ON SOME SINGULAR LIMITS OF HOMOGENEOUS SEMIGROUPS

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In memory of our friend Philippe Bénilan

"Mais là où les uns voyaient l'abstraction, d'autres voyaient la vérité"
Camus, *La Peste*

1. Introduction.

This paper is based upon some handwritten notes Philippe sent us in late 1996, following his visit to UC Berkeley. He was interested in a scaling argument from our paper with Feldman [E-F-G], and in his notes extended this trick to cover general nonlinear evolutions governed by homogeneous accretive operators. We reproduce his proof in §2 below, and add some commentary and a few PDE examples.

The basic issue is this. Consider a sequence of first–order evolution equations in a Banach space having the form

$$\begin{cases} \dot{u}_n + A_n(u_n) = 0 & \text{for } t > 0 \\ u_n(0) = x_n, \end{cases} \tag{1.1}$$

where the dot means $\frac{d}{dt}$ and A_n denotes some nonlinear operator satisfying the homogeneity condition

$$A_n(\lambda u) = \lambda^{m_n} A_n(u) \quad \text{for all } \lambda > 0. \tag{1.2}$$

We ask what at first may seem an odd question: What happens when the degrees of homogeneity m_n go to infinity? To gain some initial insight, we first note that since u_n solves (1.1), the rescaled functions $v_n(t) := \lambda^{\frac{1}{m_n-1}} u_n(\lambda t)$ solve

$$\begin{cases} \dot{v}_n + A_n(v_n) = 0 & \text{for } t > 0 \\ v_n(0) = \lambda^{\frac{1}{m_n-1}} x_n. \end{cases} \tag{1.3}$$

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We formally write

$$w_n := \frac{\partial}{\partial \lambda} v_n, \text{ evaluated at } \lambda = 1. \tag{1.4}$$

Let us differentiate (1.3) in λ, set $\lambda = 1$, and thereby obtain the linearized evolution

$$\begin{cases} \dot{w}_n + A_n'(v_n) w_n = 0 & \text{for } t > 0 \\ w_n(0) = \frac{1}{m_n - 1} x_n. \end{cases} \tag{1.5}$$

If $x_n \to x$, it presumably follows that

$$w_n \to 0 \quad \text{as} \quad n \to \infty. \tag{1.5}$$

But (1.4) also implies

$$w_n = \frac{1}{m_n - 1} u_n + t \dot{u}_n;$$

and therefore (1.5) suggests that

$$\dot{u}_n \to 0 \quad \text{as} \quad n \to \infty \tag{1.6}$$

for times $t > 0$. These purely formal computations lead us to guess that in the singular limit $m_n \to \infty$ the solutions u_n of (1.1) will converge to a constant value.

But which value? In many cases the limiting value will simply be the initial datum x, but in other situations the highly singular and nonlinear dynamics (1.1) will create an initial short-time layer during which the solution rapidly changes before settling down. The theorem proved in §2 below answers this question in many interesting cases. The point is to construct a new nonlinear evolution equation, having the general form

$$\dot{v} + A(v) \ni \frac{v}{t}, \tag{1.7}$$

such that the limiting value of $u_n(t)$ for times $t > 0$ is $v(1)$. Otherwise said, the dynamics (1.7) characterize a suitably rescaled limit of the fast changes undergone by solutions of (1.1) within a short initial time layer, as $n \to \infty$.

The paper Benilan–Crandall [B-C] on regularizing properties of nonlinear semigroups generated by homogeneous accretive operators involves similar heuristics.

2. An abstract singular limit.

Let X denote a real Banach space, with norm $|| \; ||$. If A is a nonlinear, possibly multivalued, m-accretive operator on X, we will sometimes write $(x, y) \in A$ to mean that $x \in D(A)$, the domain of A, and $y \in Ax$. We also let e^{-tA} denote the nonlinear semigroup generated by A.

