LECTURES ON SEMICLASSICAL ANALYSIS

VERSION 0.2

LAWRENCE C. EVANS AND MACIEJ ZWORSKI

DEPARTMENT OF MATHEMATICS
UC BERKELEY

Preface

These lectures originate with a course MZ taught at UC Berkeley during the spring semester of 2003, notes for which LCE took in class.

In this presentation we have tried hard to work out the full details for many proofs only sketched in class. In addition, we have reworked the order of presentation, added additional topics, and included more heuristic commentary.

We have as well introduced consistent notation, much of which is collected into Appendix A. Relevant functional analysis and other facts have been consolidated into Appendices B–D.

We should mention that two excellent treatments of mathematical semiclassical analysis have appeared recently. The book by Dimassi and Sjöstrand [D-S] starts with the WKB-method, develops the general semiclassical calculus, and then provides "high tech" spectral asymptotics. The presentation of Martinez [M] is based on a systematic development of FBI (Fourier-Bros-Iagolnitzer) transform techniques, with applications to microlocal exponential estimates and propagation estimates.

These notes are intended as a more elementary and broader introduction. Except for the general symbol calculus, where we followed Chapter 7 of [D-S], there is little overlap with these other two texts, or with the early and influential book of Robert [R]. In his study of semiclassical calculus MZ has been primarily influenced by his long collaboration with Johannes Sjöstrand and he acknowledges that with pleasure and gratitude.

Our thanks to Faye Yeager for typing a first draft and to Hans Christianson for his careful reading of earlier versions of these notes. And thanks also to Jonathan Dorfman for TeX advice.

We are quite aware that many errors remain in our exposition, and so we ask our readers to please send any comments or corrections to us at

evans@math.berkeley.edu zworski@math.berkeley.edu.

This Version 0.1 represents our first draft, the clarity of which we hope greatly to improve in later editions. We will periodically post improved versions these lectures on our websites.

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Contents

1. Introduction

1.1 BASIC THEMES

Our basic theme is understanding the relationships between dynamical systems and the behavior of solutions to various linear PDE containing a small positive parameter h.

Small parameters. Our principal source of motivation is quantum mechanics, in which case we understand h as denoting *Planck's constant*. With this interpretation in mind, we break down our basic task into these two subquestions:

• How and to what extent does classical dynamics determine the behavior as $h \to 0$ of solutions to *Schrödinger's equation*

$$ih\partial_t u = -h^2 \Delta u + V(x)u, \tag{1.1}$$

and the related *eigenvalue equation*

$$-h^2 \Delta u + V(x)u = Eu\,? \tag{1.2}$$

• Conversely, given various mathematical objects associated with classical mechanics, for instance symplectic transformations, how can we profitably "quantize" them?

In fact the techniques of semiclassical analysis apply in many other settings and for many other sorts of PDE. For example we will later study the *damped wave equation*

$$\partial_t^2 u + a(x)\partial_t u - \Delta u = 0 \tag{1.3}$$

for large times. A rescaling in time will introduce the requisite small parameter h.

Basic techniques. We will construct in Chapters 2–4 a wide variety of mathematical tools to address these issues, among them

• the apparatus of symplectic geometry (to record succinctly the behavior of classical dynamical systems);
• the Fourier transform (to display dependence upon both the position variables x and the momentum variables ξ);
• stationary phase (to describe asymptotics as $h \to 0$ of various expressions involving rescaled Fourier transforms);

• pseudodifferential operators (to "microlocalize" estimates in phase space).

1.2 CLASSICAL AND QUANTUM MECHANICS

In this section we introduce and foreshadow a bit about quantum and classical correspondences.

•**Observables.** We can think of a given function $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, $a = a(x, \xi)$, as a *classical observable* on phase space, where as above x denotes *position* and ξ *momentum*. We will also call a a *symbol*.

Now let $h > 0$ be given. We will associate with the observable a, a corresponding *quantum observable* $a^w(x, hD)$, an operator defined by the formula

$$a^w(x,hD)u(x) := \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a\left(\frac{x+y}{2},\xi\right) u(y)\, d\xi dy \tag{1.4}$$

for appropriate smooth functions u.

This is *Weyl's quantization formula.* We will later learn that if we change variables in a symbol, we preserve the principal symbol up to lower order terms (that is, terms involving high powers of the small parameter h.)

• **Equations of evolution.** We are concerned as well with the evolution of classical particles and quantum states.

Classical evolution. Our most important example will concern the symbol

$$p(x, \xi) := |\xi|^2 + V(x),$$

corresponding to the phase space flow

$$\begin{cases} \dot{\mathbf{x}} = 2\boldsymbol{\xi} \\ \dot{\boldsymbol{\xi}} = -\partial V. \end{cases}$$

We generalize by introducing the arbitrary Hamiltonian $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $p = p(x, \xi)$, and the corresponding *Hamiltonian dynamics*

$$\begin{cases} \dot{\mathbf{x}} = \partial_\xi p(\mathbf{x}, \boldsymbol{\xi}) \\ \dot{\boldsymbol{\xi}} = -\partial_x p(\mathbf{x}, \boldsymbol{\xi}). \end{cases} \tag{1.5}$$

It is instructive to change our viewpoint somewhat, by firstly introducing some more notation. Let us define $\Phi_t = \exp(tH_p)$ for the solution of (1.5), where

$$H_p q := \{p, q\}$$

is the Poisson bracket. Set

$$a_t(x,\xi) := a(\Phi_t(x,\xi)). \tag{1.6}$$

Then

$$\frac{d}{dt}a_t = \{p, a_t\}. \tag{1.7}$$

This equation tells us how the symbol a evolves in time.

Quantum evolution: Now put $P = p^w(x, hD), A = a^w(x, uD)$, and define

$$A_t := e^{\frac{itP}{n}}\ A\ e^{-\frac{itP}{n}}. \tag{1.8}$$

Then we have the evolution equation

$$\frac{d}{dt}A_t = \frac{i}{h}[P, A_t], \tag{1.9}$$

an obvious analog of (1.7).

Here then is a basic principle: any assertion about Hamiltonian dynamics, and so the Poisson bracket $\{\cdot,\cdot\}$, will involve at the quantum level the commutator $[\cdot,\cdot]$.

1.3 OVERVIEW

Chapters 2–4 and 8 develop the basic machinery, and the other chapters cover applications to PDE. Here is a quick overview, with some of the highpoints:

Chapter 2: We start with a quick introduction to symplectic geometry and its implications for classical Hamiltonian dynamical systems.

Chapter 3: This chapter provides the basics of the Fourier transform and derives also important *stationary phase* asymptotic estimates, of the sort

$$I_h = (2\pi h)^{n/2}|\det\,\partial^2\phi(x_0)|^{-1/2}e^{\frac{i\pi}{4}\operatorname{sgn}\partial^2\phi(x_0)}e^{\frac{i\phi(x_0)}{h}}a(x_0) + O\left(h^{\frac{n+2}{2}}\right)$$

as $h \to 0$, for the oscillatory integral

$$I_h := \int_{\mathbb{R}^n} e^{\frac{i\phi}{h}}a\,dx.$$

We assume here that the gradient $\partial\phi$ vanishes only at the point x_0.

Chapter 4: Next we introduce the Weyl quantization $a^w(x, hD)$ of the symbol $a(x,\xi)$ and work out various properties, among them the composition formula

$$a^w(x, hD) \circ b^w(x, hD) = c^w(x, hD),$$

where the symbol $c := a\#b$ is computed explictly in terms of a and b. We will prove as well the sharp Gårding inequality, stating that if a is a nonnegative symbol, then

$$\langle a^w(x, hD)u, u\rangle \geq -Ch\|u\|_{L^2}^2$$

for all u and sufficiently small $h > 0$.

Chapter 5: This section introduces semiclassical defect measures, and uses them to derive decay estimates for the damped wave equation

$$(\partial_t^2 + a(x)\partial_t - \Delta)u = 0,$$

where $a \geq 0$, on the flat torus $\mathbb{T}^n$.

Chapter 6: In Chapter 6 we begin our study of the eigenvalue problem

$$P(h)u(h) = E(h)u(h),$$

for the operator

$$P(h) := -h^2\Delta + V(x).$$

We prove *Weyl's Law* for the asymptotic distributions of eigenvalues as $h \to 0$, stating for all $a < b$ that

$$\begin{aligned} &\#\{E(h) \mid a \leq E(h) \leq b\} \\ &\qquad = \frac{1}{(2\pi h)^n}(\mathrm{Vol}\{a \leq |\xi|^2 + V(x) \leq b\} + o(1)), \end{aligned} \tag{1.10}$$

where "Vol" means volume.

Chapter 7: Chapter 7 continues the study of eigenfunctions, first establishing an exponential vanishing theorem in the "classically forbidden" region. We derive as well a *Carlemann-type* inequality

$$\|u(h)\|_{L^2(E)} \geq e^{-\frac{C}{h}}\|u(h)\|_{L^2(\mathbb{R}^n)}$$

where $E \subset \mathbb{R}^n$. This is a quantitative estimate for quantum mechanical *tunneling*.

Chapter 8: We return to the symbol calculus. We introduce the useful formalism of "half-densities" and use them to illustrate how changing variables in a symbol affects the Weyl quantization. We introduce the notion of the semiclassical wave front set and show how a natural localization in phase space leads to pointwise bounds on approximate solutions. We also prove a semiclassical version of Beals's Theorem, characterizing pseudodifferential operators. As an application we show how, on the level of order functions, quantization commutes with exponentiation.

Chapter 9: Chapter 9 concerns the quantum implications of ergodicity for our underlying dynamical systems. A key assertion is that if the underlying dynamical system satisfies an appropriate ergodic condition, then

$$(2\pi h)^n \sum_{a \le E_j \le b} \left| \langle A u_j, u_j \rangle - \fint_{\{a \le p \le b\}} \sigma(A)\, dx d\xi \right|^2 \longrightarrow 0$$

as $h \to 0$, for a wide class of pseudodifferential operators A. In this expression the classical observable $\sigma(A)$ denotes the symbol of A.

Chapter 10: The concluding Chapter 10 explains how to quantize symplectic transformations, with applications including local constructions of propagators, L^p bounds on eigenfuctions, and normal forms of differential operators.

Appendices: Appendix A records our notation in one convenient location, and Appendix B collects various useful functional analysis theorems (with selected proofs). Appendix C is a quick introduction/review of differential forms.

Appendix D discusses general manifolds and modifications our the symbol calculus to cover pseudodifferential operators on manifolds.

2. Symplectic geometry

Since our task in these notes is understanding some interrelationships between dynamics and PDE, we provide in this chapter a quick discussion of the symplectric geometric structure on $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ and its interplay with Hamiltonian dynamics.

The reader may wish to first review our basic notation and also the theory of differential forms, set forth respectively in Appendices A and C.

2.1 FLOWS

Let $V : \mathbb{R}^N \to \mathbb{R}^N$ denote a smooth vector field. Fix a point $m \in \mathbb{R}^N$ and solve the ODE

$$\begin{cases} \dot{m}(t) = V(m(t)) & (t \in \mathbb{R}) \\ m(0) = m. \end{cases} \tag{2.1}$$

We assume that the solution of (2.1) exists and is unique for all times $t \in \mathbb{R}$.

NOTATION. We define

$$\Phi_t m := m(t)$$

and sometimes also write

$$\Phi_t =: \exp(tV).$$

We call $\{\Phi_t\}_{t\in\mathbb{R}}$ the *exponential map.*

The following lemma records some standard assertions from theory of ordinary differential equations:

LEMMA 2.1 (Properties of flow map).
(i) $\Phi_0 m = m$.
(ii) $\Phi_{t+s} = \Phi_t \circ \Phi_s$.
(iii) *For each time* t, $\Phi_t : \mathbb{R}^N \to \mathbb{R}^N$ *is a diffeomorphism.*
(iv) $(\Phi_t)^{-1} = \Phi_{-t}$.

2.2 SYMPLECTIC STRUCTURE ON $\mathbb{R}^{2n}$

We henceforth specialize to the even-dimensional space $\mathbb{R}^N = \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$.

NOTATION. We will often write a typical point of $\mathbb{R}^{2n}$ as

$$m = (x, \xi),$$

and interpret $x \in \mathbb{R}^n$ as denoting *position*, $\xi \in \mathbb{R}^n$ as *momentum*. Alternatively, we can think of ξ as belonging to $T_x^*\mathbb{R}^n$, the cotangent space of $\mathbb{R}^n$ at x.

We let $\langle \cdot, \cdot \rangle$ denote the usual innner product on $\mathbb{R}^n$, and then define this new inner product on $\mathbb{R}^{2n}$:

DEFINITION. Given two vectors $u = (x, \xi)$, $v = (y, \eta)$ on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, define their *symplectic product*

$$\sigma(u, v) := \langle \xi, y \rangle - \langle x, \eta \rangle. \tag{2.2}$$

Observe that

$$\sigma(u, v) = \langle u, Jv \rangle \tag{2.3}$$

for the $2n \times 2n$ matrix

$$J := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \tag{2.4}$$

LEMMA 2.2 (Properties of $\boldsymbol{\sigma}$).
(i) *σ is bilinear.*
(ii) *σ is antisymmetric.*
(iii) *σ is nondegenerate; that is,*

$$\textit{if } \sigma(u, v) = 0 \textit{ for all } v, \textit{ then } u = 0. \tag{2.5}$$

These assertions are straightforward to check.

NOTATION. Using the terminology of differential forms, reviewed in Appendix C, we can write

$$\sigma = d\xi \wedge dx = \sum_{j=1}^{n} d\xi_j \wedge dx_j. \tag{2.6}$$

Observe also that

$$\sigma = d\omega \quad \text{for} \quad \omega := \xi dx = \sum_{j=1}^{n} \xi_j dx_j. \tag{2.7}$$

2.3 CHANGING VARIABLES.

Suppose next that $U, V \subseteq \mathbb{R}^{2n}$ are open sets and

$$\boldsymbol{\kappa} : U \to V$$

is a smooth mapping. We will write

$$\boldsymbol{\kappa}(x,\xi) = (y,\eta) = (\mathbf{y}(x,\xi), \boldsymbol{\eta}(x,\xi)).$$

DEFINITION. We call $\boldsymbol{\kappa}$ a *symplectomorphism* if

$$\boldsymbol{\kappa}^*\sigma = \sigma. \tag{2.8}$$

We will usually write (2.8) as

$$d\eta \wedge dy = d\xi \wedge dx. \tag{2.9}$$

NOTATION. Here the *pull-back* $\boldsymbol{\kappa}^*$ of the symplectic product is defined by

$$(\boldsymbol{\kappa}^*\sigma)(u,v) := \sigma(\boldsymbol{\kappa}_*(u), \boldsymbol{\kappa}_*(v)),$$

$\boldsymbol{\kappa}_*$ denoting the *push-forward* of vectors.

EXAMPLE 1: Lifting diffeomorphisms. Let

$$x \mapsto y = \mathbf{y}(x) \tag{2.10}$$

be a diffeomorphism, with nondegenerate Jacobian matrix

$$\frac{\partial y}{\partial x} := \left(\frac{\partial y^i}{\partial x_j}\right)_{n\times n}.$$

We propose to extend (2.10) to a symplectomorphism

$$(x,\xi) \mapsto (y,\eta) = (\mathbf{y}(x), \boldsymbol{\eta}(x,\xi)),$$

by "lifting" to the momentum variables. In other words, we want to find a function η so that

$$d\eta \wedge dy = d\xi \wedge dx.$$

Now

$$\begin{cases} d\eta = M\ dx + N\ d\xi, \\ dy = A\ dx \end{cases}$$

for $A := \frac{\partial y}{\partial x}$, $M := \frac{\partial \eta}{\partial x}$, $N := \frac{\partial \eta}{\partial \xi}$. Therefore

$$\begin{aligned} d\eta \wedge dy &= (M\ dx + N\ d\xi) \wedge (A\ dx) \\ &= (N\ d\xi \wedge A\ dx) + (M\ dx \wedge A\ dx) \\ &= (d\xi \wedge (N^T A)dx) + (dx \wedge (M^T A)dx). \end{aligned}$$

We need to construct $\boldsymbol{\eta}$ so that both

$$\text{(i) } N^T A = I \quad \text{and} \quad \text{(ii) } M^T A \text{ is symmetric,} \tag{2.11}$$

the latter condition implying, as we will see, that the term $dx \wedge (M^T A) dx$ vanishes. To do so, let us define

$$\boldsymbol{\eta}(x, \xi) := \left[\left(\frac{\partial y}{\partial x} \right)^{-1} \right]^T \xi. \tag{2.12}$$

Then $N^T = \left(\frac{\partial \eta}{\partial \xi} \right)^T = \left(\frac{\partial y}{\partial x} \right)^{-1}$ and so (i) holds. A direct calculation confirms (ii). Since $M^T A$ is symmetric and $dx_i \wedge dx_j = 0$ for all i, j, we deduce that $d\eta \wedge dy = d\xi \wedge dx$, as desired. $\square$

INTERPRETATION: This example will be useful later, when we quantize symbols in Chapter 4 and learn that the operator

$$P(h) = -h^2 \Delta \tag{2.13}$$

is associated with the symbol $p(x, \xi) = |\xi|^2$. If we change variables $y = y(x)$, then we can ask how

$$\Delta_x = \langle \partial_x, \partial_x \rangle$$

transforms. Now $\partial_x = \left(\frac{\partial y}{\partial x} \right)^T \partial_y$ and so

$$\Delta_x = \langle \partial_x, \partial_x \rangle = \left\langle \left(\frac{\partial y}{\partial x} \right)^T \partial y, \left(\frac{\partial y}{\partial x} \right)^T \partial y \right\rangle.$$

Hence

$$-h^2 \Delta_x = -h^2 \ \left(\frac{\partial y}{\partial x} \right)^T \partial_y \left(\left(\frac{\partial y}{\partial x} \right)^T \partial_y \right).$$

We will see later that this operator is associated with the symbol

$$\left\langle \left(\frac{\partial y}{\partial x} \right)^T \eta, \left(\frac{\partial y}{\partial x} \right)^T \eta \right\rangle.$$

And this is consistent with the transformation (2.12) discovered in Example 1.

Here is an instance of another general principle: " if we change variables in a symbol, we preserve the principal symbol, modulo lower order terms". $\square$

EXAMPLE 2: Generating functions. Our next example demonstrates that we can, locally at least, build a symplectic transformation from a real-valued *generating function.*

So suppose $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\phi = \phi(x, y)$, is smooth. Assume also that

$$\det(\partial^2_{xy}\phi(x_0, y_0)) \neq 0.$$

Define

$$\xi = \partial_x\phi, \;\; \eta = -\partial_y\phi, \tag{2.14}$$

and observe that the Implicit Function Theorem implies that (y, η) is a smooth function of (x, ξ) near $(x_0, \partial_x\phi(x_0, y_0))$.

