1$ and applying Jensen's inequality we obtain

$$\begin{aligned} G_\beta &\ge \frac{\rho_i}{2}\int_{A_i} \frac{\left|B^H_t - B^H_{t'}\right|^{2p-2}}{|t-t'|^q} d\vec{t} \ge \frac{\rho_i}{2}\int_{A_i}\left|B^H_t - B^H_{t'}\right|^{2p-2} d\vec{t} \\ &\ge \frac{\rho_i}{2(2\delta_i)^{p-2}}\left(\int_{A_i}\left(B^H_t - B^H_{t'}\right)^2 d\vec{t}\right)^{p-1}. \end{aligned} \tag{3.8}$$

Note that for $f \in C[a,b]$ with average $\overline{f} = \frac{1}{b-a}\int_a^b f(\xi)d\xi$, we have

$$\frac{1}{b-a}\int_a^b (f(\xi) - \overline{f})^2 d\xi \le \frac{1}{b-a}\int_a^b (f(\xi) - c)^2 d\xi$$

for any number c. Then

$$\int_{A_i}\left(B^H_t - B^H_{t'}\right)^2 d\vec{t} = \int_{t_i-\delta_i}^{t_i+\delta_i}\int_{t'_i-\delta_i}^{t'_i+\delta_i}\left(B^H_t - B^H_{t'}\right)^2 dtdt' \ge 2\delta_i\int_{t_i-\delta_i}^{t_i+\delta_i}\left(B^H_t - \overline{B^H}\right)^2 dt \tag{3.9}$$

where $\overline{B^H} = \int_{t_i-\delta_i}^{t_i+\delta_i} B^H_t dt$. Combining (3.8) and (3.9) and applying Lemma 2.1 we obtain

$$\begin{aligned} P\left(G_\beta \le \varepsilon\right) &\le P(\int_{t_i-\delta_i}^{t_i+\delta_i}\left(B^H_t - \overline{B^H}\right)^2 dt \le (\rho_i\delta_i)^{-\frac{1}{p-1}}\varepsilon^{\frac{1}{p-1}}) \\ &\le \exp\{-c_H\delta_i\left(\rho_i\delta_i\right)^{\frac{1}{2H(p-1)}}\varepsilon^{-\frac{1}{2H(p-1)}}\}. \end{aligned}$$

Then one obtains (3.6) by choosing $C_{p,H} = c_H \min_{1\le i\le m}\delta_i\left(\rho_i\delta_i\right)^{\frac{1}{2H(p-1)}}$. □

Remark: At the first glance, it seems that (3.6) can be obtained by applying (1.5) to the first inequality in (3.8). But (1.5) can only be applied to square interval on the diagonal like $[a,b]\times[a,b]$ (after applying the scaling and self-similarity property of fBm), and here the interval $A_i = [t'_i - \delta_i, t'_i + \delta_i] \times [t_i - \delta_i, t_i + \delta_i]$ is off diagonal.

Lemma 3.3. *For any integer n, the random vector $\mathbf{V} = (V_1, \dots, V_n)$ defined in (3.4) is non-degenerate.*

Proof. Denote by $M = \left(\langle DV_i, DV_j\rangle_{\mathfrak{H}}\right)$ the Malliavin matrix of $\mathbf{V}$. We want to show that $(\det M)^{-1} \in L^k$, for any $k \geq 1$. Note that $\det M \geq \gamma_1^n$, where $\gamma_1 > 0$ is the smallest eigenvalue of the positive definite matrix M. Then it suffices to show that $\gamma_1^{-1} \in L^{nk}$, for any $k \geq 1$, for which it is enough to estimate $P(\gamma_1 \leq \varepsilon)$ for ε small. We have

$$\gamma_1 = \inf_{\|\beta\|=1} (M\beta, \beta) = \inf_{\|\beta\|=1} \left\| D\left(\sum_{i=1}^n \beta_i V_i\right)\right\|_{\mathfrak{H}}^2. \tag{3.10}$$

For any $\beta = (\beta_1, \dots, \beta_n) \in S^{n-1}$, let $\Psi_\beta(\vec{t}) = \sum_{i=1}^n \beta_i \varphi_i(\vec{t})$. Then,

$$D_r\left(\sum_{i=1}^n \beta_i V_i\right) = (2p-1)\int_I \Psi_\beta(\vec{t})\frac{\left|B_t^H - B_{t'}^H\right|^{2p-2}}{|t-t'|^q}\mathbf{1}_{[t',t]}(r)d\vec{t}.$$

Applying (3.1) in the computation of the norm (3.10) yields

$$\begin{aligned}
\left\| D\left(\sum_{i=1}^n \beta_i V_i\right)\right\|_{\mathfrak{H}}^2 &= (2p-1)^2\int_0^1 dr\left(\int_I \Psi_\beta(\vec{t})\frac{\left|B_t^H - B_{t'}^H\right|^{2p-2}}{|t-t'|^q}\mathbf{1}_{[t',t]}(r)d\vec{t}\right)^2\\
&= (2p-1)^2\sum_{i\geq 1}\lambda_i\left(\int_I \varphi_i(\vec{t})\Psi_\beta(\vec{t})\frac{\left|B_t^H - B_{t'}^H\right|^{2p-2}}{|t-t'|^q}d\vec{t}\right)^2\\
&\geq (2p-1)^2\sum_{i=1}^n \lambda_i q_i^2,
\end{aligned}$$