Suppose that for $n = 1, 2, \dots,$ A_n be are m-accretive operators defined on X. Our key hypothesis is that each is also positively homogeneous of degree m_n, meaning

$$A_n(\lambda x) = \lambda^{m_n} A_n x \quad \text{for all } x \in D(A_n), \; \lambda > 0.$$

Define

$$C := \{x \in X | \text{ there exist } (x_n, y_n) \in A_n \text{ with } x_n \to x, \; y_n \to 0\}$$

and let

$$X_0 := \overline{\bigcup_{\lambda > 0} \lambda C}.$$

We will assume that the degrees of homogeneity tend to infinity:

$$\lim_{n \to \infty} m_n = \infty, \tag{2.1}$$

and furthermore that the limit

$$Px := \lim_{n \to \infty} (I + A_n)^{-1} x \tag{2.2}$$

exists in X, for all $x \in X_0$.

Theorem. *There exists a nonlinear operator $Q : X_0 \to C$ such that if $x_n \in \overline{D(A_n)}$ for $n = 1, 2, \dots$ and if $x_n \to x \in X_0$, then*

$$e^{-tA_n} x_n \to Qx,$$

uniformly for t in compact subsets of $(0, \infty)$.

More precisely, we assert

(i) *C is a closed set, $\lambda C \subseteq C$ for each $\lambda \in [0, 1]$. The mapping P is a contraction of X_0 onto C, with $Px = x$ for each $x \in C$.*

(ii) *The operator*

$$A := P^{-1} - I$$

is an accretive operator on X with $D(A) = C$, $R(I + \lambda A) \supseteq X_0$ and $(I + \lambda A)^{-1} = P$ on X_0 for each $\lambda > 0$.

(iii) *Q is a contraction of X_0 onto C, and $Qx = x$ for $x \in C$. If $x \in \lambda C$ for some $\lambda > 1$, we have*

$$Qx = v(1), \tag{2.3}$$

where v is the unique mild solution of the evolution equation

$$\begin{cases} \dot{v} + Av \ni \frac{v}{t} & \text{on } (\delta, \infty) \\ v(\delta) = \delta x \end{cases} \tag{2.4}$$

for $\delta := \lambda^{-1} \in (0,1)$.

Proof. 1. Let $x \in X_0$ and set $x_n := (I + A_n)^{-1}x$. Define $\lambda_n := e^{-\frac{1}{\sqrt{m_n}}}$ and $x_n' := \lambda_n x_n$. Then $\lim x_n' = \lim x_n = Px$, $A_n x_n' \ni \lambda_n^{m_n}(x - x_n) \to 0$. It follows that $Px \in C$, and since each operator $(I + A_n)^{-1}$ is a contraction, so is P.

Suppose next that $x \in C$, $\lambda \in (0,1]$, $(x_n, y_n) \in A_n$, $x_n \to x$, and $y_n \to 0$. We then have $\lambda x_n = (I + A_n)^{-1}(\lambda x_n + \lambda^{m_n} y_n)$, and therefore $\lambda x = P(\lambda x)$. This proves (i).

2. Next, fix $\lambda > 0$. For large n, we have $|\lambda^{-\frac{1}{m_n}} - 1| < 1$. Then

$$\begin{aligned} (I + \lambda A_n)^{-1} &= \left(I + A_n(\lambda^{\frac{1}{m_n}} \cdot)\right)^{-1} \\ &= \lambda^{-\frac{1}{m_n}} \left(\lambda^{-\frac{1}{m_n}} I + A_n\right)^{-1} \\ &= \lambda^{-\frac{1}{m_n}} (I + A_n)^{-1} \left(I + \left(\lambda^{-\frac{1}{m_n}} - 1\right)(I + A_n)^{-1}\right)^{-1}. \end{aligned}$$

Consequently

$$(I + \lambda A_n)^{-1}x \to Px \quad \text{for each } x \in X_0 \text{ and } \lambda > 0. \tag{2.5}$$

Statement (ii) follows, and one has $A \subset \liminf_{n\to\infty} A_n$.