THEOREM 2.3 (Generating functions and symplectic maps). *The mapping* $\boldsymbol{\kappa}$ *defined by*

$$(x, \partial_x\phi(x, y)) \mapsto \big(y, -\partial\phi_y(x, y)\big) \tag{2.15}$$

is a symplectomorphism near (x_0, ξ_0).

Proof. We compute

$$\begin{aligned} d\eta \wedge dy &= d(-\partial_y\phi) \wedge dy \\ &= [(-\partial^2_y\phi dy) \wedge dy] + [(-\partial^2_{xy}\phi dx) \wedge dy] \\ &= -(\partial^2_{xy}\phi) dx \wedge dy, \end{aligned}$$

since $\partial^2_y S$ is symmetric. Likewise,

$$\begin{aligned} d\xi \wedge dx &= d(\partial_x\phi) \wedge dx \\ &= [(\partial^2_x\phi \; dx) \wedge dx] + [(\partial^2_{xy}\phi \; dy) \wedge dx] \\ &= -(\partial^2_{xy}\phi) dx \wedge dy = d\eta \wedge dy. \end{aligned}$$

□

TERMINOLOGY. The word "symplectic" means "intertwined" in Greek and this nomenclature is motivated by Example 2. The generating function $\phi = \phi(x, y)$ is a function of a mixture of half of the original variables (x, ξ) and half of the new variables (y, η).

APPLICATION: Lagrangian submanifolds. A *Lagrangian submanifold* Λ of $\mathbb{R}^{2n}$ is defined by the property that

$$\sigma|_\Lambda = 0.$$

Then

$$d\omega|_\Lambda = \sigma|_\Lambda = 0;$$

and so

$$\omega = d\phi$$

according to Poincaré's Theorem C.3. We will exploit this observation in Section 10.2. □

2.4 HAMILTONIAN VECTOR FIELDS

DEFINITION. If $f \in C^\infty(\mathbb{R}^{2n})$, we define the corresponding *Hamiltonian vector field* by requiring

$$\sigma(u, H_f) = df(u). \tag{2.16}$$

This is well defined, since σ is nondegenerate. We can write explicitly that

$$H_f = \langle \partial_\xi f, \partial_x \rangle - \langle \partial_x f, \partial_\xi \rangle = \sum_{j=1}^n \frac{\partial f}{\partial \xi_j}\partial_{x_j} - \frac{\partial f}{\partial x_j}\partial_{\xi_j}. \tag{2.17}$$

LEMMA 2.4 (Differentials and Hamiltonian vector fields). *We have the relation*

$$df = -(H_f \lrcorner \sigma), \tag{2.18}$$

for the contraction $\lrcorner$ *defined in Appendix C.*

Proof. We calculate for each vector u that

$$(H_f \lrcorner \sigma)(u) = \sigma(H_f, u) = -\sigma(u, H_f) = -df(u).$$

□

DEFINITION. If $f, g \in C^\infty(\mathbb{R}^{2n})$, we define their *Poisson bracket*

$$\{f, g\} := H_f g = \sigma(H_f, H_g). \tag{2.19}$$

That is,

$$\{f, g\} = \langle \partial_\xi f, \partial_x g \rangle - \langle \partial_x f, \partial_\xi g \rangle = \sum_{j=1}^n \frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j}. \tag{2.20}$$

LEMMA 2.5 (Brackets and commutators). *We have the indentity*

$$H_{\{f,g\}} = [H_f, H_g]. \tag{2.21}$$

Proof. Calculate. □

THEOREM 2.6 (Jacobi's Theorem). *If $\boldsymbol{\kappa}$ is a symplectomorphism, then*

$$\boldsymbol{\kappa}^*(H_f) = H_{\boldsymbol{\kappa}^* f}. \tag{2.22}$$

In other words, the pull-back of a Hamiltonian vector field generated by f is the Hamiltonian vector field generated by the pull-back of f.

Proof.

$$\begin{aligned} \boldsymbol{\kappa}^*(H_f) \lrcorner \sigma &= \boldsymbol{\kappa}^*(H_f) \lrcorner \boldsymbol{\kappa}^* \sigma \\ &= \boldsymbol{\kappa}^*(H_f \lrcorner \sigma) \\ &= -\boldsymbol{\kappa}^*(df) = -d(\boldsymbol{\kappa}^* f) \\ &= H_{\boldsymbol{\kappa}^* f} \lrcorner \sigma. \end{aligned}$$

Since σ is nondegenerate, (2.22) follows. □

EXAMPLE. Define $\boldsymbol{\kappa} = J$; so that $\boldsymbol{\kappa}(x, \xi) = (-\xi, x)$. We will later in Section 10.2 interpret this transformation as the classical analog of the Fourier transform.

Observe that $\boldsymbol{\kappa}$ a symplectomorphism, since

$$d\xi \wedge dx = dx \wedge d(-\xi).$$

We explicitly compute that

$$\begin{aligned} H_f &= \langle \partial_\xi f, \partial_x \rangle - \langle \partial_x f, \partial_\xi \rangle \\ &= \langle \partial_x f, \partial_{-\xi} \rangle - \langle \partial_{-\xi} f, \partial_x \rangle = H_{\boldsymbol{\kappa}^* f}. \end{aligned}$$

□

THEOREM 2.7 (Hamiltonian flows and symplectomorphisms). *If f is smooth, then for each time t, the mapping*

$$(x, \xi) \mapsto \Phi_t(x, \xi) = \exp(tH_f(x, \xi))$$

is a symplectomorphism.

Proof. According to Cartan's formula (Theorem C.2), we have

$$\frac{d}{dt}((\Phi_t)^* \sigma) = \mathcal{L}_{H_f} \sigma = d(H_f \lrcorner \sigma) + (H_f \lrcorner d\sigma).$$

Since $d\sigma = 0$, it follows that

$$\frac{d}{dt}((\Phi_t)^* \sigma) = d(-df) = -d^2 f = 0.$$

Thus $(\Phi_t)^* \sigma = \sigma$ for all times t. □

THEOREM 2.8 (Darboux's Theorem). *Let U be a neighborhood of $(0,0)$ and suppose η is a nondegenerate 2-form defined on U, satisfying*

$$d\eta = 0.$$

Then near $(0,0)$ there exists a diffeomorphism $\boldsymbol{\kappa}$ such that

$$\boldsymbol{\kappa}^*\eta = \sigma. \tag{2.23}$$

INTERPRETATION. The assertion is that, locally, all symplectic structures are identical, in the sense that all are equivalent to that generated by σ. □

Proof. 1. We first find a linear mapping L so that

$$L^*\eta(0,0) = \sigma(0,0).$$

This means that we find a basis $\{e_k, f_k\}_{k=1}^n$ of $\mathbb{R}^{2n}$ such that

$$\begin{cases} \eta(f_l, e_k) = \delta_{kl} \\ \eta(e_k, e_l) = 0 \\ \eta(f_k, f_l) = 0 \end{cases}$$

for all $1 \le k, l \le n$. Then if $u = \sum_{i=1}^n x_i e_i + \xi_i f_i$, $v = \sum_{j=1}^n y_j e_j + \eta_j f_j$, we have

$$\begin{aligned} &\eta(u,v) \\ &\quad = \sum_{i,j=1}^n x_i y_j \eta(e_i, e_j) + \xi_i \eta_j \eta(f_i, f_j) + x_i \eta_j \sigma(e_i, f_j) + \xi_i y_j \sigma(f_i, e_j) \\ &\quad = \langle \xi, y\rangle - \langle x, \eta\rangle = \sigma(u,v). \end{aligned}$$

2. Next, define $\eta_t := t\eta + (1-t)\sigma$ for $0 \le t \le 1$. Our intention is to find $\boldsymbol{\kappa}_t$ so that $\boldsymbol{\kappa}_t^*\eta_t = \sigma$ near $(0,0)$; then $\boldsymbol{\kappa} := \boldsymbol{\kappa}_1$ solves our problem. We will construct $\boldsymbol{\kappa}_t$ by solving the flow

$$\begin{cases} \dot{\mathbf{m}}(t) = V_t(\mathbf{m}(t)) \qquad (0 \le t \le 1) \\ \mathbf{m}(0) = m, \end{cases} \tag{2.24}$$

and setting $\boldsymbol{\kappa}_t := \Phi_t$.

For this to work, we must design the vector fields V_t in (2.24) so that $\frac{d}{dt}(\boldsymbol{\kappa}_t^*\eta_t) = 0$. Let us therefore calculate

$$\begin{aligned} \frac{d}{dt}(\boldsymbol{\kappa}_t^*\eta_t) &= \boldsymbol{\kappa}_t^*\left(\frac{d}{dt}\eta_t\right) + \boldsymbol{\kappa}_t^*\mathcal{L}_{V_t}\eta_t \\ &= \boldsymbol{\kappa}_t^*[(\eta - \sigma) + d(V_t \lrcorner \eta_t) + V_t \lrcorner d\eta_t], \end{aligned}$$

where we used Cartan's formula, Theorem C.2.

Note that $d\eta_t = t d\eta + (1-t)d\sigma$. Hence $\frac{d}{dt}(\boldsymbol{\kappa}_t^* \eta_t) = 0$ provided

$$(\eta - \sigma) + d(V_t \lrcorner \eta_t) = 0. \tag{2.25}$$

According to Poincaré's Lemma, Theorem C.3, we can write

$$\eta - \sigma = d\alpha \quad \text{near } (0,0).$$

So (2.25) will hold, provided

$$V_t \lrcorner \eta_t = -\alpha \qquad (0 \leq t \leq 1). \tag{2.26}$$

Using the nondegeneracy of η and σ, we can solve this equation for the vector field V_t. $\square$

3. Fourier transform, stationary phase

We discuss in this chapter how to define the Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ on various classes of smooth functions and nonsmooth distributions. We introduce also the rescaled semiclassical transforms $\mathcal{F}_h, \mathcal{F}_h^{-1}$ depending on the small parameter h, and develop stationary phase asymptotics to help us understand various formulas involving $\mathcal{F}_h$ in the limit as $h \to 0$.

3.1 FOURIER TRANSFORM ON $\mathcal{S}$

We begin by defining and studying the Fourier transform of smooth functions that decay rapidly as $|x| \to \infty$.

DEFINITIONS (i) The *Schwartz space* is

$$\mathcal{S} = \mathcal{S}(\mathbb{R}^n) := \{\phi \in C^\infty(\mathbb{R}^n) \mid \sup_{\mathbb{R}^n} |x^\alpha \partial^\beta \phi| < \infty \text{ for all multiindices } \alpha, \beta\}.$$

(ii) We say

$$\phi_j \to \phi \quad \text{in } \mathcal{S}$$

provided

$$\sup_{\mathbb{R}^n} |x^\alpha \partial^\beta (\phi_j - \phi)| \to 0$$

for all multiindices α, β.

DEFINITION. If $\phi \in \mathcal{S}$, define the *Fourier transform*

$$\mathcal{F}\phi(\xi) = \hat{\phi}(\xi) := \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} \phi(x)\, dx \qquad (\xi \in \mathbb{R}^n). \tag{3.1}$$

The Fourier transform $\mathcal{F}$ in effect lets us move from the position variables x to the momentum variables ξ.

EXAMPLE: Exponential of a real quadratic form.

LEMMA 3.1 (Transform of a real exponential). *Let A be a real, symmetric, positive definite $n \times n$ matrix. Then*

$$\mathcal{F}(e^{-\langle Ax,x\rangle}) = \frac{\pi^{n/2}}{(det\ A)^{1/2}} e^{-\frac{1}{4}\langle A^{-1}\xi,\xi\rangle}. \tag{3.2}$$

We can of course replace A by $\frac{1}{2}A$, to derive the equivalent formula

$$\mathcal{F}(e^{-\frac{1}{2}\langle Ax,x\rangle}) = \frac{(2\pi)^{n/2}}{(\det\ A)^{1/2}} e^{-\frac{1}{2}\langle A^{-1}\xi,\xi\rangle}, \tag{3.3}$$

which we will need later.

Proof. Let us calculate

$$\begin{aligned} \mathcal{F}(e^{-\langle Ax,x\rangle}) &= \int_{\mathbb{R}^n} e^{-\langle Ax,x\rangle - i\langle x,\xi\rangle} dx \\ &= \int_{\mathbb{R}^n} e^{-\langle A(x+\frac{iA^{-1}\xi}{2}),\, x+\frac{iA^{-1}\xi}{2}\rangle} e^{-\frac{1}{4}\langle A^{-1}\xi,\xi\rangle} dx \\ &= e^{-\frac{1}{4}\langle A^{-1}\xi,\xi\rangle} \int_{\mathbb{R}^n} e^{-\langle Ay,y\rangle} dy. \end{aligned}$$

We compute the last integral by making an orthogonal change of variables that converts A into diagonal form $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Then

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-\langle Ay,y\rangle} dy &= \int_{\mathbb{R}^n} e^{-\sum_{k=1}^n \lambda_k w_k^2}\, dw = \prod_{k=1}^n \int_{-\infty}^{\infty} e^{-\lambda_k w^2} dw \\ &= \prod_{k=1}^n \frac{1}{\lambda_k^{1/2}} \int_{-\infty}^{\infty} e^{-y^2} dy \\ &= \frac{\pi^{n/2}}{(\lambda_1 \cdots \lambda_n)^{1/2}} = \frac{\pi^{n/2}}{(\det\ A)^{1/2}}. \end{aligned}$$

□

THEOREM 3.2 (Properties of Fourier transform).

(i) *The mapping $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is an isomorphism.*

(ii) *We have*

$$\mathcal{F}^{-1} = \frac{1}{(2\pi)^n} R \circ \mathcal{F}, \tag{3.4}$$

where $Rf(x) := f(-x)$. In other words,

$$\mathcal{F}^{-1}\psi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \psi(\xi)\, d\xi; \tag{3.5}$$

and therefore

$$\phi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \hat{\phi}(\xi)\, d\xi. \tag{3.6}$$

(iii) *In addition,*

$$D_\xi^\alpha(\mathcal{F}\phi) = \mathcal{F}((-x)^\alpha \phi) \tag{3.7}$$

and

$$\mathcal{F}(D_x^\alpha \phi) = \xi^\alpha \mathcal{F}\phi. \tag{3.8}$$

(iv) *Furthermore,*

$$\mathcal{F}(\phi\psi) = \frac{1}{(2\pi)^n} \mathcal{F}(\phi) * \mathcal{F}(\psi). \tag{3.9}$$

REMARKS. (i) In these formulas we employ the notation from Appendix A that

$$D^\alpha = \frac{1}{i^{|\alpha|}} \partial^\alpha. \tag{3.10}$$

(ii) We will later interpret the Fourier inversion formula (3.5) as saying that

$$\delta_{\{y=x\}} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle}\, d\xi \quad \text{in } \mathcal{S}', \tag{3.11}$$

δ denoting the Dirac measure. $\square$

Proof. 1. Let us calculate for $\phi \in \mathcal{S}$ that

$$\begin{aligned} D_\xi^\alpha(\mathcal{F}\phi) &= D_\xi^\alpha \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} \phi(x)\, dx = \frac{1}{i^\alpha} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} (-ix)^\alpha \phi(x)\, dx \\ &= \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} (-x)^\alpha \phi(x)\, dx = \mathcal{F}((-x)^\alpha \phi). \end{aligned}$$

Likewise,

$$\begin{aligned} \mathcal{F}(D_x^\alpha \phi) &= \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} D_x^\alpha \phi\, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} D_x^\alpha (e^{-i\langle x,\xi\rangle}) \phi\, dx \\ &= (-1)^{|\alpha|} \int_{\mathbb{R}^n} \frac{1}{i^{|\alpha|}} (-i\xi)^\alpha e^{-i\langle x,\xi\rangle} \phi\, dx = \xi^\alpha (\mathcal{F}\phi). \end{aligned}$$

This proves (iii).

2. Recall from Appendix A the notation

$$\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}.$$

Then for all multiindices α, β, we have

$$\begin{aligned}
\sup_\xi |\xi^\beta D_\xi^\alpha \hat{\phi}| &= \sup_\xi |\xi^\beta \mathcal{F}((-x)^\alpha \phi)| \\
&= \sup_\xi |\mathcal{F}(D_x^\beta((-x)^\alpha \phi)| \\
&= \sup_\xi \left| \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} \frac{1}{\langle x\rangle^{n+1}} \langle x\rangle^{n+1}\ D_x^\beta((-x)^\alpha \phi)\, dx \right| \\
&\leq \sup_x |\langle x\rangle^{n+1} D_x^\beta((-x)^\alpha \phi)| \int_{\mathbb{R}^n} \frac{dx}{\langle x\rangle^{n+1}} < \infty.
\end{aligned}$$

Hence $\mathcal{F} : \mathcal{S} \to \mathcal{S}$, and a similar calculation shows that $\phi_i \to \phi$ in $\mathcal{S}$ implies $\mathcal{F}(\phi_j) \to \mathcal{F}(\phi)$.

3. To show $\mathcal{F}$ is invertible, note that

$$\begin{aligned}
R \circ \mathcal{F} \circ \mathcal{F} \circ D_{x_j} &= R \circ \mathcal{F} \circ M_{\xi_j} \circ \mathcal{F} \\
&= R \circ (-D_{x_j}) \circ \mathcal{F} \circ \mathcal{F} \\
&= D_{x_j} \circ R \circ \mathcal{F} \circ \mathcal{F},
\end{aligned}$$

where M_{ξ_j} denotes multiplication by ξ_j. Thus $R \circ \mathcal{F} \circ \mathcal{F}$ commutes with D_{x_j} and it likewise commutes with the multiplication operator M_λ. According to Lemma 3.3, stated and proved below, $R \circ \mathcal{F} \circ \mathcal{F}$ is a multiple of the identity operator:

$$R \circ \mathcal{F} \circ \mathcal{F} = cI. \tag{3.12}$$

From the example above, we know that

$$\mathcal{F}(e^{-\frac{|x|^2}{2}}) = (2\pi)^{n/2} e^{-\frac{|\xi|^2}{2}}.$$

Thus $\mathcal{F}(e^{-\frac{|\xi|^2}{2}}) = (2\pi)^{n/2} e^{-\frac{|x|^2}{2}}$. Consequently $c = (2\pi)^n$, and hence

$$\mathcal{F}^{-1} = \frac{1}{(2\pi)^n} R \circ \mathcal{F}.$$

4. Lastly, since

$$\phi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \hat{\phi}(\xi)\, d\xi, \quad \psi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\eta\rangle} \hat{\psi}(\eta)\, d\eta,$$

we have

$$\begin{aligned}
\phi\psi &= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi+\eta\rangle} \hat{\phi}(\xi) \hat{\psi}(\eta)\, d\xi d\eta \\
&= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n} e^{i\langle x,\rho\rangle} \left(\int_{\mathbb{R}^n} \hat{\phi}(\xi) \hat{\psi}(\rho - \xi)\, d\rho \right) d\xi \\
&= \frac{1}{(2\pi)^n} \mathcal{F}^{-1}(\hat{\phi} * \hat{\psi}).
\end{aligned}$$

But $\phi\psi = \mathcal{F}^{-1}\mathcal{F}(\phi\psi)$, and so assertion (iv) follows. $\square$

LEMMA 3.3 (Commutativity). *Suppose $L : \mathcal{S} \to \mathcal{S}$ is linear, and*

$$L \circ M_\lambda = M_\lambda \circ L, \;\; L \circ D_{x_j} = D_{x_j} \circ L \tag{3.13}$$

for $\lambda \in \mathbb{R}$ and $j = 1, \ldots, n$. Then

$$L = cI$$

for some constant c, where I denotes the identity operator.