where $q_i = \int_I \varphi_i(\vec{t})\Psi_\beta(\vec{t})\frac{\left(B_t^H - B_{t'}^H\right)^{2p-2}}{|t-t'|^q}d\vec{t}$. The definition (3.5) implies $G_\beta = \sum_{i=1}^n \beta_i q_i$. Since $\lambda_1 \geq \dots \geq \lambda_n > 0$, we obtain

$$\sum_{i=1}^n \lambda_i q_i^2 \geq \lambda_n \sum_{i=1}^n q_i^2 \geq \lambda_n \sum_{i=1}^n \beta_i^2 q_i^2 \geq \frac{\lambda_n}{n} G_\beta^2,$$

where in the third inequality we used the fact that $\sum_{i=1}^n a_i^2 \geq \frac{1}{n}(\sum_{i=1}^n a_i)^2$. Therefore

$$\left\| D\left(\sum_{i=1}^n \beta_i V_i\right)\right\|_{\mathfrak{H}}^2 \geq (2p-1)^2\frac{\lambda_n}{n}G_\beta^2. \tag{3.11}$$

Combining (3.10) and (3.11) we have

$$\gamma_1 = \inf_{\|\beta\|=1}(M\beta, \beta) \geq (2p-1)^2\frac{\lambda_n}{n}\inf_{\|\beta\|=1} G_\beta^2. \tag{3.12}$$

For any $\varepsilon > 0$ and $0 < \alpha < \frac{1}{2H(p-1)}$, let

$$W_\beta = \{G_\beta \geq \varepsilon\},$$

and

$$W_n = \left\{\|DV_i\|_{\mathfrak{H}}^2 \leq \exp \varepsilon^{-\alpha}, i = 1, \dots, n\right\}.$$

On W_n, for any $\beta, \beta' \in S^{n-1}$ we have

$$|(M\beta, \beta) - (M\beta', \beta')| \leq C_n \|\beta - \beta'\| \exp\frac{1}{\varepsilon^\alpha},$$

where C_n is a constant independent of β, β' and ε.

Note that we can find a finite cover $\cup_{i=1}^m B(\beta^i, \exp(-\frac{2}{\varepsilon^\alpha}))$ of S^{n-1} with $\beta^i \in S^{n-1}$ and

$$m \leq C \exp \frac{2n}{\varepsilon^\alpha}.$$

Then on W_n, for any $\beta \in S^{n-1}$, there exists a β^i such that

$$(M\beta, \beta) \geq \left(M\beta^i, \beta^i\right) - C_n \exp \frac{1}{\varepsilon^\alpha} \exp(-\frac{2}{\varepsilon^\alpha}).$$

On $W_{\beta^i} \cap W_n$, applying (3.12) with $A_n = (2p-1)^2 \frac{\lambda_n}{n}$ and taking ε small enough,

$$(M\beta, \beta) \geq A_n \varepsilon^2 - C_n \exp(-\frac{1}{\varepsilon^\alpha}) \geq \frac{A_n}{2} \varepsilon^2.$$

Hence, on $\cap_{i=1}^m W_{\beta^i} \cap W_n$,

$$\gamma_1 = \inf_{\|\beta\|=1} (M\beta, \beta) \geq \frac{A_n}{2} \varepsilon^2 > 0. \tag{3.13}$$

On the other hand, applying Lemma 3.2, we have

$$\begin{aligned} P(\cup_{i=1}^m W_{\beta^i}^c) &\leq \sum_{i=1}^m P(\cup_{i=1}^m W_{\beta^i}^c) \leq m\sqrt{2} \exp(-\frac{C_{p,H}}{\varepsilon^{1/2H(p-1)}}) \\ &\leq C \exp \frac{2n}{\varepsilon^\alpha} \exp(-\frac{C_{p,\alpha}}{\varepsilon^{1/2H(p-1)}}) \leq C \exp(-\frac{C}{\varepsilon^{1/2H(p-1)}}). \end{aligned} \tag{3.14}$$

Also, by Chebyshev's inequality, we can write

$$P(W_n^c) \leq C \exp(-\frac{1}{\varepsilon^\alpha}). \tag{3.15}$$

Then it follows from (3.13)–(3.15) that for ε small,

$$P(\gamma_1 < \frac{A_n}{2} \varepsilon^2) \leq C \exp(-\frac{1}{\varepsilon^\alpha}).$$

This completes the proof of the lemma. □

Proof of Theorem 1.1. Note that

$$\|DF\|_{\mathfrak{H}}^2 = 16p^2 \sum_{i \geq 1} \lambda_i V_i^2 \geq 16p^2 \lambda_n \sum_{i=1}^n V_i^2, \tag{3.16}$$

for any integer n. Then, denoting $|\mathbf{V}|^2 = \sum_{i=1}^n V_i^2$ we have

$$P(\|DF\|_{\mathfrak{H}} < \varepsilon) \leq P\left(|\mathbf{V}| < \frac{\varepsilon}{4p\sqrt{\lambda_n}}\right).$$

Since $\mathbf{V} = (V_1, \dots, V_n)$ is non-degenerate, then it has a smooth density $f_{V_n}(x)$. Then we have

$$P\left(|\mathbf{V}| < \frac{\varepsilon}{4p\sqrt{\lambda_n}}\right) \leq C_{n,p} \varepsilon^n,$$

where $C_{n,p} = \frac{2\pi^{n/2}}{n\Gamma(\frac{n}{2})} \left(4p\sqrt{\lambda_n}\right)^{-n} \max_{|x| \leq 1} f_{V_n}(x)$. Now the theorem follows. □

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