3. Let $x_n \in \overline{D(A_n)}$, $x_n \to x$. If $x \in C = D(A)$, then usual limit theory shows that $e^{-tA_n}x_n \to e^{-tA}x = x$, uniformly for t in bounded subsets of $[0, \infty)$.

Assume now $x \in \lambda C$ with $\lambda > 1$ and set $\delta := \lambda^{-1}$. We first demonstrate that

$$e^{-t_n A_n} x_n \to x \quad \text{for} \quad t_n := \frac{\delta^{m_n}}{m_n}. \tag{2.6}$$

By definition of C, we can find $(x_n', y_n) \in A_n$ such that $x_n' \to \delta x$, $y_n \to 0$. Since $\lambda^{m_n} y_n \in A_n(\lambda x_n')$, we have $\|e^{-t_n A_n}(\lambda x_n') - \lambda x_n'\| \le t_n \lambda^{m_n} \|y_n\| = \frac{\|y_n\|}{m_n}$. On the other hand,

$$\|e^{-t_n A_n} x_n - e^{-t_n A_n}(\lambda x_n')\| \le \|x_n - \lambda x_n'\|;$$

and then (2.6) follows.

4. Let now $v_n(t) := te^{-\frac{t^{m_n}}{m_n}A_n}x_n$. Using the homogeneity of A_n and the definition of mild solution, it is clear that v_n is a mild solution of

$$\dot{v}_n + A_n v_n \ni \frac{v_n}{t} \qquad \text{on } (0,\infty).$$

Using (2.6), we see that $v_n(\delta) \to \delta x \in C = D(A)$. According to (ii), there exists a unique mild solution v of (3). Regular limit theory implies then that $v_n \to v$ in $\mathcal{C}([\delta,\infty); X)$.

Finally, $e^{-tA_n}x_n = \frac{v_n(\tau_m)}{\tau_m}$ with $\tau_m := (m_n t)^{\frac{1}{m_n}}$. Then $\tau_m \to 1$, and so $e^{-tA_n}x_n \to v(1)$, uniformly for t in compact subsets of $(0,\infty)$. This proves the existence of Q on $\bigcup_{\lambda>0}\lambda C$. Since each mapping e^{-tA_n} is a contraction, we can extend the definition of Q to X_0. □

Remark. We note here explicitly that the foregoing construction of $Qx = v(1)$ does not depend upon the choice of $\lambda > 1$ for which $x \in \lambda C$. □

3. Example 1: homogeneous Hamilton–Jacobi PDE.

As a first, heuristic example, consider the initial–value problem

$$\text{(3.1)} \qquad \begin{cases} u_t + \frac{1}{p}|Du|^p = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\}, \end{cases}$$

where g is bounded and Lipschitz continuous.

We will work out for this problem the formal implications of the Theorem in §2, without providing complete justification. Careful proofs would entail some rather subtle issues concerning viscosity solutions for multivalued Hamiltonians, which we do not wish to address here. We will however later be able to check that the formalism in fact predicts the correct answer.

Let us take $X = BUC(\mathbb{R}^N)$, the space of bounded, uniformly continuous functions with the sup-norm, and define A_p to be the operator $\frac{1}{p}|Du|^p$ in the sense of viscosity solutions. Therefore

$$C = \{u \in W^{1,\infty}(\mathbb{R}^N) \mid |Du| \le 1 \ a.e.\}$$

and so $X_0 = X$. Now $u_p = (I + A_p)^{-1} f$ means

$$u_p + \frac{1}{p}|Du_p|^p = f \quad \text{in } \mathbb{R}^n$$

in the viscosity sense, and so also almost everywhere. We guess that as $p \to \infty$, we have $u_p \to u$ uniformly, where $|Du| \le 1$ a.e. and u is a solution of

$$\text{(3.2)} \qquad u + \gamma(|Du|) \ni f \quad \text{in } \mathbb{R}^n,$$

for the multivalued graph

$$\gamma(z) := \begin{cases} \{0\} & \text{if } \ z < 1 \\ [0, \infty) & \text{if } \ z = 1 \\ \emptyset & \text{if } \ z > 1. \end{cases}$$

Then $Pf = u$.