Proof. 1. Choose $\phi \in \mathcal{S}$, fix $y \in \mathbb{R}^n$, and write

$$\phi(x) - \phi(y) = \sum_{j=1}^n (x_j - y_j)\psi_j(x)$$

for

$$\psi_j(x) := \int_0^1 \phi_{x_j}(y + t(x-y))\, dt.$$

Since possibly $\psi_j \notin \mathcal{S}$, we select a smooth function χ with compact support such that $\chi \equiv 1$ for x near y. Write

$$\phi_j(x) := \chi(x)\psi_j(x) + \frac{(x_j - y_j)}{|x-y|^2}(1 - \chi(x))\phi(x).$$

Then

$$\phi(x) - \phi(y) = \sum_{j=1}^n (x_j - y_j)\phi_j(x) \tag{3.14}$$

with $\phi_j \in \mathcal{S}$.

2. We claim next that if $\phi(y) = 0$, then $L\phi(y) = 0$. This follows from (3.14), since

$$L\phi(x) = \sum_{j=1}^n (x_j - y_j)L\phi_j = 0$$

at $x = y$.

Therefore $L\phi(x) = c(x)\phi(x)$ for some function c. Taking $\phi(x) = e^{-|x|^2}$, we deduce that $c \in C^\infty$. Finally, since L commutes with differentiation, we conclude that c must be a constant. $\square$

THEOREM 3.4 (Integral identities). *If $\phi, \psi \in \mathcal{S}$, then*

$$\int_{\mathbb{R}^n} \hat{\phi}\psi \; dx = \int_{\mathbb{R}^n} \phi\hat{\psi} \; dy \tag{3.15}$$

and

$$\int_{\mathbb{R}^n} \phi\bar{\psi} \; dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{\phi}\bar{\hat{\psi}} \; d\xi. \tag{3.16}$$

In particular,

$$\|\phi\|_{L^2}^2 = \frac{1}{(2\pi)^n} \|\hat{\phi}\|_{L^2}^2. \tag{3.17}$$

Proof. Note first that

$$\begin{aligned}
\int_{\mathbb{R}^n} \hat{\phi}\psi \; dx &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{-i\langle x,y\rangle} \phi(y) \; dy \right) \psi(x) \; dx \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{-i\langle y,x\rangle} \psi(x) \; dx \right) \phi(y) \; dy = \int_{\mathbb{R}^n} \hat{\psi}\phi \; dy.
\end{aligned}$$

Replace ψ by $\bar{\hat{\psi}}$ in (3.15):

$$\int_{\mathbb{R}^n} \hat{\phi}\bar{\hat{\psi}} \; d\xi = \int_{\mathbb{R}^n} \phi(\bar{\hat{\psi}})^{\wedge} \; dx.$$

But $\bar{\hat{\psi}} = \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \bar{\psi}(x) \; dx = (2\pi)^n \mathcal{F}^{-1}(\bar{\psi})$ and so $(\bar{\hat{\psi}})^{\wedge} = (2\pi)^n \bar{\psi}$. $\square$

We record next some elementary estimates that we will need later:

LEMMA 3.5 (Useful estimates).
(i) *We have the bounds*

$$\|\hat{u}\|_{L^\infty} \leq \|u\|_{L^1} \tag{3.18}$$

and

$$\|u\|_{L^\infty} \leq \frac{1}{(2\pi)^n} \|\hat{u}\|_{L^1}. \tag{3.19}$$

(ii) *There exists a constant C such that*

$$\|\hat{u}\|_{L^1} \leq C \sup_{|\alpha| \leq n+1} \|\partial^\alpha u\|_{L^1}. \tag{3.20}$$

Proof. Estimates (3.18) and (3.19) follow easily from (3.1) and (3.6). Then

$$\begin{aligned}
\|\hat{u}\|_{L^1} &= \int_{\mathbb{R}^n} |\hat{u}|\langle\xi\rangle^{n+1}\langle\xi\rangle^{-n-1} \; d\xi \; \leq C \sup_{\xi}(|\hat{u}|\langle\xi\rangle^{n+1}) \\
&\leq C \sup_{|\alpha| \leq n+1} |\xi^\alpha \hat{u}| = C \sup_{|\alpha| \leq n+1} |(\partial^\alpha u)^{\wedge}| \leq C \sup_{|\alpha| \leq n+1} \|\partial^\alpha u\|_{L^1}.
\end{aligned}$$

This proves (3.20). $\square$

We close this section with an application showing that we can sometimes use the Fourier transform to solve PDE with variable coefficients.

EXAMPLE: Solving a PDE. Consider the initial-value problem

$$\begin{cases} \partial_t u = x\partial_y u + \partial_x^2 u & \text{on } \mathbb{R}^2 \times (0,\infty) \\ u = \delta_{(x_0,y_0)} & \text{on } \mathbb{R}^2 \times \{t=0\}. \end{cases}$$

Let $\hat{u} := \mathcal{F}u$ denote the Fourier transform of u in the variables x, y (but not in t). Then

$$(\partial_t + \eta\partial_\xi)\hat{u} = -\xi^2\hat{u}.$$

This is a linear first-order PDE we can solve by characteristics:

$$\begin{aligned} \hat{u}(t,\xi+t\eta,\eta) &= \hat{u}(0,\xi,\eta)e^{-\int_0^t(\xi+s\eta)^2ds} \\ &= \hat{u}(0,\xi,\eta)e^{-\xi^2t-\xi\eta t^2-\frac{\eta^2t^3}{3}} \\ &= \hat{u}(0,\xi,\eta)e^{-\frac{1}{2}\langle B_t(\xi,\eta),(\xi,\eta)\rangle} \end{aligned}$$

for

$$B_t := \begin{pmatrix} 2t & t^2 \\ t^2 & \frac{2t^3}{3} \end{pmatrix}.$$

Furthermore, $\hat{u}(0,\xi,\eta) = \hat{\delta}_{(x_0,y_0)}$. Taking $\mathcal{F}^{-1}$, we find

$$\begin{aligned} u(t,x,y-tx) &= \delta_{(x_0,y)} * \mathcal{F}^{-1}(e^{-\frac{1}{2}\langle B_t(\xi,\eta),(\xi,\eta)\rangle}) \\ &= \frac{\sqrt{3}}{2\pi t^3}\exp(-\frac{(x-x_0)^2}{t} + \frac{3(x-x_0)(y-y_0)}{t^2} - \frac{3(y-y_0)^2}{t^3}); \end{aligned}$$

and hence

$$\begin{aligned} &u(t,x,y) \\ &= \frac{\sqrt{3}}{2\pi t^3}\exp(-\frac{(x-x_0)^2}{t} + \frac{3(x-x_0)(y+tx-y_0)}{t^2} - \frac{3(y+tx-y_0)^2}{t^3}). \end{aligned}$$

$\square$

3.2 FOURIER TRANSFORM ON $\mathcal{S}'$

Next we extend the Fourier transform to $\mathcal{S}'$, the dual space of $\mathcal{S}$. We will then be able to study the Fourier transforms of various interesting, but nonsmooth, expressions.

DEFINITIONS.

(i) We write $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ for the space of *tempered distributions*, which is the dual of $\mathcal{S}$. That is, $u \in \mathcal{S}'$ provided $u : \mathcal{S} \to \mathbb{C}$ is linear and $\phi_j \to \phi$ in $\mathcal{S}$ implies $u(\phi_j) \to u(\phi)$.

(ii) We say

$$u_j \to u \quad \text{in } \mathcal{S}'$$

if

$$u_j(\phi) \to u(\phi) \quad \text{for all } \phi \in \mathcal{S}.$$

DEFINITION. If $u \in \mathcal{S}'$, we define

$$D^\alpha u, x^\alpha u, \mathcal{F}u \in \mathcal{S}'$$

by the rules

$$\begin{aligned} D^\alpha u(\phi) &:= (-1)^{|\alpha|} u(D^\alpha \phi) \\ (x^\alpha u)(\phi) &:= u(x^\alpha \phi) \\ (\mathcal{F}u)(\phi) &:= u(\mathcal{F}\phi) \end{aligned}$$

for $\phi \in \mathcal{S}$.

EXAMPLE 1: Dirac measure. It follows from the definitions that

$$\hat{\delta}_0(\phi) = \delta_0(\hat{\phi}) = \hat{\phi}(0) = \int_{\mathbb{R}^n} \phi \, dx.$$

We interpret this calculation as saying that

$$\hat{\delta}_0 \equiv 1 \quad \text{in } \mathbb{R}^n. \tag{3.21}$$

□

EXAMPLE 2: Exponential of an imaginary quadratic form. The *signature* of a real, symmetric, nonsingular matrix Q is

$$\begin{aligned} \operatorname{sgn} Q := {} & \text{number of positive eigenvalues of } Q \\ & - \text{number of negative eigenvalues of } Q. \end{aligned} \tag{3.22}$$

LEMMA 3.6 (Transform of an imaginary exponential). *Let Q be a real, symmetric, nonsingular $n \times n$ matrix. Then*

$$\mathcal{F}\left(e^{\frac{i}{2}\langle Qx, x\rangle}\right) = \frac{(2\pi)^{n/2} e^{\frac{i\pi}{4} \operatorname{sgn}(Q)}}{|\det Q|^{1/2}} e^{-\frac{i}{2}\langle Q^{-1}\xi, \xi\rangle}. \tag{3.23}$$

Compare this carefully with the earlier formula (3.3). The extra *phase shift* term $e^{\frac{i\pi}{4} \operatorname{sgn} Q}$ in (3.23) arises from the complex exponential.

Proof. 1. Let $\epsilon > 0$, $Q_\epsilon := Q + \epsilon i I$. Then

$$\begin{aligned}
\mathcal{F}\left(e^{\frac{i}{2}\langle Q_\epsilon x, x\rangle}\right) &= \int_{\mathbb{R}^n} e^{\frac{i}{2}\langle Q_\epsilon x, x\rangle - i\langle x, \xi\rangle}\, dx \\
&= \int_{\mathbb{R}^n} e^{\frac{i}{2}\langle Q_\epsilon (x - Q_\epsilon^{-1}\xi), x - Q_\epsilon^{-1}\xi\rangle} e^{-\frac{i}{2}\langle Q_\epsilon^{-1}\xi, \xi\rangle}\, dx \\
&= e^{-\frac{i}{2}\langle Q_\epsilon^{-1}\xi, \xi\rangle} \int_{\mathbb{R}^n} e^{\frac{i}{2}\langle Q_\epsilon y, y\rangle}\, dy.
\end{aligned}$$

Now change variables, to write Q in the form $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$, with $\lambda_1, \dots, \lambda_r > 0$ and $\lambda_{r+1}, \dots, \lambda_n < 0$. Then

$$\int_{\mathbb{R}^n} e^{\frac{i}{2}\langle Q_\epsilon y, y\rangle}\, dy = \int_{\mathbb{R}^n} e^{\sum_{k=1}^n \frac{1}{2}(i\lambda_k - \epsilon) w_k^2}\, dw = \prod_{k=1}^n \int_{-\infty}^{\infty} e^{\frac{1}{2}(i\lambda_k - \epsilon) w^2}\, dw.$$

2. If $1 \le k \le r$, then $\lambda_k > 0$ and we set $z = (\epsilon - i\lambda_k)^{1/2} w$, and we take the branch of the square root so that $\operatorname{Im}(\epsilon - i\lambda_k)^{1/2} < 0$. Then

$$\int_{-\infty}^{\infty} e^{\frac{1}{2}(i\lambda_k - \epsilon) w^2}\, dw = \frac{1}{(\epsilon - i\lambda_k)^{1/2}} \int_{\Gamma_k} e^{-\frac{z^2}{2}}\, dz,$$

for the contour Γ_k as drawn.

Since $e^{-\frac{z^2}{2}} = e^{\frac{y^2 - x^2}{2} - ixy}$ and $x^2 > y^2$ on Γ_k, we can deform Γ_k into the real axis.

Hence

$$\int_{\Gamma_k} e^{-\frac{z^2}{2}}\, dz = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}.$$

Thus

$$\prod_{k=1}^r \int_{-\infty}^{\infty} e^{\frac{1}{2}(i\lambda_k - \epsilon) w^2} dw = (2\pi)^{r/2} \prod_{k=1}^r \frac{1}{(\epsilon - i\lambda_k)^{1/2}}.$$

Also for $1 \le k \le r$:

$$\lim_{\epsilon \to 0^+} \frac{1}{(\epsilon - i\lambda_k)^{1/2}} = \frac{1}{(-i)^{1/2} \lambda_k^{1/2}} = \frac{e^{\frac{i\pi}{4}}}{\lambda_k^{1/2}},$$

since we take the branch of the square root with $(-i)^{1/2} = e^{-i\pi/4}$.

3. Similarly for $r + 1 \le k \le n$, we set $z = (\epsilon - i\lambda_k)^{1/2} w$, but now take the branch of square root with $\operatorname{Im}(\epsilon - i\lambda_k)^{1/2} > 0$. Hence

$$\prod_{k=r+1}^n \int_{-\infty}^{\infty} e^{\frac{1}{2}(i\lambda_k - \epsilon) w^2}\, dw = (2\pi)^{\frac{n-r}{2}} \prod_{k=r+1}^n \frac{1}{(\epsilon - i\lambda_k)^{1/2}};$$

and for $r+1 \le k \le n$

$$\lim_{\epsilon\to 0^+} \frac{1}{(\epsilon - i\lambda_k)^{1/2}} = \frac{1}{(-i\lambda_k)^{1/2}} = \frac{e^{-\frac{i\pi}{4}}}{|\lambda_k|^{1/2}},$$

since we take the branch of the square root with $i^{1/2} = e^{\frac{i\pi}{4}}$.

4. Combining the foregoing calculations gives us

$$\begin{aligned} \mathcal{F}\left(e^{\frac{i}{2}\langle Qx,x\rangle}\right) &= \lim_{\epsilon\to 0} \mathcal{F}\left(e^{\frac{i}{2}\langle Q_\epsilon x,x\rangle}\right) \\ &= e^{-\frac{i}{2}\langle Q^{-1}\xi,\xi\rangle} \frac{(2\pi)^{n/2} e^{\frac{i\pi}{4}(r-(n-r))}}{|\lambda_1\lambda_2\dots\lambda_n|^{1/2}} \\ &= e^{-\frac{i}{2}\langle Q^{-1}\xi,\xi\rangle} \frac{(2\pi)^{n/2} e^{\frac{i\pi}{4}\operatorname{sgn} Q}}{|\det Q|^{1/2}}. \end{aligned}$$

□

3.3 SEMICLASSICAL FOURIER TRANSFORM

We will later need for $h > 0$ the *semiclassical Fourier transform*

$$\hat{\phi}(\xi) = \mathcal{F}_h \phi(\xi) := \int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle x,\xi\rangle} \phi(x)\,dx \tag{3.24}$$

and its inverse

$$\mathcal{F}_h^{-1}\psi(x) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x,\xi\rangle} \psi(\xi)\,d\xi. \tag{3.25}$$

We record for future reference some formulas involving the parameter h:

THEOREM 3.7 (Properties of semiclassical Fourier transform). *We have*

$$(hD_\xi)^\alpha \mathcal{F}_h \phi = \mathcal{F}_h((-x)^\alpha \phi), \tag{3.26}$$

$$\mathcal{F}_h((hD_x)^\alpha \phi) = \xi^\alpha \mathcal{F}_h \phi, \tag{3.27}$$

and

$$\|\phi\|_{L^2} = \frac{1}{(2\pi h)^{n/2}} \|\mathcal{F}_h \phi\|_{L^2}. \tag{3.28}$$

Consequently

$$\delta_{\{y=x\}} = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle}\,d\xi \quad \text{in } \mathcal{S}'. \tag{3.29}$$

The reader should compare this identity with the unscaled form (3.11).

THEOREM 3.8 (Uncertainty principle). *We have*

$$\frac{h}{2}\|f\|_{L^2}\,\|\mathcal{F}_h f\|_{L^2} \leq \|x_j f\|_{L^2}\,\|\xi_j \mathcal{F}_h f\|_{L^2} \quad (j = 1, \cdots, n). \tag{3.30}$$

The uncertainty principle in its various guises limits the extent to which we can simultaneously "localize" our calculations in both the x and ξ variables.

Proof. To see this, note first that

$$\xi_j \mathcal{F}_h f(\xi) = \mathcal{F}_h(hD_{x_j} f).$$

Also, if A, B are self-adjoint opertors, then

$$\operatorname{Im}\langle Af, Bf\rangle = \frac{1}{2i}\langle [B, A]f, f\rangle.$$

Let $A = hD$, $B = x$. Therefore

$$[x, hD]f = \frac{h}{i}[\langle x, \partial f\rangle - \partial(xf)] = inhf.$$

Thus

$$\begin{aligned}
\|x_j f\|_{L^2}\,\|\xi_j \mathcal{F}_h f\|_{L^2} &= \|x_j f\|_{L^2}\,\|\mathcal{F}_h(hD_{x_j} f)\|_{L^2} \\
&= (2\pi h)^{n/2}\|x_j f\|_{L^2}\,\|hD_{x_j} f\|_{L^2} \\
&\geq (2\pi h)^{n/2}|\langle hD_{x_j} f, x_j f\rangle| \\
&\geq (2\pi h)^{n/2}|\operatorname{Im}\langle hD_{x_j} f, x_j f\rangle| \\
&= \frac{(2\pi h)^{n/2}}{2}|\langle [x_j, hD_{x_j}]f, f\rangle| \\
&= \frac{(2\pi h)^{n/2}}{2} h\|f\|_{L^2}^2 \\
&= \frac{h}{2}\|f\|_{L^2}\,\|\mathcal{F}_h f\|_{L^2}.
\end{aligned}$$

□

3.4 STATIONARY PHASE IN ONE DIMENSION

Understanding the right hand side of (3.24) in the limit $h \to 0$ requires our studying integral expressions with rapidly oscillating integrands.

We start with one dimensional problems.

DEFINITION. Define for $h > 0$ the *oscillatory integral*

$$I_h = I_h(a, \phi) := \int_{-\infty}^{\infty} e^{\frac{i\phi}{h}} a\, dx, \tag{3.31}$$

where $a \in C_c^\infty(\mathbb{R})$, $\phi \in C^\infty(\mathbb{R})$.

LEMMA 3.9 (Rapid decay). *If $\phi' \neq 0$ on $K := \operatorname{spt}(a)$, then*

$$I_h = O(h^\infty) \quad \text{as } h \to 0. \tag{3.32}$$

NOTATION. The identity (3.32) means that for each positive integer N, there exists a constant C_N such that

$$|I_h| \leq C_N h^N \quad \text{for all } 0 < h \leq 1. \tag{3.33}$$

Proof. We will in effect integrate by parts N times to achieve (3.33). For this, observe that the operator

$$L := \frac{h}{i}\frac{1}{\phi'(x)}\partial_x$$

is defined for $x \in K$, since $\phi' \neq 0$ there. Notice also that

$$L\left(e^{\frac{i\phi}{h}}\right) = e^{\frac{i\phi}{h}}.$$

Hence $L^N(e^{i\phi/h}) = e^{i\phi/h}$, for $N = 1, 2, \ldots$. Consequently

$$|I_h| = \left|\int_{-\infty}^{\infty} L^N\left(e^{\frac{i\phi}{h}}\right) a\, dx\right| = \left|\int_{-\infty}^{\infty} e^{i\phi/h}(L^*)^N a\, dx\right|,$$

L^* denoting the adjoint of L. Since a is smooth, $L^* a = -\frac{h}{i}\partial_x\left(\frac{a}{\phi'}\right)$ is of order h. Therefore we deduce that $|I_h| \leq C_N h^N$. $\square$

THEOREM 3.10 (Stationary phase asymptotics). *Let $a \in C_c^\infty(\mathbb{R})$. Suppose $x_0 \in K = \operatorname{spt}(a)$ and*

$$\phi'(x_0) = 0,\ \ \phi''(x_0) \neq 0.$$

Assume further that $\phi'(x) \neq 0$ on $K - \{x_0\}$.

(i) *Then there exist for $k = 0, 1, \ldots$ differential operator $A_{2k}(x, D)$, of order less than or equal to $2k$, such that for each N*

$$\begin{aligned}\left|I_h - \left(\sum_{k=0}^{N-1} A_{2k}(x,D)a(x_0)h^{k+\frac{1}{2}}\right)e^{\frac{i}{h}\phi(x_0)}\right|& \\ \leq C_N h^{N+\frac{1}{2}} \sum_{0\leq m\leq 2N+2} \sup_{\mathbb{R}} |a^{(m)}|.&\end{aligned} \tag{3.34}$$

(i) *In particular, we see that*

$$A_0 = (2\pi)^{1/2}|\phi''(x_0)|^{-1/2} e^{\frac{i\pi}{4}\operatorname{sgn}\phi''(x_0)}; \tag{3.35}$$

and consequently

$$I_h = (2\pi h)^{1/2}|\phi''(x_0)|^{-1/2} e^{\frac{i\pi}{4}\operatorname{sgn}\phi''(x_0)} e^{\frac{i\phi(x_0)}{h}} a(x_0) + O(h^{3/2}) \tag{3.36}$$

as $h \to 0$.

We will provide two proofs of this important theorem.

First proof of Theorem 3.10. 1. We may without loss assume $x_0 = 0$, $\phi(0) = 0$. Then $\phi(x) = \frac{1}{2}\Phi(x)x^2$, for

$$\Phi(x) := 2\int_0^1 (1-t)\phi''(tx)\, dt.$$

Notice that $\Phi(0) = \phi''(0) \neq 0$. We change variables by writing

$$y := |\Phi(x)|^{1/2} x$$

for x near 0. Thus

$$\frac{\partial x}{\partial y} = |\phi''(0)|^{-1/2} \quad \text{at } x = y = 0.$$

Now select a smooth function $\chi : \mathbb{R} \to \mathbb{R}$ such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ near 0, and $\operatorname{sgn}\phi''(x) = \operatorname{sgn}\phi''(0) \neq 0$ on the support of χ. Then Lemma 3.9 implies

$$\begin{aligned} I_h &= \int_{-\infty}^{\infty} e^{\frac{i\phi(x)}{h}} \chi(x) a(x)\, dx + \int_{-\infty}^{\infty} e^{\frac{i\phi(x)}{h}} (1-\chi(x)) a(x)\, dx \\ &= \int_{-\infty}^{\infty} e^{\frac{i\epsilon}{2h}y^2} u(y)\ dy + O(h^\infty), \end{aligned}$$

for $\epsilon := \operatorname{sgn}\phi''(0) = \pm 1$, $u(y) := \chi(x(y)) a(x(y)) |\det \partial_y x|$.

2. Note that $e^{\frac{i\epsilon}{2h}y^2} = (e^{-\frac{i\epsilon}{2h}y^2})^{-1}$. Also, the Fourier transform formula (3.23) tells us that

$$\mathcal{F}\left(e^{-\frac{i\epsilon y^2}{2h}}\right) = (2\pi h)^{1/2} e^{-\frac{i\pi\epsilon}{4}} e^{\frac{i\epsilon h\xi^2}{2}}.$$

Applying (3.16), we see that consequently

$$I_h = \left(\frac{h}{2\pi}\right)^{1/2} e^{\frac{i\pi\epsilon}{4}} \int_{-\infty}^{\infty} e^{-\frac{i\epsilon h\xi^2}{2}} \hat{u}(\xi)\ d\xi + O(h^\infty).$$

The advantage is that the small parameter h, and not h^{-1}, occurs in the exponential.

Next, write

$$J(h,u) := \int_{-\infty}^{\infty} e^{-\frac{i\epsilon h\xi^2}{2}} \hat{u}(\xi)\ d\xi.$$

Then

$$\partial_h J(h,u) = \int_{-\infty}^{\infty} e^{-\frac{i\epsilon h\xi^2}{2}} \left(-\frac{i\epsilon\xi^2}{2}\hat{u}(\xi) \right) d\xi = J(h, Pu)$$

for $P := -\frac{i\epsilon}{2}\partial^2$. Continuing, we discover

$$\partial_h^k J(h,u) = J(h, P^k u).$$

Therefore

$$J(h,u) = \sum_{k=0}^{N-1} \frac{h^k}{k!} J(0, P^k u) + \frac{h^N}{N!} R_N(h,u),$$

for the remainder term

$$R_N(h,u) := N \int_0^1 (1-t)^{N-1} J(th, P^N u)\, dt.$$

Thus Lemma 3.5 implies

$$|R_N| \leq C_N \|\widehat{P^N u}\|_{L^1} \leq C_N \sum_{0\leq k\leq 2} \sup_x |\partial^k (P^N u)|.$$

□

The second proof of stationary phase will employ this

LEMMA 3.11 (More on rapid decay). *For each positive integer k, there exists a constant C_k such that*

$$\left| \int_{-\infty}^{\infty} e^{\frac{i\phi(x)}{h}} a(x)\ dx \right| \leq C_k h^k \sum_{0\leq m\leq k} \sup_{\mathbb{R}} (|a^{(m)}||\phi'|^{m-2k}). \tag{3.37}$$

This inequality will be useful at points x where $\phi'(x)$ is small, provided $a^{(m)}(x)$ is also small.

Proof. The proof is an induction on k, the case $k = 0$ being obvious. Assume the assertion for $k-1$. Then

$$\begin{aligned} \int_{-\infty}^{\infty} e^{\frac{i\phi}{h}} a\ dx &= \frac{h}{i} \int_{-\infty}^{\infty} \left(e^{\frac{i\phi}{h}} \right)' \frac{a}{\phi'} dx \\ &= -\frac{h}{i} \int_{-\infty}^{\infty} e^{\frac{i\phi}{h}} \left(\frac{a}{\phi'} \right)' dx = -\frac{h}{i} \int_{-\infty}^{\infty} e^{\frac{i\phi}{h}} \tilde{a}\ dx, \end{aligned}$$

for

$$\tilde{a} := \left(\frac{a}{\phi'} \right)'.$$

By the induction hypothesis,

$$\begin{aligned}\left|\int_{-\infty}^{\infty} e^{\frac{i\phi}{h}} a\, dx\right| &\leq h\left|\int_{-\infty}^{\infty} e^{\frac{i\phi}{h}} \tilde{a}\, dx\right| \\ &\leq C_{k-1}h^k \sum_{0\leq m\leq k-1} \sup_{\mathbb{R}}(|\tilde{a}^{(m)}||\phi'|^{m-2(k-1)}) \\ &\leq C_k h^k \sum_{0\leq m\leq k} \sup_{\mathbb{R}}(|a^{(m)}||\phi'|^{m-2k}).\end{aligned}$$

$\square$

Second proof of Theorem 3.10. 1. As before, we may assume $x_0 = 0$, $\phi(0) = 0$. Then

$$\phi'(x) = \phi''(0)x + O(x^2).$$

Therefore

$$|x| \leq |\phi''(0)|^{-1}|\phi'(x) + O(x^2)| \leq 2|\phi''(0)|^{-1}|\phi'(x)|$$

for sufficiently small x. Consequently,

$$\frac{x}{\phi'(x)} \text{ is bounded near } 0.$$

Hence if $|a^{(m)}| \leq C|x|^{2N-m}$ for $m = 0, \dots, N$, Lemma 3.11 implies

$$\begin{aligned}\left|\int_{-\infty}^{\infty} e^{\frac{i\phi}{h}} a\, dx\right| &\leq Ch^N \sum_{0\leq m\leq N} \sup_{x}(|a^{(m)}||\phi'|^{m-2N}) \\ &\leq Ch^N \sum_{0\leq m\leq k} \left|\frac{x}{\phi'}\right|^{2N-m} = O(h^N).\end{aligned} \tag{3.38}$$

2. Return now to our integral

$$I_h = \int_{-\infty}^{\infty} e^{\frac{i\phi}{h}} a\, dx.$$

We write

$$a = \sum_{k=0}^{2N} \frac{a^{(k)}(0)}{k!} x^k + a_{2N},$$

where

$$|a_{2N}^{(m)}| \leq C|x|^{2N-m} \quad \text{for } m = 0, \dots, N. \tag{3.39}$$

Then

$$\int_{-\infty}^{\infty} e^{\frac{i\phi}{h}} a\, dx = \sum_{k=0}^{2N} \frac{a^{(k)}(0)}{k!} \int_{-\infty}^{\infty} e^{\frac{i\phi}{h}} x^k\, dx + \int_{-\infty}^{\infty} e^{\frac{i\phi}{h}} a_{2N}\, dx.$$

Estimates (3.39) and (3.38) demonstrate that the last term is $O(h^N)$ as $h \to 0$. $\square$

3.5 STATIONARY PHASE IN HIGHER DIMENSIONS

We turn next to n-dimensional oscillatory integrals.

DEFINITION. We call the expression

$$I_h = I_h(a, \phi) = \int_{\mathbb{R}^n} e^{\frac{i\phi}{h}} a \, dx \tag{3.40}$$

an *oscillatory integral*, where $a \in C_c^\infty(\mathbb{R}^n)$, $\phi \in C^\infty(\mathbb{R}^n)$.

LEMMA 3.12 (Rapid decay again). *If $\partial \phi \neq 0$ on $K := \operatorname{spt}(a)$, then*

$$I_h = O(h^\infty).$$

In particular, for each positive integer N

$$|I_h| \leq C h^N \sum_{|\alpha| \leq N} \sup_{\mathbb{R}^n} |\partial^\alpha a|, \tag{3.41}$$

where C depends on K and n only.

Proof. Define the operator

$$L := \frac{h}{i} \frac{1}{|\partial \phi|^2} \langle \partial \phi, \partial \rangle$$

for $x \in K$, and observe that

$$L\left(e^{\frac{i\phi}{h}}\right) = e^{\frac{i\phi}{h}}.$$

Hence $L^N\left(e^{\frac{i\phi}{h}}\right) = e^{\frac{i\phi}{h}}$. Consequently

$$|I_h| = \left| \int_{\mathbb{R}^n} L^N \left(e^{\frac{i\phi}{h}}\right) a \; dx \right| = \left| \int_{\mathbb{R}^n} e^{\frac{i\phi}{h}} (L^*)^N a \; dx \right| \leq C h^N.$$

$\square$

DEFINITION. We say $\phi : \mathbb{R}^n \to \mathbb{R}$ has a *nondegenerate critical point* at x_0 if

$$\partial \phi(x_0) = 0, \; \det \partial^2 \phi(x_0) \neq 0.$$

Next we change variables locally to convert the phase function ϕ into a quadratic:

THEOREM 3.13 (Morse Lemma). *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be smooth, with a nondegenerate critical point at x_0. Then there exist neighborhoods U of 0 and V of x_0 and a diffeomorphism*

$$\boldsymbol{\kappa} : V \to U$$

such that

$$(\phi \circ \boldsymbol{\kappa}^{-1})(x) = \phi(x_0) + \frac{1}{2}(x_1^2 + \cdots + x_r^2 - x_{r+1}^2 \cdots - x_r^2), \tag{3.42}$$

where r is the number of positive eigenvalues of $\partial^2\phi(x_0)$.

Proof. 1. As usual, we suppose $x_0 = 0$, $\phi(0) = 0$. After a linear change of variables, we have

$$\phi(x) = \frac{1}{2}(x_1^2 + \cdots + x_r^2 - x_{r+1}^2 \cdots - x_r^2) + O(|x|^3);$$

and so the problem is to design a further change of variables that removes the cubic and higher terms.

2. Now

$$\phi(x) = \int_0^1 (1-t)^2 \partial_t^2 \phi(tx)\, dt = \frac{1}{2}\langle x, Q(x)x\rangle,$$

where

$$Q(0) = \partial^2\phi(0) = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}.$$

We want to find a smooth mapping A from $\mathbb{R}^n$ to $\mathbb{M}^{n\times n}$ such that

$$\langle A(x)x, Q(0)A(x)x\rangle = \langle x, Q(x)x\rangle. \tag{3.43}$$

Then

$$\boldsymbol{\kappa}(x) = A(x)x$$

is the desired change of variable.

Formula (3.43) will hold provided

$$A^T(x)Q(0)A(x) = Q(x). \tag{3.44}$$

Let $F : \mathbb{M}^{n\times n} \to \mathbb{S}^{n\times n}$ be defined by

$$F(A) = A^T Q(0) A.$$

We want to find a *right inverse* $G : \mathbb{S}^{n\times n} \to \mathbb{M}^{n\times n}$, so that

$$F \circ G = I \quad \text{near } Q(0).$$

Then

$$A(x) := G(Q(x))$$

will solve (3.44).

3. We will apply a version of the Implicit Function Theorem (Theorem B.7). To do so, it suffices to find $A \in L(\mathbb{S}^{n\times n}, \mathbb{M}^{n\times n})$ such that

$$\partial F(I)A = I.$$

Now

$$\partial F(I)(C) = C^T Q(0) + Q(0)C.$$

Define

$$A(D) := \frac{1}{2} Q(0)^{-1} D$$

for $D \in \mathbb{S}^{n\times n}$. Then

$$\begin{aligned} \partial F(I)A(D) &= \frac{1}{2}\partial F(I)(Q^{-1}(0)D) \\ &= \frac{1}{2}[(Q(0)^{-1}D)^T Q(0) + Q(0)(Q(0)^{-1}D)] \\ &= D. \end{aligned}$$

□

THEOREM 3.14 (Stationary phase asymptotics). *Assume that* $a \in C_c^\infty(\mathbb{R}^n)$. *Suppose* $x_0 \in K := \operatorname{spt}(a)$ *and*

$$\partial\phi(x_0) = 0, \ \det \partial^2\phi(x_0) \neq 0.$$

Assume further that $\partial\phi(x) \neq 0$ *on* $K - \{x_0\}$.

(i) *Then there exist for* $k = 0, 1, \dots$ *differential operators* $A_{2k}(x, D)$ *of order less than or equal to* $2k$, *such that for each* N

$$\begin{aligned} &\left| I_h - \left(\sum_{k=0}^{N-1} A_{2k}(x,D)a(x_0)h^{k+\frac{n}{2}} \right) e^{\frac{i\phi(x_0)}{h}} \right| \\ &\qquad\qquad \leq C_N h^{N+\frac{n}{2}} \sum_{|\alpha|\leq 2N+n+1} \sup_{\mathbb{R}^n} |\partial^\alpha a|. \end{aligned} \tag{3.45}$$

(ii) *In particular,*

$$A_0 = (2\pi)^{n/2} |\det \partial^2\phi(x_0)|^{-1/2} e^{\frac{i\pi}{4}\operatorname{sgn}\partial^2\phi(x_0)}; \tag{3.46}$$

and therefore

$$\begin{aligned} &I_h = \\ &(2\pi h)^{n/2} |\det \partial^2\phi(x_0)|^{-1/2} e^{\frac{i\pi}{4}\operatorname{sgn}\partial^2\phi(x_0)} e^{\frac{i\phi(x_0)}{h}} a(x_0) + O\left(h^{\frac{n+2}{2}}\right) \end{aligned} \tag{3.47}$$

as $h \to 0$.