The dynamics (2.4) therefore read

$$\begin{cases} v_t + \gamma(|Dv|) \ni \frac{v}{t} & \text{in } \mathbb{R}^n \times (\delta, \infty) \\ v = \delta g & \text{on } \mathbb{R}^n \times \{t = \delta\}. \end{cases} \tag{3.3}$$

This PDE is sufficiently simple that we can guess the solution

$$v(x,t) = \min_{y \in \mathbb{R}^n} \{|x-y| + tg(y)\}\,; \tag{3.4}$$

so that

$$Qg = v(\cdot, 1) = \min_{y \in \mathbb{R}^n} \{|x-y| + g(y)\}\,. \tag{3.5}$$

Remark. We can quickly check this assertion, since the Hopf–Lax formula (cf. [E, Chapter 3]) provides us with a formula for the solution of (3.1):

$$u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ \frac{t}{q} \frac{|x-y|^q}{t^q} + g(y) \right\},$$

with $\frac{1}{p} + \frac{1}{q} = 1$. Hence for each time $t > 0$, the solutions of (3.1) do in fact converge as $p \to \infty$ to $\min_{y \in \mathbb{R}^n} \{|x-y| + g(y)\}$, as predicted. □

4. Example 2: the parabolic mesa problem.

We next investigate the limiting behavior as $m \to \infty$ of solutions to the porous medium equation

$$\begin{cases} u_{m,t} - \Delta(|u_m|^{m-1} u_m) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u_m = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{4.1}$$

We take $X = L^1(\mathbb{R}^n)$ and define the operator A_m by $A_m u := -\Delta(|u|^{m-1}u)$ for u belonging to $D(A_m) = \{u \in L^1(\mathbb{R}^n) \mid \Delta(|u|^{m-1}u) \in L^1(\mathbb{R}^n)\}$. The operator A_m is m-accretive and is homogeneous of degree m. Furthermore,

$$C = \{u \in L^\infty(\mathbb{R}^n) \mid |u| \leq 1 \ a.e.\},$$

and therefore $X_0 = X$.

We compute P and Q. Firstly, $u_m = (I + A_m)^{-1} f$ means

$$u_m - \Delta(|u_m|^{m-1} u_m) = f \quad \text{in } \mathbb{R}^n. \tag{4.2}$$

We will need some compactness:

Lemma. *The set of functions $\{u_m\}_{m=1}^{\infty}$ is precompact in $L^1(\mathbb{R}^n)$.*

Proof. 1. Assume first that f is bounded, nonnegative and has compact support, in which case $u \geq 0$. We simplify notation by removing the subscript m, and so write

$$u - \Delta(u^m) = f. \tag{4.3}$$

Standard L^1-contraction estimates imply the bounds

$$||u||_{L^1} \leq ||f||_{L^1}, \quad ||u(\cdot) - u(\cdot + h)||_{L^1} \leq ||f(\cdot) - f(\cdot + h)||_{L^1}$$

for each $h \in \mathbb{R}^n$, and so $\{u_m\}_{m=1}^{\infty}$ is precompact in $L^1(K)$ for each compact set $K \subset \mathbb{R}^n$.

2. We must show that $\{u_m\}_{m=1}^{\infty}$ is tight, which is to say, that no mass moves to infinity as $m \to \infty$. For this, note first that our multiplying by u^m gives

$$\int_{\mathbb{R}^n} u^{m+1} + Du^m \cdot Du^m \, dx = \int_{\mathbb{R}^n} f u^m \, dx.$$

According to the Sobolev inequality, the second term on the left is greater than or equal to

$$C \left(\int_{\mathbb{R}^n} u^{\frac{2mn}{n-2}} \, dx \right)^{\frac{n-2}{n}}$$

for some positive constant C. Making some elementary estimates, we derive the inequality

$$\int_{\mathbb{R}^n} u^{m+1} \, dx \leq C \left(\int_{\mathbb{R}^n} f^{\frac{2n}{n+4}} \, dx \right)^{\frac{n+4}{n}},$$

where C does not depend on m.