Proof. Without loss $x_0 = 0$, $\phi(x_0) = 0$. Introducing a cutoff function χ and applying the Morse Lemma, Theorem 3.13, and then Lemma 3.12, we can write

$$I_h = \int_{\mathbb{R}^n} e^{\frac{i\phi}{h}} a \, dx = \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle Qx,x\rangle} u \, dx + O(h^\infty),$$

where

$$Q = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}$$

and u is smooth. In this expression the upper indentity matrix is $r \times r$ and the lower identity matrix is $(n-r) \times (n-r)$. Note that $\operatorname{sgn} Q = \operatorname{sgn} \partial^2 \phi(x_0)$ and $|\det Q| = 1$. Hence the Fourier transform formulas (3.23) and (3.16) give

$$I_h = \left(\frac{h}{2\pi}\right)^{n/2} e^{\frac{i\pi}{4} \operatorname{sgn} Q} \int_{\mathbb{R}^n} e^{-\frac{ih}{2}\langle Q^{-1}\xi,\xi\rangle} \hat{u}(\xi) \, d\xi.$$

Write

$$J(h,u) := \int_{\mathbb{R}^n} e^{-\frac{ih}{2}\langle Q^{-1}\xi,\xi\rangle} \hat{u}(\xi) \, d\xi;$$

then

$$\partial_h J(h,u) = \int_{\mathbb{R}^n} e^{-\frac{ih}{2}\langle Q^{-1}\xi,\xi\rangle} \left(-\frac{i}{2}\langle Q^{-1}\xi,\xi\rangle \hat{u}(\xi)\right) d\xi = J(h, Pu)$$

for

$$P := -\frac{i}{2}\langle Q^{-1} D_x, D_x\rangle. \tag{3.48}$$

Therefore

$$J(h,u) = \sum_{k=0}^{N-1} \frac{h^k}{k!} J(0, P^k u) + \frac{h^N}{N!} R_N(h,u),$$

for the remainder term

$$R_N(h,u) := N \int_0^1 (1-t)^{N-1} J(th, P^n u) \, dt.$$

Then Lemma 3.5 implies

$$|R_N| \le C_N \|\widehat{P^N u}\|_{L^1} \le C_N \sup_{|\alpha| \le 2N+n+1} |\partial^\alpha a|.$$

$\square$

3.6 AN IMPORTANT EXAMPLE.

In Chapter 4 we will be primarily interested in the particular phase function

$$\phi(x,y) = \langle x, y\rangle \tag{3.49}$$

on $\mathbb{R}^{2n}$:

THEOREM 3.15 (A simple phase function). *Assume that a belongs to $C_c^\infty(\mathbb{R}^{2n})$. Then for each postive integer N, we have*

$$\begin{aligned}\int_{\mathbb{R}^{2n}} & e^{-\frac{i}{h}\langle x,y\rangle} a(x,y)\,dxdy = \\ & (2\pi h)^{n/2}\left(\sum_{k=0}^{N-1}\frac{h^k}{k!}\left(\frac{\langle D_x, D_y\rangle}{i}\right)^k a(0,0) + O(h^N)\right).\end{aligned} \tag{3.50}$$

REMARK. It will be convenient later for us to rewrite this identity in the form

$$\begin{aligned}\int_{\mathbb{R}^{2n}} & e^{-\frac{i}{h}\langle x,y\rangle} a(x,y)\,dxdy \\ & \sim (2\pi h)^{n/2} e^{-ih\langle D_x, D_y\rangle} a(0,0).\end{aligned} \tag{3.51}$$

□

Proof. We write (x,y) to denote a typical point of $\mathbb{R}^{2n}$, and let

$$Q := -\begin{pmatrix} O & I \\ I & O\end{pmatrix}.$$

Then $Q = Q^{-1}$, $|\text{det}Q| = 1$, $\text{sgn}(Q) = 0$ and $Q(x,y) = (-y,-x)$.

Consequently $\frac{1}{2}\langle Q(x,y),(x,y)\rangle = -\langle x,y\rangle$. Furthermore the operator P, introduced at (3.48) in the previous proof, becomes

$$P = -\frac{i}{2}\langle Q^{-1}D_{(x,y)}, D_{(x,y)}\rangle = \frac{1}{i}\langle D_x, D_y\rangle.$$

Hence

$$\begin{aligned} J(0, P^k a) &= \int_{\mathbb{R}^{2n}} e^{-ih\langle \xi,\eta\rangle}\left(\left(\frac{1}{i}\langle D_x, D_y\rangle\right)^k a\right)^{\wedge} d\xi d\eta \\ &= (2\pi)^n \left(\frac{1}{i}\langle D_x, D_y\rangle\right)^k a(0,0).\end{aligned}$$

□

REMARK. We can similarly write

$$\int_{\mathbb{R}^n} e^{\frac{i}{2h}\langle Q^{-1}x,x\rangle} a(x)\,dx \sim \left(\frac{h}{2\pi}\right)^{n/2} e^{\frac{i\pi}{4}\operatorname{sgn} Q} |\det Q|^{1/2} e^{ih\langle QD,D\rangle} a(0), \tag{3.52}$$

if Q is nonsingular and symmetric. $\square$

4. Quantization of symbols

The Fourier transform and its inverse allow us move at will between the position x and momentum ξ variables, but what we really want is to deal with both sets of variables simultaneously. This chapter therefore introduces the quantization of "symbols", that is, of appropriate functions of both x and ξ. The resulting operators applied to functions entail information in the full (x, ξ) phase space, and particular choices of the symbol will later prove very useful, allowing us for example to "localize" in phase space.

The plan is to introduce quantization and then to work out the resulting *symbol calculus*, meaning the systematic rules for manipulating symbols and their associated operators.

4.1 QUANTIZATION FORMULAS

NOTATION. For this section we take $a \in \mathcal{S} = \mathcal{S}(\mathbb{R}^{2n})$, $a = a(x, \xi)$. We hereafter call a a *symbol.*

DEFINITIONS.

(i) We define the *Weyl quantization* to be the operator $a^w(x, hD)$ acting on $u \in \mathcal{S}(\mathbb{R}^n)$ by the formula

$$a^w(x,hD)u(x) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a\left(\frac{x+y}{2}, \xi\right) u(y)\, dy d\xi. \tag{4.1}$$

(ii) We define also the *standard quantization*

$$a(x,hD)u(x) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a(x,\xi) u(y)\, dy d\xi \tag{4.2}$$

for $u \in \mathcal{S}$.

(iii) More generally, each $0 \le t \le 1$, we set

$$\mathrm{Op}_t(a)u(x) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a(tx+(1-t)y,\xi) u(y)\, dy d\xi \tag{4.3}$$

for $u \in \mathcal{S}$.

NOTATION. We will often for notational simplicity just write "Op(a)" for $\mathrm{Op}_{1/2}(a)$. Therefore

$$\mathrm{Op}(a) = a^w(x, hD).$$

EXAMPLES:

(i) If $a(x, \xi) = \xi^\alpha$, then

$$\mathrm{Op}_t(a)u = (hD)^\alpha u \qquad (0 \leq t \leq 1).$$

(ii) If $a(x, \xi) = V(x)$, then

$$\mathrm{Op}_t(a)u = V(x)u \qquad (0 \leq t \leq 1).$$

(iii) If $a(x, \xi) = \langle x, \xi \rangle$, then

$$\mathrm{Op}_t(a)u = (1-t)\langle hD, xu \rangle + t\langle x, hDu \rangle \qquad (0 \leq t \leq 1).$$

(iv) If $a(x, \xi) = \sum_{|\alpha| \leq N} a_\alpha(x)\xi^\alpha$ and $t = 1$, then

$$a(x, hD) = \sum_{|\alpha| \leq N} a_\alpha(x)(hD)^\alpha u.$$

These formulas follow straightforwardly from the definitions.

THEOREM 4.1 (Schwartz class symbols). *If $a \in \mathcal{S}$, then $\mathrm{Op}_t(a)$ can be defined as an operator mapping $\mathcal{S}'$ to $\mathcal{S}$; and furthermore*

$$\mathrm{Op}_t(a) : \mathcal{S}' \to \mathcal{S} \qquad (0 \leq t \leq 1)$$

is continuous.

Proof. We have

$$\mathrm{Op}_t(a)u(x) = \int_{\mathbb{R}^n} K(x, y)u(y)\, dy$$

for the kernel

$$\begin{aligned} K(x, y) &:= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y, \xi \rangle} a(tx + (1-t)y, \xi)\, d\xi \\ &= \mathcal{F}_h^{-1}(a(tx + (1-t)y, \cdot))(x - y). \end{aligned}$$

Thus $K \in \mathcal{S}$, and so

$$\mathrm{Op}_t(a)u(x) = u(K(x, \cdot))$$

maps $\mathcal{S}'$ continuously into $\mathcal{S}$. $\square$

THEOREM 4.2 (Adjoints). *Assume* $a \in \mathcal{S}$

(i) *We have*

$$\mathrm{Op}_t(a)^* = \mathrm{Op}_{1-t}(\bar{a}) \qquad (0 \le t \le 1).$$

(ii) *Consequently,*

$$a^w(x, hD)^* = a^w(x, hD) \quad \textit{if } a \textit{ is real.}$$

In particular, the Weyl quantization of a real symbol is self-adjoint.

Proof. The kernel of $\mathrm{Op}_t(a)^*$ is $K^*(x,y) := \overline{K(y,x)}$, which is the kernel of $\mathrm{Op}_{1-t}(\bar{a})$. □

We next observe that the formulas (4.1)–(4.3) make sense if a is merely a distribution:

THEOREM 4.3 (Distributional symbols). *If* $a \in \mathcal{S}'$, *then* $\mathrm{Op}_t(a)$ *can be defined as an operator mapping* $\mathcal{S}$ *to* $\mathcal{S}'$*; and furthermore*

$$\mathrm{Op}_t(a) : \mathcal{S} \to \mathcal{S}' \qquad (0 \le t \le 1)$$

is continuous.

Proof. The formula for the distibutional kernel of $\mathrm{Op}_t(a)$ shows that it is an element K of $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$. Hence $\mathrm{Op}_t(a)$ is well defined as an operator from $\mathcal{S}$ to $\mathcal{S}$: if $u, v \in \mathcal{S}$ then

$$(Op_t(a)u)(v) := K(u \otimes v).$$

□

4.2 COMPOSITION

We begin now a careful study of the properties of the quantized operators defined above. Our particular goal in this section is showing that if a and b are symbols, then there exists a symbol $c = a\#b$ such that

$$a^w(x, hD) \circ b^w(x, hD) = c^w(x, hD).$$

4.2.1 Linear symbols. We begin with linear symbols.

LEMMA 4.4 (Quantizing linear symbols). *Fix* $(x^*, \xi^*) \in \mathbb{R}^{2n}$ *and define the linear symbol*

$$l(x,\xi) := \langle x^*, x\rangle + \langle \xi^*, \xi\rangle. \tag{4.4}$$

Then

$$\mathrm{Op}_t(l)u = \langle x^*, x\rangle u + \langle \xi^*, hD\rangle u \qquad (0 \le t \le 1). \tag{4.5}$$

NOTATION. In view of this result, we hereafter write $l(x, hD)$ for $l^w(x, hD)$.

Proof. Compute the derivative

$$\begin{aligned}
\frac{d}{dt}\mathrm{Op}_t(l)u &= \frac{1}{(2\pi h)^n}\frac{d}{dt}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle}(\langle x^*, tx+(1-t)y\rangle \\
&\qquad\qquad + \langle \xi^*, \xi\rangle)u(y)\,dyd\xi \\
&= \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle}\langle x^*, x-y\rangle u\,dyd\xi \\
&= \frac{h}{(2\pi h)^n}\int_{\mathbb{R}^n}\left\langle x^*, D_\xi\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle}u(y)\,dy\right\rangle d\xi \\
&= \frac{h}{(2\pi h)^n}\int_{\mathbb{R}^n}\left\langle x^*, D_\xi(e^{\frac{i}{h}\langle x,\xi\rangle}\hat{u}(\xi))\right\rangle d\xi.
\end{aligned}$$

Since $\hat{u}(\xi) \to 0$ rapidly as $|\xi| \to \infty$, the last expression vanishes. Therefore $\mathrm{Op}_t(l)$ does not in fact depend upon t; and consequently for all $0 \le t \le 1$, $\mathrm{Op}_t(l)u = \mathrm{Op}_1(l)u = \langle x^*, x\rangle u + \langle \xi^*, hD\rangle u$.

□

THEOREM 4.5 (Composition with a linear symbol). *Let $b \in \mathcal{S}$. Then*

$$l(x, hD)b^w(x, hD) = c^w(x, hD), \tag{4.6}$$

for

$$c := lb + \frac{h}{2i}\{l, b\}. \tag{4.7}$$

NOTATION. Here we use the notation

$$\{l, b\} = \langle \partial_\xi l, \partial_x b\rangle - \langle \partial_x l, \partial_\xi b\rangle = \langle \xi^*, \partial_x b\rangle - \langle x^*, \partial_\xi b\rangle. \tag{4.8}$$

Proof. According to Lemma 4.4,

$$l(x, hD) = \langle x^*, x\rangle + \langle \xi^*, hD\rangle.$$

Now

$$\begin{aligned}
&\langle x^*, x\rangle b^w(x, hD)u \\
&\qquad = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\langle x^*, x\rangle e^{\frac{i}{h}\langle x-y,\xi\rangle} b\left(\frac{x+y}{2}, \xi\right)u(y)\,dyd\xi.
\end{aligned}$$

We write

$$\langle x^*, x\rangle = \langle x^*, \frac{x+y}{2}\rangle + \langle x^*, \frac{x-y}{2}\rangle$$

and observe then that

$$\frac{x-y}{2}e^{\frac{i}{h}\langle x-y,\xi\rangle} = \frac{h}{2i}\partial_\xi\left(e^{\frac{i}{h}\langle x-y,\xi\rangle}\right).$$

An integration by parts shows that consequently

$$\langle x^*, x\rangle b^w(x,hD)u =$$
$$\frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{n}\langle x-y,\xi\rangle}\left(\langle x^*, \frac{x-y}{2}\rangle b - \frac{h}{2i}\langle x^*, \partial_\xi b\rangle\right) u \;\; dyd\xi.$$

Furthermore

$$\begin{aligned}
&\langle \xi^*, hD_x\rangle b^w(x,hD)u \\
&\quad = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\langle \xi^*, hD_x\rangle\left(e^{\frac{i}{h}\langle x-y,\xi\rangle} b\left(\frac{x+y}{2},\xi\right)\right)u(y)\, dyd\xi \\
&\quad = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle}\left(\langle \xi^*, \xi\rangle b + \frac{h}{2i}\langle \xi^*, \partial_x b\rangle\right) u \;\; dyd\xi.
\end{aligned}$$

Adding the last two equations shows us that

$$\begin{aligned}
&l(x,hD)b^w(x,hD) \\
&\quad = (\langle x^*, x\rangle + \langle \xi^*, hD\rangle)b^w(x,hD) \\
&\quad = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} \\
&\qquad\qquad\qquad \left(\left(\langle \xi^*, \xi\rangle + \langle x^*, \frac{x+y}{2}\rangle\right) b + \frac{h}{2i}\{l,b\}\right) u\, dyd\xi,
\end{aligned}$$

in view of (4.8). This proves (4.6), (4.7). □

THEOREM 4.6 (Quantizing the exponential of a linear symbol). *We have the identity*

$$\operatorname{Op}\left(e^{-\frac{i}{h}l}\right) = e^{-\frac{i}{h}l(x,hD)}. \tag{4.9}$$

Proof. Consider the differential equation

$$\begin{cases} ih\partial_t u = l(x,hD)u & (t\in\mathbb{R}) \\ \;u(0) = v; \end{cases} \tag{4.10}$$

the solution of which is

$$u := \operatorname{Op}\left(e^{-\frac{it}{h}l}\right)v,$$

since

$$ih\partial_t u = \operatorname{Op}\left(le^{-\frac{it}{h}l}\right)v = \operatorname{Op}(la)v$$

for $a := e^{-\frac{it}{h}l}$. Observe that then $\{l, a\} = 0$, since a is a function of l. Hence Theorem 4.5 implies

$$\mathrm{Op}(la)v = l(x, hD)\mathrm{Op}(a)v = l(x, hD)u.$$

As the solution of (4.10) is $e^{-\frac{it}{h}l(x,hD)}v$, assertion (4.9) holds. $\square$

LEMMA 4.7 (Translation and quantization). *We have*

$$\mathrm{Op}\left(e^{-\frac{il}{h}}\right) = e^{-\frac{i}{2h}\langle x^*,x\rangle} \circ T_{\xi^*} \circ e^{-\frac{i}{2h}\langle x^*,x\rangle}, \tag{4.11}$$

where T_{ξ^*} is the translation operator.

See Appendix A for the definition of the translation operator. If we write out (4.11) explicitly, we find

$$e^{-\frac{i}{h}l(x,hD)}u(x) = e^{-\frac{i}{h}\langle x^*,x\rangle + \frac{i}{2h}\langle x^*,\xi^*\rangle}u(x - \xi^*). \tag{4.12}$$

Proof. To check this, observe that

$$\begin{aligned}
\mathrm{Op}\left(e^{-\frac{il}{h}}\right)u &= \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} e^{-\frac{i}{h}\left(\langle \xi^*,\xi\rangle + \langle x^*, \frac{x+y}{2}\rangle\right)}u(y)\,dyd\xi \\
&= \frac{e^{-\frac{i}{2h}\langle x^*,x\rangle}}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y-\xi^*,\xi\rangle}\left(e^{-\frac{i}{2h}\langle x^*,y\rangle}u(y)\right)\,dyd\xi \\
&= \frac{e^{-\frac{i}{2h}\langle x^*,x\rangle}}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle}\left(e^{-\frac{i}{2h}\langle x^*,y-\xi^*\rangle}u(y-\xi^*)\right)\,dyd\xi \\
&= \left(e^{-\frac{i}{2h}\langle x^*,x\rangle} \circ T_{\xi^*} \circ e^{-\frac{i}{2h}\langle x^*,x\rangle}\right)u,
\end{aligned}$$

since

$$\delta_{\{y=x\}} = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle}d\xi \quad \text{in } \mathcal{S}',$$

according to (3.29) $\square$

NOTATION. To simplify calculations later on, we henceforth identify the linear symbol

$$l(x,\xi) := \langle x^*, x\rangle + \langle \xi^*, \xi\rangle$$

with the point $(x^*, \xi^*) \in \mathbb{R}^{2n}$.