Now fix $S > R > 0$, where the radius R is selected so large that supt $f \subseteq B(0, R)$. Take a smooth function ζ such that $0 \leq \zeta \leq 1$, $\zeta \equiv 0$ on $B(0, R)$, $\zeta \equiv 1$ on $\mathbb{R}^n - B(0, S)$, $|D\zeta| \leq \frac{C}{S}, |D^2\zeta| \leq \frac{C}{S^2}$. Our multiplying (4.3) by ζ gives

$$\int_{\mathbb{R}^n} u\zeta \, dx = \int_{\mathbb{R}^n} u^m \Delta\zeta \, dx \leq \frac{C}{S^2} \left(\int_{\mathbb{R}^n} u^{m+1} \, dx \right)^{\frac{m}{m+1}} |B(0, S)|^{\frac{1}{m+1}};$$

and this implies

$$\int_{\{|x| \geq S\}} u \, dx = O(S^{-2+\frac{n}{m+1}}) = o(1), \tag{4.4}$$

uniformly for $m \geq n$.

This proves tightness if f is bounded, nonnegative and has compact support. If f is bounded and has compact support, but can change sign, we compare u with solutions $u^{\pm}$ of (4.2) with $f^{\pm}$ replacing f. Note finally that bounded functions with compact support are dense in L^1 and the mapping $f \mapsto u$ is a contraction. From these observations it follows that for each $f \in L^1$, the sequence $\{u_m\}_{m=1}^{\infty}$ is precompact. $\square$

Using the Lemma, we can find a subsequence $m_j \to \infty$ for which $u_{m_j} \to u$ in L^1. We check that u is the unique solution of

$$u - \Delta\phi(u) \ni f \quad \text{in } \mathbb{R}^n,$$

for the multivalued graph

$$\phi(z) := \begin{cases} \emptyset & \text{if } z < -1 \\ (-\infty, 0] & \text{if } z = -1 \\ \{0\} & \text{if } -1 < z < 1 \\ [0, \infty) & \text{if } z = 1 \\ \emptyset & \text{if } z > 1. \end{cases}$$

By uniqueness, in fact $u_m \to u$ in L^1. We can restate our conclusion by noting that

$$Pf = u = f\chi_{\{w=0\}} + \chi_{\{w>0\}} - \chi_{\{w<0\}},$$

where $w \in \phi(u)$ is the solution of the mesa problem

$$\operatorname{sgn}(w) - \Delta w \ni f \quad \text{in } \mathbb{R}^n$$

and we write $\operatorname{sgn} = \phi^{-1}$. (See Benilan–Igbida [B-I] for more details.)

In our case the evolution (2.4) becomes

$$\text{(4.5)} \qquad \begin{cases} v_t - \Delta\phi(v) \ni \frac{v}{t} & \text{in } \mathbb{R}^n \times (\delta, \infty) \\ \qquad\quad v = \delta g & \text{on } \mathbb{R}^n \times \{t = \delta\}, \end{cases}$$

and according to the theory from §2, we have $Qg = v(\cdot, 1)$.

Several authors, including Benilan, Boccardo, and Herrero [B-B-H], have studied the situation that $g \geq 0$, and have shown that in this case the sequence of solutions of (4.1) converges for $t > 0$ to Pg. We now check that this conclusion accords with our theory.

Proposition. *1. We have*

$$Q = P \quad \text{on } L^1(\mathbb{R}^n)^+.$$

2. In general, however

$$Q \neq P \quad \text{on } L^1(\mathbb{R}^n).$$

Proof. 1. We modify some ideas from [B-B-H]. Assume that g is bounded and

$$\text{(4.6)} \qquad g \geq 0.$$

We assert that

$$t \mapsto v(\cdot,t) \quad \text{is nondecreasing.} \tag{4.7}$$

This follows since (4.6) implies firstly that $v \geq 0$, and then that pointwise either $v = 1$ or else $v_t = \frac{v}{t}$.