LEMMA 4.8 (Two linear symbols). *Suppose $l, m \in \mathbb{R}^{2n}$. Then*

$$e^{-\frac{i}{h}l(x,hD)}e^{-\frac{i}{h}m(x,hD)} = e^{\frac{i}{2h}\{l,m\}}e^{-\frac{i}{h}(l+m)(x,hD)}. \tag{4.13}$$

Proof. We have $l(x,\xi) = \langle x_1^*, x\rangle + \langle \xi_1^*, \xi\rangle$, and $m(x,\xi) = \langle x_2^*, x\rangle + \langle \xi_2^*, \xi\rangle$. Then

$$\{l, m\} = \langle \partial_\xi l, \partial_x m\rangle - \langle \partial_x l, \partial_\xi m\rangle = \langle \xi_1^*, x_2^*\rangle - \langle x_1^*, \xi_2^*\rangle = \sigma(l,m). \tag{4.14}$$

According to (4.12)

$$e^{-\frac{i}{h}m(x,hD)}u(x) = e^{-\frac{i}{h}\langle x_2^*, x\rangle + \frac{i}{2h}\langle x_2^*, \xi_2^*\rangle}u(x-\xi_2^*);$$

and consequently

$$\begin{aligned} &e^{-\frac{i}{h}l(x,hD)}e^{-\frac{i}{h}m(x,hD)}u(x) = \\ &e^{-\frac{i}{h}\langle x_1^*, x\rangle + \frac{i}{2h}\langle x_1^*, \xi_1^*\rangle}e^{-\frac{i}{h}\langle x_2^*, x-\xi_1^*\rangle + \frac{i}{2h}\langle x_2^*, \xi_2^*\rangle}u(x-\xi_1^*-\xi_2^*). \end{aligned} \tag{4.15}$$

On the other hand, (4.12) implies also that

$$\begin{aligned} e^{-\frac{i}{h}(l+m)(x,hD)}u(x) &= e^{-\frac{i}{h}\langle x_1^*+x_2^*, x\rangle + \frac{i}{2h}\langle x_1^*+x_2^*, \xi_1^*+\xi_2^*\rangle}u(x-\xi_1^*-\xi_2^*) \\ &= e^{\frac{i}{2h}(\langle x_1^*, \xi_2^*\rangle - \langle x_2^*, \xi_1^*\rangle)}e^{-\frac{i}{h}l(x,hD)}e^{-\frac{i}{h}m(x,hD)}u(x), \end{aligned}$$

the last equality following from (4.15). This proves (4.13). □

4.2.2 Fourier decomposition of a^w. Suppose now $a \in \mathcal{S}$ and $l \in \mathbb{R}^{2n}$. We define

$$\hat{a}(l) := \int_{\mathbb{R}^{2n}} e^{-\frac{i}{h}l(x,\xi)}a(x,\xi)\,dxd\xi;$$

so that by the Fourier inversion formula

$$a(x,\xi) = \frac{1}{(2\pi h)^{2n}}\int_{\mathbb{R}^{2n}} e^{\frac{i}{h}l(x,\xi)}\hat{a}(l)\,dl.$$

This is a decomposition of the symbol a into linear symbols of the form treated above. Therefore Theorem 4.6 provides the useful representation formula

$$a^w(x,hD) = \frac{1}{(2\pi h)^{2n}}\int_{\mathbb{R}^{2n}} \hat{a}(l)e^{\frac{i}{h}l(x,hD)}\,dl. \tag{4.16}$$

4.2.3 Composing symbols. Next we establish the fundamental formula:

$$a^w b^w = (a\#b)^w, \tag{4.17}$$

along with a recipe for computing the new symbol $a\#b$:

THEOREM 4.9 (Composition for Weyl quantization).
(i) *Suppose that $a, b \in \mathcal{S}$. Then*

$$a^w(x,hD)\circ b^w(x,hD) = c^w(x,hD)$$

for the symbol

$$c = a\#b,$$

where

$$a\#b(x,\xi) := e^{\frac{ih}{2}\sigma(D_x,D_\xi,D_y,D_\eta)}\big(a(x,\xi)b(y,\eta)\big)\Big|_{\substack{x=y\\ \xi=\eta}} \tag{4.18}$$

for

$$\sigma(D_x,D_\xi,D_y,D_\eta) := \langle D_\xi, D_y\rangle - \langle D_x, D_\eta\rangle.$$

(ii) *We also have the integral representation formula*

$$\begin{aligned} a\#b(x,\xi) = \\ \frac{1}{(\pi h)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(x+z,\xi+\zeta)b(x+y,\xi+\eta) \\ e^{\frac{2i}{h}\sigma(y,\eta;z,\zeta)}\,dyd\eta dzd\zeta, \end{aligned} \tag{4.19}$$

for

$$\sigma(y,\eta;z,\zeta) = \langle \eta, z\rangle - \langle y,\zeta\rangle.$$

Proof. 1. Similarly to (4.16), we have

$$b^w(x,hD) = \frac{1}{(2\pi h)^{2n}}\int_{\mathbb{R}^{2n}} \hat{b}(m)e^{\frac{i}{h}m(x,hD)}dm.$$

Therefore Lemma 4.8 lets us compute

$$\begin{aligned} a^w(x,hD)b^w(x,hD) &= \frac{1}{(2\pi h)^{4n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} \hat{a}(l)\hat{b}(m)e^{\frac{i}{h}l(x,hD)}e^{\frac{i}{h}m(x,hD)}\,dmdl \\ &= \frac{1}{(2\pi h)^{4n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} \hat{a}(l)\hat{b}(m)e^{\frac{i}{2h}\{l,m\}}e^{\frac{i}{h}(l+m)(x,hD)}\,dldm \\ &= \frac{1}{(2\pi h)^{2n}}\int_{\mathbb{R}^{2n}} \hat{c}(r)e^{\frac{i}{h}r(x,hD)}dr \end{aligned}$$

for

$$\hat{c}(r) := \frac{1}{(2\pi h)^{2n}}\int_{\{l+m=r\}} \hat{a}(l)\hat{b}(m)e^{\frac{i\{l,m\}}{2h}}\,dl. \tag{4.20}$$

To get this, we changed variables by setting $r = m + l$.

2. We will show that $\hat{c}$ defined by (4.20) is the Fourier transform of the symbol c defined by the right hand side of (4.18).

To see this, we simplify notation write $z = (x,\xi)$, $w = (y,\eta)$. Then

$$c(z) = e^{\frac{ih}{2}\sigma(D_z,D_w)}a(z)b(w)|_{z=w} = e^{\frac{i}{2h}\sigma(hD_z,hD_w)}a(z)b(w)|_{z=w}.$$

Now

$$\begin{aligned} a(z) &= \frac{1}{(2\pi h)^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h} l(z)} \hat{a}(l)\, dl, \\ b(w) &= \frac{1}{(2\pi h)^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h} m(w)} \hat{b}(m)\, dm. \end{aligned}$$

Furthermore, a direct calculation, the details of which we leave to the reader, demonstrates that

$$e^{\frac{i}{2h}\sigma(hD_z, hD_w)} e^{\frac{i}{h}(l(z)+m(w))} = e^{\frac{i}{h}(l(z)+m(w)) + \frac{i}{2h}\sigma(l,m)}.$$

Consequently

$$\begin{aligned} c(z) &= \frac{1}{(2\pi h)^{4n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{i}{2h}\sigma(hD_z, hD_w)} e^{\frac{i}{h}(l(z)+m(w))} \Big|_{z=w} \hat{a}(l)\hat{b}(m)\, dldm \\ &= \frac{1}{(2\pi h)^{4n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(l(z)+m(z)) + \frac{i}{2h}\sigma(l,m)} \hat{a}(l)\hat{b}(m)\, dldm. \end{aligned}$$

The semiclassical Fourier transform of c is therefore

$$\frac{1}{(2\pi h)^{2n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \left(\frac{1}{(2\pi h)^{2n}} \int_{\mathbb{R}^{n}} e^{\frac{i}{h}(l+m-r)(z)} dz \right) e^{\frac{i}{2h}\sigma(l,m)} \hat{a}(l)\hat{b}(m)\, dldm.$$

According to (3.29), the term inside the parentheses is $\delta_{\{l+m=r\}}$ in $\mathcal{S}'$. Thus the foregoing equals

$$\frac{1}{(2\pi h)^{2n}} \int_{\{l+m=r\}} e^{\frac{i}{2h}\{l,m\}} \hat{a}(l)\hat{b}(m)\, dl = \hat{c}(r),$$

in view of (4.20). We have made use of the rule $\sigma(l,m) = \{l,m\}$, established earlier at (4.14).

3. We begin the proof of (4.19) by first introducing the more convenient variables $z = (x,\xi), w_1 = (y,\eta), w_2 = (z,\zeta) \in \mathbb{R}^{2n}$. In these variables, formula (4.18) says

$$\begin{aligned} &a\#b(z) \\ &\quad = \frac{1}{(2\pi h)^{4n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(\langle z_1, z-w_1\rangle + \langle z_2, z-w_2\rangle)} e^{\frac{i}{2h}\sigma(z_1,z_2)} \\ &\qquad\qquad\qquad\qquad a(w_1)b(w_2)\, dz_1dz_2dw_1dw_2 \\ &\quad = \frac{1}{(2\pi h)^{4n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(\langle z_1, w_3\rangle + \langle z_2, w_4\rangle)} e^{\frac{i}{2h}\sigma(z_1,z_2)} \\ &\qquad\qquad\qquad\qquad a(z-w_3)b(z-w_4)\, dz_1dz_2dw_3dw_4. \end{aligned}$$

Next, observe that

$$\begin{aligned}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} & e^{\frac{i}{h}(\langle z_1,w_3\rangle+\langle z_2,w_4\rangle)}e^{\frac{i}{2h}\sigma(z_1,z_2)}\,dz_1dz_2\\ &=\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(\langle z_1,w_3\rangle+\langle z_2,w_4\rangle+\frac{1}{2}\langle z_1,Jz_2\rangle)}\,dz_1dz_2\\ &=\int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle z_2,w_4\rangle}\left(\int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(\langle z_1,w_3+\frac{1}{2}Jz_2\rangle)}dz_1\right)dz_2\\ &=(2\pi h)^{2n}\int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle z_2,w_4\rangle}\delta_{\{w_3+\frac{1}{2}Jz_2\}}dz_2\\ &=(2\pi h)^{2n}2^{2n}\int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle 2J(w_3-z_3),w_4\rangle}\delta_{\{0\}}\,dz_3\\ &=(2\pi h)^{2n}2^{2n}e^{\frac{i}{h}\langle 2Jw_3,w_4\rangle}=(2\pi h)^{2n}2^{2n}e^{-\frac{2i}{h}\sigma(w_3,w_4)}.\end{aligned}$$

We changed variables above, by setting $z_3=w_3+\frac{1}{2}Jz_2$.

Insert this calculation into the previous formula, to discover

$$\begin{aligned}a\#b(z)&=\frac{1}{(\pi h)^{2n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} e^{-\frac{2i}{h}\sigma(w_3,w_4)}a(z-w_3)b(z-w_4)\,dw_3dw_4\\ &=\frac{1}{(\pi h)^{2n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} e^{\frac{2i}{h}\sigma(w_3,w_4)}a(z+w_4)b(z+w_3)\,dw_3dw_4;\end{aligned}$$

this is (4.19). □

REMARK. For future reference, we record this alternative expression for (4.19) that we just derived:

$$\begin{aligned}&a\#b(z)=\\ &\quad\frac{1}{(\pi h)^{2n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} e^{-\frac{2i}{h}\sigma(w_1,w_2)}a(z-w_1)b(z-w_2)\,dw_1dw_2.\end{aligned}\tag{4.21}$$

□

4.2.4 Asymptotics. We next apply stationary phase to derive a useful asymptotic expansion for $a\#b$:

THEOREM 4.10 (Semiclassical expansions). *Assume $a,b\in\mathcal{S}$.*

(i) *We can write for* $N = 0, 1, \dots,$

$$a\#b(x,\xi) = \tag{4.22}$$
$$\sum_{k=0}^{N} \frac{1}{k!} \left(\frac{ih}{2} \sigma(D_x, D_\xi, D_y, D_\eta) \right)^k a(x,\xi) b(y,\eta) \Big|_{\substack{x=\xi \\ y=\eta}} + O(h^{N+1}),$$

the error taken in $\mathcal{S}$.

(ii) *In particular,*

$$a\#b = ab + \frac{h}{2i}\{a,b\} + O(h^2). \tag{4.23}$$

as $h \to 0$.

(iii) *Furthermore, if* $\operatorname{spt}(a) \cap \operatorname{spt}(b) = \emptyset$, *then*

$$a\#b = O(h^\infty).$$

as $h \to 0$.

Proof. 1. Apply the stationary phase Theorem 3.15 to prove (4.22).

2. We can compute

$$\begin{aligned} a\#b &= ab + \frac{ih}{2}\sigma(D_x, D_\xi, D_y, D_\eta) a(x,\xi) b(y,\eta) \Big|_{\substack{x=y \\ \xi=\eta}} + O(h^2) \\ &= ab + \frac{ih}{2}(\langle D_\xi a, D_y b\rangle - \langle D_x a, D_\eta b\rangle) \Big|_{\substack{x=y \\ \xi=\eta}} + O(h^2) \\ &= ab + \frac{h}{2i}(\langle \partial_\xi a, \partial_x b\rangle - \langle \partial_x a, \partial_\xi b\rangle) + O(h^2) \\ &= ab + \frac{h}{2i}\{a,b\} + O(h^2). \end{aligned}$$

3. If $\operatorname{spt}(a) \cap \operatorname{spt}(b) = \emptyset$, each term in the expansion (4.22) vanishes. $\square$

As a quick application, we record

THEOREM 4.11 (Commutators and brackets). *Assume that* $a, b \in \mathcal{S}$. *If* $A = a^w$ *and* $B = b^w$, *then*

$$[A,B] = \frac{h}{i}\{a,b\}^w + O(h^2), \tag{4.24}$$

the error taken in $\mathcal{S}$.

Proof. We have

$$\begin{aligned}[A,B] &= a^w b^w - b^w a^w = (a\#b - b\#a)^w \\ &= \left(ab + \frac{h}{2i}\{a,b\} - \left(ba + \frac{h}{2i}\{b,a\}\right) + O(h^2)\right)^w \\ &= \frac{h}{i}\{a,b\}^w + O(h^2).\end{aligned}$$

□

Next we replace Weyl ($t = \frac{1}{2}$) by standard ($t = 1$) quantization in our composition formulas.

THEOREM 4.12 (Composition for standard quantization). *Let $a, b \in \mathcal{S}$. Then*

$$a(x,hD) \circ b(x,hD) = c(x,hD)$$

for

$$c(x,\xi) := e^{ih\langle D_\xi, D_y\rangle} a(x,\xi) b(y,\eta)\Big|_{\substack{x=y\\ \xi=\eta}}.$$

Proof. We have

$$\begin{aligned}&a(x,hD) \circ b(x,hD)u(x) \\ &\quad = \frac{1}{(2\pi h)^{2n}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}(\langle x,\eta\rangle + \langle y,\xi-\eta\rangle)} a(x,\eta) b(y,\xi) \hat{u}(\xi)\, dy d\eta d\xi \\ &\quad = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} c(x,\xi) e^{\frac{i}{h}\langle x,\xi\rangle} \hat{u}(\xi)\, d\xi,\end{aligned}$$

for

$$c(x,\xi) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle x-y,\xi-\eta\rangle} a(x,\eta) b(y,\xi)\, dy d\eta.$$

Then

$$\frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle z,w\rangle} u(x,w)\, dz dw \sim e^{ih\langle D_z, D_w\rangle} u(0,0)$$

by stationary phase. □

4.2.3 Transforming between different quantizations. We lastly record an interesting conversion formula:

THEOREM 4.13 (Changing quantizations). *If*

$$A = \mathrm{Op}_t(a_t) \qquad (0 \le t \le 1),$$

then

$$a_t(x,\xi) = e^{i(t-s)h\langle D_x, D_\xi\rangle} a_s(x,\xi). \tag{4.25}$$

Proof. The decomposition formula (4.16) demonstrates that

$$\mathrm{Op}_t(a_t) = \frac{1}{(2\pi h)^{2n}} \int_{\mathbb{R}^{2n}} \hat{a}_t(l)\mathrm{Op}_t(e^{\frac{i}{h}l})\, dl.$$

Denoting the Fourier transform used there by $\mathcal{F}_h$, we have

$$\mathcal{F}_h\left(e^{i(t-s)h\langle D_x, D_\xi\rangle} a_s(x,\xi)\right)(l) = e^{\frac{i}{h}(t-s)\langle x^*, \xi^*\rangle}\mathcal{F}_h a_s(l);$$

and as before we identify $l = (x^*, \xi^*) \in \mathbb{R}^{2n}$ with the linear function $l(x,\xi) = \langle x^*, x\rangle + \langle \xi^*, \xi\rangle$.

The theorem will be a consequence of the identity

$$\mathrm{Op}_t\left(e^{\frac{i}{h}l(x,\xi)}\right) = e^{\frac{i}{h}(s-t)\langle x^*, \xi^*\rangle}\mathrm{Op}_s\left(e^{\frac{i}{h}l(x,\xi)}\right). \tag{4.26}$$

Proceeding as in the proof of Lemma 4.7 shows that

$$\mathrm{Op}_t\left(e^{\frac{i}{h}l(x,\xi)}\right) = e^{\frac{i}{h}t\langle x, x^*\rangle} T_{\xi^*} e^{\frac{i}{h}(1-t)\langle x, x^*\rangle},$$

from which (4.26) follows. □

4.3 GENERAL SYMBOL CLASSES

We propose next to extend our calculus to symbols $a = a(x,\xi,h)$, depending on the parameter h, which can grow as $|x|, |\xi| \to \infty$.

4.3.1 More definitions.

DEFINITION. A function $m : \mathbb{R}^{2n} \to (0,\infty)$ is called an *order function* if there exist constants C, N such that

$$m(z) \le C\langle z - w\rangle^N m(w) \tag{4.27}$$

for all $w, z \in \mathbb{R}^n$.

Observe that if m_1, m_2 are order functions, so is $m_1 m_2$.

EXAMPLES. Standard examples are $m(z) \equiv 1$ and $m(z) = \langle z\rangle = (1 + |z|^2)^{1/2}$. □

DEFINITIONS.

(i) Given an order function m on $\mathbb{R}^{2n}$, we define the corresponding class of symbols:

$$\begin{aligned} S(m) := \{a \in C^\infty \mid & \text{ for each multiindex } \alpha \\ & \text{there exists a constant } C_\alpha \text{ so that } |\partial^\alpha a| \leq C_\alpha m\}. \end{aligned}$$

(ii) We as well define

$$S^k(m) := \{a \in C^\infty \mid |\partial^\alpha a| \leq C_\alpha h^{-k} m \ \text{ for all multiindices } \alpha\}$$

and

$$S^k_\delta(m) := \{a \in C^\infty \mid |\partial^\alpha a| \leq C_\alpha h^{-\delta|\alpha|-k} m \ \text{ for all multiindices } \alpha\}.$$

The index k indicates how singular is the symbol a as $h \to 0$; the index δ allows for increasing singularity of the higher derivatives. Notice that the more *negative* k is, the more rapidly a and its derivatives vanish as $h \to 0$.

(iii) Write also

$$S^{-\infty}(m) := \{a \in C^\infty \mid \text{for each } \alpha \text{ and } N, \ \ |\partial^\alpha a| \leq C_{\alpha,N} h^N m\}.$$

So if a is a symbol belonging to $S^{-\infty}(m)$, then a and all of its derivatives are $O(h^\infty)$ as $h \to 0$.

NOTATION. If the order function is the constant function $m \equiv 1$, we will usually not write it:

$$S^k := S^k(1), \ S^k_\delta := S^k_\delta(1).$$

We will also omit zero superscripts. Thus

$$\begin{aligned} S &:= \{a \in C^\infty(\mathbb{R}^{2n}) \mid |\partial^\alpha a| \leq C_\alpha \ \text{ for all multiindices } \alpha\} \\ S_\delta &:= \{a \in C^\infty \mid |\partial^\alpha a| \leq C_\alpha h^{-\delta|\alpha|} \ \text{ for all multiindices } \alpha\}. \end{aligned}$$

REMARKS: rescaling in h.