2. Take $w \in \phi(v)$ so that

$$v_t - \Delta w = \frac{v}{t}; \tag{4.8}$$

and observe that since $v \geq 0$, we have $w \geq 0$. Define next

$$\hat{w} := \int_\delta^1 \frac{w}{t}\, dt.$$

Then (4.8) implies

$$\Delta \hat{w} = \int_\delta^1 \frac{\Delta w}{t}\, dt = \int_\delta^1 \frac{d}{dt}\left(\frac{v}{t}\right)\, dt = v(\cdot,1) - g. \tag{4.9}$$

We claim that

$$\hat{w} \in \phi(v(\cdot,1)). \tag{4.10}$$

Indeed, if $0 \leq v(x,1) < 1$, then (4.7) implies $0 \leq v(x,t) < 1$ for all $\delta \leq t \leq 1$. Since $w \in \phi(v)$, we deduce that $w(x,t) = 0$ for $\delta \leq t \leq 1$. Consequently $\hat{w}(x) = 0 \in \phi(v(x,1))$. On the other hand, if $v(x,1) = 1$, then $0 \leq \hat{w}(x) \in \phi(v(x,1))$. This proves (4.10).

3. According to (4.9) and (4.10), we have

$$v(\cdot,1) - \Delta\phi(v(\cdot,1)) \ni g,$$

and so

$$Qg = v(\cdot,1) = Pg.$$

4. In the next section we will discuss an example showing for $n = 1$ that in general $P \neq Q$. □

5. Example 3: collapsing sandpiles.

This example is based upon our paper with Feldman [E-F-G], where we suggest that the following provides a (very crude) model for sand dynamics.

We ask about the behavior as $p \to \infty$ of solutions to the parabolic p-Laplacian equation

$$
(5.1) \qquad \begin{cases} u_{p,t} - \operatorname{div}(|Du_p|^{p-2} Du_p) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u_p = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}
$$

We take $X = L^2(\mathbb{R}^n)$ and define the m-accretive operator A_p by $A_p u := -\operatorname{div}(|Du|^{p-1}Du)$ for u belonging to

$$
D(A_p) = \{u \in L^2(\mathbb{R}^n) \mid \operatorname{div}(|Du|^{p-2}Du) \in L^2(\mathbb{R}^n)\}.
$$

Then

$$
C = \{u \in L^2(\mathbb{R}^n) \mid Du \in L^\infty, \ |Du| \le 1 \ a.e.\},
$$

and hence $X_0 = X$.

Now $u_p = (I + A_p)^{-1} f$ means

$$
u_p - \operatorname{div}(|Du_p|^{p-2}Du_p) = f \quad \text{in } \mathbb{R}^n.
$$

The following assertion provides compactness:

Lemma. *The set of functions $\{u_p\}_{p=1}^\infty$ is precompact in $L^2(\mathbb{R}^n)$.*

Proof. Assume firstly that f is bounded and the support of f lies in the ball $B(0,R)$. We drop the subscript p and so write

$$
(5.2) \qquad u - \operatorname{div}(|Du|^{p-2}Du) = f.
$$

The estimates $||u||_{L^\infty} \le ||f||_{L^\infty}$ and $\int_{\mathbb{R}^n} |Du|^p \, dx \le ||f||_{L^2}$ follow. Also, L^2-contraction estimates imply the bounds

$$
||u||_{L^2} \le ||f||_{L^2}, \quad ||u(\cdot) - u(\cdot + h)||_{L^2} \le ||f(\cdot) - f(\cdot + h)||_{L^2}
$$

for each $h \in \mathbb{R}^n$, and consequently $\{u_p\}_{p=1}^\infty$ is precompact in $L^2(K)$ for each compact set $K \subset \mathbb{R}^n$.