(i) We will in the next subsection show that if $a \in S_\delta$, then the quantization formula

$$a^w(x, hD)u(x) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a\left(\frac{x+y}{2}, \xi\right) u(y) d\xi dy$$

makes sense for $u \in \mathcal{S}$. It is often convenient to rescale to the case $h = 1$, by changing to the new variables

$$\tilde{x} := h^{-\frac{1}{2}} x, \ \tilde{y} := h^{-\frac{1}{2}} y, \ \tilde{\xi} := h^{-\frac{1}{2}} \xi. \tag{4.28}$$

Then

$$
\begin{aligned}
a^w(x, hD)u(x) & \\
&= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a\left(\frac{x+y}{2}, \xi\right) e^{\frac{i}{h}\langle x-y, \xi\rangle} u(y)\, dy d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a_h\left(\frac{\tilde{x}+\tilde{y}}{2}, \tilde{\xi}\right) e^{i\langle \tilde{x}-\tilde{y}, \tilde{\xi}\rangle} \tilde{u}(\tilde{y})\, d\tilde{y} d\tilde{\xi} \\
&= a_h^w(\tilde{x}, D)\tilde{u}(\tilde{x}),
\end{aligned}
$$

for

$$
\tilde{u}(\tilde{x}) := u(x) = u(h^{\frac{1}{2}}\tilde{x}), \quad a_h(\tilde{x}, \tilde{\xi}) := a(x, \xi) = a(h^{\frac{1}{2}}\tilde{x}, h^{\frac{1}{2}}\tilde{\xi}). \tag{4.29}
$$

Notice that (4.28) is the only homogeneous change of variables that converts the term $e^{\frac{i}{h}\langle x-y, \xi\rangle}$ into $e^{i\langle \tilde{x}-\tilde{y}, \tilde{\xi}\rangle}$.

(ii) Observe also that if $a \in S_\delta$, then

$$
|\partial^\alpha a_h| = h^{\frac{|\alpha|}{2}} |\partial^\alpha a| \leq C_\alpha h^{|\alpha|\left(\frac{1}{2}-\delta\right)} \tag{4.30}
$$

for each multiindex α. If $\delta > \frac{1}{2}$, the last term is unbounded as $h \to 0$; and consequently we will henceforth always assume

$$
0 \leq \delta \leq \frac{1}{2}.
$$

We see also that the case

$$
\delta = \frac{1}{2}
$$

is critical, in that we then do not get decay as $h \to 0$ for the terms on the right hand side of (4.30) when $|\alpha| > 0$. □

4.3.2 Quantization. Next we discuss the Weyl quantization of symbols in the class $S_\delta(m)$:

THEOREM 4.14 (Quantizing general symbols). *If $a \in S_\delta(m)$, then*

$$
\operatorname{Op}(a) : \mathcal{S} \to \mathcal{S}.
$$

Proof. 1. We take $h = 1$ for simplicity, and set

$$
\operatorname{Op}(a)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y, \xi\rangle} a\left(\frac{x+y}{2}, \xi\right) u(y) d\xi dy.
$$

where $u \in \mathcal{S}$.

Observe next that $L_1 e^{i\langle x-y,\xi\rangle} = e^{i\langle x-y,\xi\rangle}$ for

$$
L_1 := \frac{1 + \langle x-y, D_\xi\rangle}{1 + \langle x-y\rangle^2};
$$

and $L_2 e^{i\langle x-y,\xi\rangle} = e^{i\langle x-y,\xi\rangle}$ for

$$L_2 := \frac{1-\langle \xi, D_y\rangle}{1+\langle \xi\rangle^2}.$$

We as usual employ these operators, to show that $\mathrm{Op}(a) : \mathcal{S} \to L^\infty$.

2. Also

$$x_j \mathrm{Op}(a)u = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(D_{\xi_j}+y_j)e^{i\langle x-y,\xi\rangle} au \, d\xi dy.$$

Integrate by parts, to conclude that $x^\alpha \mathrm{Op}(a) : \mathcal{S} \to L^\infty$.

Furthermore, since

$$\mathrm{Op}_t(a)\left(e^{-i\left(\frac{1}{2}-t\right)D_x D_\xi}a\right) = \mathrm{Op}(a),$$

we have

$$\begin{aligned} D_{x_j}\mathrm{Op}(a)u &= D_{x_j}\mathrm{Op}_0\left(e^{-\frac{i}{2}D_x D_\xi}a\right)u \\ &= D_{x_j}\left(\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\frac{i}{2}D_x D_\xi}a(y,\xi)e^{i\langle x-y\xi\rangle}u(y)\ d\xi dy\right) \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\frac{i}{2}D_x D_\xi}a(y,\xi)(-D_{y_j}e^{i\langle x-y,\xi\rangle})u(y)\ d\xi dy. \end{aligned}$$

Again integrate by parts, to deduce $D^\beta \mathrm{Op}(a) : \mathcal{S} \to L^\infty$.

3. The estimates in Step 2 together show that $D^\beta x^\alpha \mathrm{Op}(a) : \mathcal{S} \to L^\infty$, for all multiindices α, β. It follows that $\mathrm{Op}(a) : \mathcal{S} \to \mathcal{S}$. $\square$

4.3.3 Asymptotic series. Next we consider infinite sums of terms in various symbol classes.

DEFINITION. Let $a \in S_\delta^{k_0}(m)$ and $a_j \in S_\delta^{k_j}(m)$, where $k_{j+1} < k_j$, $k_j \to -\infty$. We say that a is *asymptotic* to $\sum a_j$, and write

$$a \sim \sum_{j=0}^{\infty} a_j, \tag{4.31}$$

provided for each $N = 1, 2, \dots$

$$a - \sum_{j=0}^{N-1} a_j \in S_\delta^{k_N}(m). \tag{4.32}$$

INTERPRETATION. Observe that for each $h > 0$, the series $\sum_{j=0}^{\infty} a_j$ need not converge in any sense. We are requiring rather in (4.32) that for each N, the difference $a - \sum_{j=0}^{N-1} a_j$, and its derivatives, vanish at appropriate rates as $h \to 0$.

Perhaps surprisingly, we can always construct such an asymptotic sum of symbols:

THEOREM 4.15 (Borel's Theorem).
(i) *Assume* $a_j \in S^{k_j}_\delta(m)$, *where* $k_{j+1} < k_j$, $k_j \to -\infty$. *Then there exists a symbol* $a \in S^{k_0}_\delta(m)$ *such that*

$$a \sim \sum_{j=0}^\infty a_j.$$

(ii) *If also* $\hat{a} \sim \sum_{j=0}^\infty a_j$, *then*

$$a - \hat{a} \in S^{-\infty}(m).$$

Proof. Choose a C^∞ function χ such that

$$0 \le \chi \le 1, \quad \chi \equiv 1 \text{ on } [0,1], \;\; \chi \equiv 0 \text{ on } [2,\infty).$$

We define

$$a := \sum_{j=0}^\infty a_j \chi(\lambda_j h), \tag{4.33}$$

where the sequence $\lambda_j \to \infty$ must be selected. Since $\lambda_j \to \infty$, there are for each $h > 0$ at most finitely many nonzero terms in the sum (4.33).

Now for each multiindex α, with $|\alpha| \le j$, we have

$$\begin{aligned} |\partial^\alpha(a_j\chi(\lambda_j h))| &= |(\partial^\alpha a_j)\chi(\lambda_j h)| \\ &\le C_{j,\alpha} h^{-k_j-\delta|\alpha|} m\chi(\lambda_j h) \\ &= C_{j,\alpha} h^{-k_j-\delta|\alpha|} \frac{\lambda_j h}{\lambda_j h} m\chi(\lambda_j h) \\ &\le 2C_{j,\alpha} \frac{h^{-k_j-1-\delta|\alpha|}}{\lambda_j} m \\ &\le h^{-k_j-1-|\alpha|\delta} 2^{-j} m \end{aligned} \tag{4.34}$$

if λ_j is selected sufficiently large. We can accomplish this for all j and multiindices α with $|\alpha| \le j$. We may assume also $\lambda_{j+1} \ge \lambda_j$, for all j. Now

$$a - \sum_{j=0}^N a_j = \sum_{j=N+1}^\infty a_j\chi(\lambda_j h) + \sum_{j=0}^N a_j(\chi(\lambda_j h) - 1).$$

Fix any multiindex α. Then taking $N \geq |\alpha|$, we have

$$\begin{aligned}\left|\partial^\alpha\left(a - \sum_{j=0}^{N} a_j\right)\right| &\leq \sum_{j=N+1}^{\infty} |(\partial^\alpha a_j)|\chi(\lambda_j h) \\ &\quad + \sum_{j=0}^{N} |\partial^\alpha a_j|(1-\chi(\lambda_j h)) \\ &=: A + B.\end{aligned}$$

According to estimate (4.34),

$$A \leq \sum_{j=N+1}^{\infty} h^{-k_j-1-\delta|\alpha|}2^{-j}m \leq mh^{-k_{N+1}-1-\delta|\alpha|}.$$

Also

$$B \leq \sum_{j=0}^{N} C_{\alpha,j} h^{-k_j-\delta|\alpha|}m(1-\chi(\lambda_j h)).$$

Since $\chi \equiv 1$ on $[0,1]$, $B = 0$ if $0 < h \leq \lambda_N^{-1}$. If $\lambda_N^{-1} \leq h \leq 1$, we have

$$\begin{aligned} B &\leq m\sum_{j=0}^{N} C_{\alpha,j} \leq m\sum_{j=0}^{N} C_{\alpha,j}\lambda_N^{-k_{N+1}} h^{-k_{N+1}-\delta|\alpha|} \\ &= mC_\alpha h^{-k_{N+1}-\delta|\alpha|} \leq mC_\alpha h^{-k_N}.\end{aligned}$$

Thus

$$\left|\partial^\alpha\left(a - \sum_{j=0}^{N} a_j\right)\right| \leq C_\alpha h^{-k_N-\delta|\alpha|}m$$

if $N \geq |\alpha|$, and therefore

$$\left|\partial^\alpha\left(a - \sum_{j=0}^{N-1} a_j\right)\right| \leq C_\alpha h^{-k_N-\delta|\alpha|}m.$$

$\square$

4.3.4 Semiclassical expansions in S_δ. Next we need to reexamine some of our earlier asymptotic expansions, deriving improved estimates on the error terms:

THEOREM 4.16 (More on semiclassical expansions). *Let*

$$A(x) = \frac{1}{2}\langle Qx, x\rangle$$

where Q is symmetric and nonsingular.

(i) *If* $0 \leq \delta \leq \frac{1}{2}$*, then*

$$e^{ihA(D)} : S_\delta(m) \to S_\delta(m).$$

(ii) *If* $0 \leq \delta < \frac{1}{2}$*, we furthermore have the expansion*

$$e^{ihA(D)}a \sim \sum_{k=0}^{\infty} \frac{1}{k!}(ihA(D))^k a \quad \text{in } S_\delta(m). \tag{4.35}$$

REMARK. Since we can for $\delta = 1/2$ always rescale to the case $h = 1$, there cannot exist an expansion like (4.35). □

Proof. 1. First, let $0 \leq \delta < \frac{1}{2}$. Recall from (3.52) in §3.6 the stationary phase expansion

$$\int_{\mathbb{R}^n} e^{\frac{i}{h}\phi(x)} a(x)\,dx \sim \left(\frac{h}{2\pi}\right)^{n/2} e^{\frac{i\pi}{4}\operatorname{sgn} Q} |\det Q|^{1/2} e^{ihA(D)} a(0)$$

for the quadratic phase

$$\phi(x) := \frac{1}{2}\langle Q^{-1}x, x\rangle.$$

Let $\chi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with $\chi \equiv 1$ on $B(0,1)$, $\chi \equiv 0$ on $\mathbb{R}^n - B(0,2)$. Shifting the origin, we have

$$\begin{aligned}
e^{ihA(D)}a(z) &\sim \frac{C_n}{h^{n/2}} \int_{\mathbb{R}^n} e^{\frac{i\phi(w)}{h}} a(z-w)\,dw \\
&= \frac{C_n}{h^{n/2}} \int_{\mathbb{R}^n} e^{\frac{i\phi(w)}{h}} \chi(w) a(z-w)\,dw \\
&\quad + \frac{C_n}{h^{n/2}} \int_{\mathbb{R}^n} e^{\frac{i\phi(w)}{h}} (1-\chi(w)) a(z-w)\,dw \\
&=: A + B,
\end{aligned}$$

for the constant

$$C_n := \frac{(2\pi)^{n/2} e^{-\frac{i\pi}{4}\operatorname{sgn} Q}}{|\det Q|^{1/2}}.$$

2. *Estimate of A.* Since $\chi(w)a(z-w)$ has compact support,

$$A \sim \sum_{k=0}^{\infty} \frac{1}{k!}(ihA(D))^k a(z).$$

3. *Estimate of B.* Let $L := \frac{\langle \partial\phi, hD\rangle}{|\partial\phi|^2}$ and note $Le^{\frac{i\phi}{h}} = e^{\frac{i\phi}{h}}$. Consequently,

$$\begin{aligned}
|B| &= \frac{C_n}{h^{n/2}} \left| \int_{\mathbb{R}^n} \left(L^N e^{\frac{i\phi}{h}} \right) (1-\chi(w)) a(z-w)\, dw \right| \\
&= \frac{C_n}{h^{n/2}} \left| \int_{\mathbb{R}^n} e^{\frac{i\phi}{h}} (L^T)^N ((1-\chi)a)\, dw \right| \\
&\leq Ch^{N-\frac{n}{2}} \sup_{|\alpha|\leq N} \int_{\mathbb{R}^n} |\partial^\alpha a(z-w)| \langle z-w\rangle^{-N}\, dw \\
&\leq Ch^{N-\frac{n}{2}-\delta N} m(z).
\end{aligned}$$

We similarly check also the higher derivatives, to conclude that $B \in S_\delta^{-N}(m)$ for all N.

4. Now assume $\delta = 1/2$. In this case we can rescale, by setting

$$\tilde{w} = wh^{-1/2}.$$

Then

$$e^{ihA(D)}a(z) = C_n \int_{\mathbb{R}^n} e^{i\phi(\tilde{w})} a(z - \tilde{w}h^{1/2})\, d\tilde{w}.$$

We use χ to break the integral into two pieces A and B, as above. $\square$

THEOREM 4.17 (Symbol class of $a\#b$).

(i) *If $a \in S_\delta(m_1)$ and $b \in S_\delta(m_2)$, then*

$$a\#b \in S_\delta(m_1 m_2). \tag{4.36}$$

Furthermore,

$$\operatorname{Op}(a) \circ \operatorname{Op}(b) = \operatorname{Op}(a\#b)$$

in the sense of operators mapping $\mathcal{S}$ to $\mathcal{S}$.

(ii) *Also*

$$a\#b - ab \in S_\delta^{2\delta-1}(m_1 m_2). \tag{4.37}$$

Proof. 1. Clearly

$$c(w,z) := a(w)b(z) \in S_\delta(m_1(w)m_2(z))$$

in $\mathbb{R}^{4n}$. If we put

$$A(D_{w,z}) := \sigma(D_x, D_\xi; D_y, D_\eta)/2,$$

for $w = (x,\xi)$ and $z = (y,\eta)$, then according to Theorem 4.16, we have

$$\exp(ihA(D))c \in S_\delta(m_1(w)m_2(z)).$$

Since (4.18) says

$$a\#b(w) = \left(\exp(ihA(D))c\right)(w, w),$$

assertion (4.36) follows.

The second statement of assertion (i) follows from the density of $\mathcal{S}$ in $S_\delta(m)$.

2.

$\square$

4.4 OPERATORS ON $\mathbf{L^2}$

So far our symbol calculus has built operators acting on either the Schwartz space $\mathcal{S}$ of smooth functions or its dual space $\mathcal{S}'$. But for applications we would like to handle functions in more convenient spaces, most notably L^2.

Our next goal is therefore showing that if $a \in S_{1/2}$, then $\mathrm{Op}(a)$ extends to become a bounded linear operator acting on L^2.

Throughout this section, we always take

$$h = 1.$$

Decomposition. We select $\chi \in C_c^\infty(\mathbb{R}^{2n})$ such that

$$0 \leq \chi \leq 1, \quad \chi \equiv 0 \quad \text{on } \mathbb{R}^{2n} - B(0, 2),$$

and

$$\sum_{\alpha \in \mathbb{Z}^{2n}} \chi_\alpha \equiv 1, \tag{4.38}$$

where $\chi_\alpha := \chi(\cdot - \alpha)$ is χ shifted by the lattice point $\alpha \in \mathbb{Z}^{2n}$. Write

$$a_\alpha := \chi_\alpha a;$$

then

$$a = \sum_{\alpha \in \mathbb{Z}^{2n}} a_\alpha.$$

We also define

$$b_{\alpha\beta} := \bar{a}_\alpha \# a_\beta \quad (\alpha, \beta \in \mathbb{Z}^{2n}).$$

LEMMA 4.18 (Decay of mixed terms). *For each N and each multiindex γ, we have the estimate*

$$|\partial^\gamma b_{\alpha\beta}(z)| \leq C_{\gamma,N} \langle \alpha - \beta \rangle^{-N} \langle z - \frac{\alpha + \beta}{2} \rangle^{-N} \tag{4.39}$$

for $z = (x, \xi) \in \mathbb{R}^{2n}$.