We must show tightness. For this, select a function ζ as in the proof of the Lemma in §4. Multiplying the PDE (5.2) by ζu and integrating by parts gives

$$
\begin{aligned}
\int_{\mathbb{R}^n} \zeta u^2 + \zeta |Du|^p \, dx &= \int_{\mathbb{R}^n} u |Du|^{p-2} Du \cdot D\zeta \, dx \le \frac{C}{S} \int_{B(0,S)} |Du|^{p-1} \, dx \\
&\le \frac{C}{S} \left(\int_{\mathbb{R}^n} |Du|^p \, dx \right)^{\frac{p-1}{p}} |B(0,S)|^{\frac{1}{p}} \\
&= O(S^{-1+\frac{n}{p}}) = o(1),
\end{aligned}
$$

uniformly in p.

This estimate establishes compactness in L^2 provided f is bounded and has compact support, and the general case follows, as such functions are dense. $\square$

Utilizing the Lemma, we deduce as in [E-F-G] that as $p \to \infty$, $u_p \to u$ in L^2, where u is the unique solution of

$$u + \partial I_C u \ni f.$$

Here ∂I_C denotes the subdifferential of the convex function

$$I_C[v] = \begin{cases} 0 & \text{if } v \in C \\ \infty & \text{otherwise} \end{cases}$$

In other words, $u = Pf$ is the L^2–projection onto the closed, convex set C. In this situation the evolution (2.4) becomes

$$\begin{cases} v_t + \partial I_C v \ni \frac{v}{t} & \text{in } \mathbb{R}^n \times (\delta, \infty) \\ v = \delta g & \text{on } \mathbb{R}^n \times \{t = \delta\}, \end{cases} \tag{5.3}$$

and $Qg = v(\cdot, 1)$.

An example where $\mathbf{P \neq Q}$. In the appendix of [E-F-G] we constructed a initial function g with compact support, for which $Qg \neq Pg$. We can as follows convert this into an example of $P \neq Q$ for the parabolic mesa problem, discussed earlier in §4.

We take $n = 1$ and consider the two problems

$$\begin{cases} u_t - (|u|^{m-1}u)_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = h & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{5.4}$$

and

$$\begin{cases} \hat{u}_t - (|\hat{u}_x|^{p-2}\hat{u}_x)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ \hat{u} = g & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{5.5}$$

For notational simplicity, we do not index u or $\hat{u}$ with a subscript m or p. Observe next that if $m = p - 1$ we have the transformation

$$\hat{u}_x = u, \tag{5.6}$$

provided

$$g_x = h. \tag{5.7}$$

The limit dynamics read, respectively,

$$\begin{cases} v_t - (\phi(v))_{xx} \ni \frac{v}{t} & \text{in } \mathbb{R} \times (\delta, \infty) \\ v = \delta h & \text{on } \mathbb{R} \times \{t = \delta\}. \end{cases} \tag{5.8}$$

and

(5.9) $$\begin{cases} \hat{v}_t + \partial I_C \hat{v} \ni \frac{\hat{v}}{t} & \text{in } \mathbb{R} \times (\delta, \infty) \\ \hat{v} = \delta g & \text{on } \mathbb{R} \times \{t = \delta\}. \end{cases}$$

But according to the theory in §2, $\hat{v}_x = v$ and so

$$(\hat{Q}g)_x = Qh.$$

Likewise

$$(\hat{P}g)_x = Ph.$$

Since we have constructed in [E-F-G] initial data g for which

$$\hat{Q}g \neq \hat{P}g + C$$

for any constant C, it follows that $Qh \neq Ph$. □

Remark. Notice that in Examples 1 and 3 the sets C are basically the same, comprising functions with Lipschitz constant less than or equal to one. However the operators P and consequently the dynamics governed by A differ. In Example 1, the mapping P is not the L^2–projection onto C. □

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