Proof. 1. We can rewrite formula (4.21) as

$$b_{\alpha\beta}(z) = \frac{1}{\pi^{2n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{i\phi(w_1,w_2)} \bar{a}_\alpha(z-w_1) a_\beta(z-w_2)\, dw_1 dw_2,$$

for

$$\phi(w_1, w_2) = -2\sigma(x,\xi,y,\eta) = 2\langle x, \eta\rangle - 2\langle \xi, y\rangle \tag{4.40}$$

and

$$w = (w_1, w_2) \quad \text{for } w_1 = (x,\xi), \quad w_2 = (y,\eta). \tag{4.41}$$

2. Select $\zeta : \mathbb{R}^{4n} \to \mathbb{R}$ such that

$$0 \le \zeta \le 1,\ \zeta \equiv 1 \text{ on } \ B(0,1),\ \zeta \equiv 0 \text{ on } \mathbb{R}^{4n} - B(0,2).$$

Then

$$\begin{aligned} b_{\alpha\beta}(z) &= c_n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\phi} \zeta(w) \bar{a}_\alpha(z-w_1) a_\beta(z-w_2)\, dw_1 dw_2 \\ &\quad + c_n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\phi} (1-\zeta(w)) \bar{a}_\alpha(z-w_1) a_\beta(z-w_2)\, dw_1 dw_2 \\ &=: A + B. \end{aligned}$$

3. *Estimate of A.* We have

$$|A| \le \iint_{\{|w|\le 2\}} |\bar{a}_\alpha(z-w_1)||a_\beta(z-w_2)|\, dw_1 dw_2.$$

The integrand equals

$$\chi(z-w_1-\alpha)\chi(z-w_2-\beta)|a(z-w_1)||a(z-w_2)|$$

and thus vanishes, unless

$$|z-w_1-\alpha| \le 2 \text{ and } |z-w_2-\beta| \le 2.$$

But then

$$|\alpha-\beta| \le 4 + |w_1| + |w_2| \le 8$$

and

$$\left| z - \frac{\alpha+\beta}{2} \right| \le 4 + |w_1| + |w_2| \le 8.$$

Hence

$$|A| \le C_N \langle \alpha-\beta\rangle^{-N} \langle z - \frac{\alpha+\beta}{2}\rangle^{-N}$$

for any N. Similarly,

$$|\partial^\gamma A| \le C_{N,\gamma} \langle \alpha-\beta\rangle^{-N} \langle z - \frac{\alpha+\beta}{2}\rangle^{-N}.$$

4. *Estimate of B.* Observe from (4.40) and (4.41) that

$$\partial\phi(w_1, w_2) = 2(\eta, -y, -\xi, x),$$

and so

$$|\partial\phi(w)| = 2|w|.$$

Also, $Le^{i\phi} = e^{i\phi}$ for

$$L := \frac{\langle \partial\phi, D\rangle}{|\partial\phi|^2}.$$

Since the integrand of B vanishes unless $|w| \geq 1$, the usual integration-by-parts argument shows that

$$|B| \leq C_M \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{n2}} \langle w\rangle^{-M} \bar{A}_\alpha(z - w_1) A_\beta(z - w_2)\, dw_1 dw_2$$

and $\operatorname{spt} A_\alpha \subseteq B(\alpha, 2)$, $\operatorname{spt} A_\beta \subseteq B(\beta, 2)$. Thus the integrand vanishes, unless

$$\frac{1}{c}\langle w\rangle \leq \langle \alpha - \beta\rangle, \langle z - \frac{\alpha+\beta}{2}\rangle \leq C\langle w\rangle.$$

Hence

$$\begin{aligned} |B| &\leq C_M \langle \alpha - \beta\rangle^{-N} \langle z - \frac{\alpha+\beta}{2}\rangle^{-N} \iint \langle w\rangle^{2N-M}\, dw_1 dw_2 \\ &\leq C_M \langle \alpha - \beta\rangle^{-N} \langle z - \frac{\alpha+\beta}{2}\rangle^{-N} \end{aligned}$$

if M is large enough. Similarly,

$$|\partial^\gamma B| \leq C_{N,\gamma} \langle \alpha - \beta\rangle^{-N} \langle z - \frac{\alpha+\beta}{2}\rangle^{-N}.$$

$\square$

LEMMA 4.19 (Operator norms). *For each N,*

$$\|\mathrm{Op}(b_{\alpha\beta})\|_{L^2 \to L^2} \leq C_N \langle \alpha - \beta\rangle^{-N}.$$

Proof. Recall that

$$\mathrm{Op}(a) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} \hat{a}(l) \mathrm{Op}(e^{il})\, dl$$

and that $\mathrm{Op}(e^{il})$ is a unitary operator on L^2. Consequently

$$\|\mathrm{Op}(a)\|_{L^2 \to L^2} \leq C \int_{\mathbb{R}^{2n}} |\hat{a}(l)|\, dl.$$

Therefore for $M > 2n$ we can estimate

$$\begin{aligned}
\|O(b_{\alpha\beta})\|_{L^2\to L^2} &\leq C\|\hat{b}_{\alpha\beta}\|_{L^1} \leq C\|\langle\xi\rangle^M \hat{b}_{\alpha\beta}\|_{L^\infty} \\
&\leq C \max_{|\gamma|\leq M} \|\widehat{D^\gamma b_{\alpha\beta}}\|_{L^\infty} \\
&\leq C \max_{|\gamma|\leq M} \|D^\gamma b_{\alpha\beta}\|_{L^1} \\
&\leq C \sup_{|\gamma|\leq M} \|\langle z\rangle^M D^\gamma b_{\alpha\beta}\|_{L^1} \\
&\leq C\langle\alpha-\beta\rangle^{-N},
\end{aligned}$$

according to Lemma 4.18. □

THEOREM 4.20 (Boundedness on $\mathbf{L^2}$). *If $0 \leq \delta \leq 1/2$ and the symbol a belongs to S_δ, then*

$$\mathrm{Op}(a) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is bounded, with the estimate

$$\|\mathrm{Op}(a)\|_{L^2\to L^2} \leq C \sum_{|\alpha|\leq M} |\partial^\alpha a|. \tag{4.42}$$

REMARK. We again emphasize that the stated estimate is for the case $h = 1$. If instead $0 < h < 1$, we can rescale, as will be demonstrated in the proof of Theorem 5.1. □

Proof. We have $\mathrm{Op}(b_{\alpha\beta}) = A_\alpha^* A_\beta$. Thus Lemma 4.19 says

$$\|A_\alpha^* A_\beta\|_{L^2\to L^2} \leq C\langle\alpha-\beta\rangle^{-N}.$$

Therefore

$$\sup_\alpha \sum_\beta \|A_\alpha A_\beta^*\|^{1/2} \leq C \sum_\beta \langle\alpha-\beta\rangle^{-N/2} \leq C;$$

and similarly

$$\sup_\alpha \sum_\beta \|A_\alpha^* A_\beta\|^{1/2} \leq C.$$

We now apply the Cotlar–Stein Theorem B.6. □

As a first application, we record the useful

THEOREM 4.21 (Composition and multiplication). *Suppose that $a, b \in S_\delta$ for $0 \leq \delta < \frac{1}{2}$.*

Then

$$\|a^w b^w - (ab)^w\|_{L^2\to L^2} = O(h^{1-2\delta}) \tag{4.43}$$

as $h \to 0$.

Proof. 1. In light of (4.22), we see that

$$a\#b - ab \in S_{\delta}^{2\delta-1}.$$

Hence Theorem 4.20 impliles

$$a^w b^w - (ab)^w = (a\#b - ab)^w = O(h^{1-2\delta}).$$

□

For the borderline case $\delta = \frac{1}{2}$, we have this assertion:

THEOREM 4.22 (Disjoint supports). *Suppose that* $a, b \in S_{\frac{1}{2}}$. *Assume also that* $\operatorname{spt}(a), \operatorname{spt}(b) \subset K$ *and*

$$\operatorname{dist}(\operatorname{spt}(a), \operatorname{spt}(b)) \geq \gamma > 0,$$

where the compact set K *and the constant* γ *are independent of* h. *Then*

$$\|a^w b^w\|_{L^2 \to L^2} = O(h^{\infty}). \tag{4.44}$$

Proof. Remember from (4.21) that

$$a\#b(z) = \frac{1}{(h\pi)^{2n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\phi(w_1,w_2)} a(z - w_1) b(z - w_2)\, dw_1 dw_2,$$

for for $w = (x, \xi), w_1 = (y, \eta), w_2 = (z, \zeta)$, and

$$\phi(w_1, w_2) = -2\sigma(x, \xi, y, \eta) = 2\langle x, \eta\rangle - 2\langle \xi, y\rangle.$$

We then proceed as in the proof of Lemma 4.18: $|\partial\phi| = 2|w|$ and thus the operator

$$L := \frac{\langle \partial\phi, D\rangle}{|\partial\phi|^2}$$

has smooth coefficients on the support of $a(w - w_1)b(w - w_2)$. From our assumption that $a, b \in S_{\frac{1}{2}}$, we see that

$$L^N(a(w - w_1)b(w - w_2)) = O(h^{\frac{N}{2}}).$$

The uniform bound on the support shows that $a\#b \in S^{-\infty}$. Its Weyl quantization is therefore bounded on L^2, with norm of order $O(h^{\infty})$. □

4.5 INVERSES

At this stage we have constructed in appropriate generality the quantizations $\operatorname{Op}(a)$ of various symbols a. We turn therefore to the practical problem of understanding how the algebraic and analytic behavior of the function a dictates properties of the corresponding quantized operators.

In this section we suppose that $a : \mathbb{R}^{2n} \to \mathbb{C}$ is nonvanishing; so that the function a is pointwise invertible. Can we draw the same conclusion about $\mathrm{Op}(a)$?

DEFINITION. We say the symbol a is *elliptic* if there exists a constant $\gamma > 0$ such that

$$|a| \geq \gamma > 0 \quad \text{ on } \mathbb{R}^{2n}.$$

THEOREM 4.23 (Inverses for elliptic symbols). *Assume that $a \in S_\delta$ for $0 \leq \delta < \frac{1}{2}$ and that a is elliptic.*

Then for some constant $h_0 > 0$,

$$\mathrm{Op}(a)^{-1}$$

exists as a bounded linear operator on $L^2(\mathbb{R}^n)$, provided $0 < h \leq h_0$.

Proof. Let $b := \frac{1}{a}$, $b \in S_\delta$. Then

$$a\#b = 1 + r_1, \text{ with } r_1 \in S_\delta^{2\delta-1}.$$

Likewise

$$b\#a = 1 + r_2, \text{ with } r_2 \in S_\delta^{2\delta-1}.$$

Hence if $A := \mathrm{Op}(a)$, $B := \mathrm{Op}(b)$, $R_1 := \mathrm{Op}(r_1)$ and $R_2 := \mathrm{Op}(r_2)$, we have

$$\begin{aligned} A \cdot B &= I + R_1 \\ B \cdot A &= I + R_2, \end{aligned}$$

with

$$\|R_1\|_{L^2 \to L^2}, \|R_2\|_{L^2 \to L^2} = O(h^{1-2\delta}) \leq \frac{1}{2}$$

if $0 < h \leq h_0$.

Thus $A = \mathrm{Op}(a)$ has an approximate left inverse and an approximate right inverse. Applying then Theorem B.2, we deduce that A^{-1} exists. $\square$

4.6 GÅRDING INEQUALITIES

We continue studying how properties of the symbol a translate into properties of the corresponding quantized operators. In this section we suppose that a is real-valued and nonnegative, and ask the consequences for $A = \mathrm{Op}(a)$.

THEOREM 4.24 (Easy Gårding inequality). *Assume $a = a(x, \xi)$ is a real-valued symbol in S and*

$$a \geq \gamma > 0 \quad \text{on } \mathbb{R}^{2n}. \tag{4.45}$$

Then for each $\epsilon > 0$ there exists $h_0 = h_0(\epsilon) > 0$ such that

$$\langle a^w(x, hD)u, u \rangle \geq (\gamma - \epsilon)\|u\|_{L^2}^2 \tag{4.46}$$

for all $0 < h \leq h_0$, $u \in C_c^\infty(\mathbb{R}^n)$.

Proof. We will show that

$$(a - \lambda)^{-1} \in S \quad \text{if } \lambda < \gamma - \epsilon. \tag{4.47}$$

Indeed if $b := (a - \lambda)^{-1}$, then

$$(a - \lambda)\#b = 1 + \frac{h}{2i}\{a - \lambda, b\} + O(h^2) = 1 + O(h^2),$$

the bracket term vanishing since b is a function of $a - \lambda$. Therefore

$$(a^w - \lambda) \circ b^w = I + O(h^2)_{L^2 \to L^2},$$

and so b^w is an approximate right inverse of $a^w - \lambda$. Likewise b^w is an approximate left inverse.

Hence Theorem B.2 implies $a^w - \lambda$ is invertible for each $\lambda < \gamma - \epsilon$. Consequently,

$$\operatorname{spec}(a^w) \subset [\gamma - \epsilon, \infty).$$

According then to Theorem B.1,

$$\langle a^w u, u \rangle \geq (\gamma - \epsilon)\|u\|_{L^2}^2$$

for all $u \in C_c^\infty(\mathbb{R}^n)$. □

We next improve the preceding estimate:

THEOREM 4.25 (Sharp Gårding inequality). *Assume $a = a(x, \xi)$ is a symbol in S and*

$$a \geq 0 \quad \text{on } \mathbb{R}^{2n}. \tag{4.48}$$

Then there exist constants $h_0 > 0$, $C \geq 0$ such that

$$\langle a^w(x, hD)u, u \rangle \geq -Ch\|u\|_{L^2}^2 \tag{4.49}$$

for all $0 < h \leq h_0$ and $u \in C_c^\infty(\mathbb{R}^n)$.

REMARK. The estimate (4.49) is in fact true for each quantization $\mathrm{Op}_t(a)$ $(0 \leq t \leq 1)$. And for the Weyl quantization, the stronger *Fefferman–Phong inequality* holds:

$$\langle a^w(x, hD)u, u\rangle \geq -Ch^2\|u\|_{L^2}^2$$

for $0 < h \leq h_0$, $u \in C_c^\infty(\mathbb{R}^n)$. □

We will need

LEMMA 4.26 (Gradient estimate). *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be* C^2, *with*

$$|\partial^2 f| \leq A.$$

Suppose also $f \geq 0$. *Then*

$$|\partial f| \leq (2Af)^{1/2} \quad (x \in \mathbb{R}^n). \tag{4.50}$$

Proof. By Taylor's Theorem,

$$f(x+y) = f(x) + \langle \partial f(x), y\rangle + \int_0^1 (1-t)\langle \partial^2 f(x+ty)y, y\rangle\, dt.$$

Let $y = -\lambda \partial f(x)$, $\lambda > 0$ to be selected. Then since $f \geq 0$, we have

$$\begin{aligned}
\lambda|\partial f(x)|^2 &\leq f(x) + \lambda^2 \int_0^1 (1-t)\langle \partial^2 f(x - \lambda t \partial f(x))\partial f(x), \partial f(x)\rangle\, dt \\
&\leq f(x) + \frac{\lambda^2}{2} A|\partial f(x)|^2.
\end{aligned}$$

Let $\lambda = \frac{1}{A}$. Then $|\partial f(x)|^2 \leq 2Af(x)$. □

Proof of Theorem 4.25 1. The primary goal is to show that if

$$\lambda = \mu h \tag{4.51}$$

and μ is fixed sufficiently large, then

$$h(a+\lambda)^{-1} \in S_{1/2}\left(\frac{1}{\mu}\right), \tag{4.52}$$

with estimates independent of μ.

To begin the proof of (4.52) we consider for any multiindex $\alpha = (\alpha_1, \dots, \alpha_{2n})$ the partial derivative ∂^α in the variables x and ξ.

We claim that $\partial^\alpha(a+\lambda)^{-1}$ has the form

(4.53)

$$\partial^\alpha(a+\lambda)^{-1} = (a+\lambda)^{-1} \sum_{k=1}^{|\alpha|} \sum_{\alpha = \beta^1 + \dots + \beta^k |\beta^j| \geq 1} C_{\beta^1, \dots, \beta^k} \prod_{j=1}^k (a+\lambda)^{-1}\partial^{\beta^j} a,$$

for appropriate constants $C_{\beta^1,\dots,\beta^k}$. To see this, observe that when we compute $\partial^\alpha (a+\lambda)^{-1}$ a typical term involves k differentiations of $(a+\lambda)^{-1}$ with the remaining derivatives falling on a. In obtaining (4.53) we for each $k \le |\alpha|$ partition α into multiindex $\beta^1, \dots, \beta^k$, each of which corresponds to one derivative falling on $(a+\lambda)^{-1}$ and the remaining derivatives falling on a.

2. Now Lemma 4.26 implies for $|\beta^j| = 1$ that

$$|\partial^{\beta_j} a|(a+\lambda)^{-1} \le C\lambda^{1/2} \tag{4.54}$$

since $\lambda^{1/2}|\partial a| \le C\lambda^{1/2} a^{1/2} \le C(\lambda + a)$. Furthermore,

$$|\partial^{\beta_j} a|(a+\lambda)^{-1} \le C\lambda^{-1} \tag{4.55}$$

if $|\beta_j| \ge 2$, since $a \in S$.

Consequently, for each partition $\alpha = \beta^1 + \cdots + \beta^k$ and $0 < \lambda \le 1$:

$$\left|\prod_{j=1}^{k} (a+\lambda)^{-1} \partial^{\beta_j} a\right| \le C \prod_{|\beta_j|\ge 2} \lambda^{-1} \prod_{|\beta_j|=1} \lambda^{-1/2} \le C \prod_{j=1}^{k} \lambda^{-\frac{|\beta_j|}{2}} = C\lambda^{-\frac{|\alpha|}{2}}.$$

Therefore

$$|\partial^\alpha (a+\lambda)^{-1}| \le C_\alpha (a+\lambda)^{-1} \lambda^{-\frac{|\alpha|}{2}}. \tag{4.56}$$

But since $\lambda = \mu h$, this implies

$$(a+\lambda)^{-1} \in S_{1/2}\left(\frac{1}{\mu h}\right);$$

that is,

$$h(a+\lambda)^{-1} \in S_{1/2}\left(\frac{1}{\mu}\right),$$

with estimates independent of μ.

3. Since $a + Ch \in S \subseteq S_{\frac{1}{2}}$, we can define $(a+Ch)\#b$, for $b = (a+Ch)^{-1}$. Using Taylor's formula, we compute

$$\begin{aligned}
(a+Ch)\#b &= e^{\frac{ih}{2}\sigma(D_x,D_\xi,D_y,D_\eta)}(a+Ch)(x,\xi)b(y,\eta))\Big|_{\substack{x=y\\ \xi=\eta}} \\
&= 1 + \frac{h}{2i}\{a+Ch, b\} \\
&\quad + \int_0^1 (1-t)e^{\frac{ith}{2}\sigma(\dots)}\left(\frac{ih}{2}\sigma(\dots)\right)^2 (a+Ch)b \, dt \\
&=: 1 + r.
\end{aligned}$$

Now according to (4.56), $hb \in S_{1/2}(1/\mu)$ and so $h^2\partial^\alpha b \in S_{1/2}(1/\mu)$ for $|\alpha| = 2$. The operation $e^{\frac{ith}{2}\sigma(\dots)}$ preserves the symbol class. Hence

$$\|r^w(x, hD)\|_{L^2 \to L^2} \le \frac{C}{\mu} \le \frac{1}{2}$$

if μ is now fixed large enough. Consequently b^w is an approximate right inverse of $(a^w + Ch)$, and is similarly an approximate left inverse.

So $(a^w + Ch)^{-1}$ exists. Likewise $(a^w + \gamma + Ch)^{-1}$ exists for all $\gamma \ge 0$. Thus

$$\operatorname{spec}(a^w) \subseteq [-Ch, \infty).$$

According then to Theorem B.1,

$$\langle a^w(x, hD)u, u\rangle \ge -Ch\|u\|_{L^2}^2$$

for all $u \in C_c^\infty(\mathbb{R}^n)$. $\square$

5. Semiclassical defect measures

One way to understand limits as $h \to 0$ of a collection of functions $\{u(h)\}_{0<h\leq h_0}$ bounded in L^2 is to construct corresponding *semiclassical defect measures* μ, which record the limiting behavior of certain quadratic forms acting on $u(h)$. If in addition these functions solve certain operator equations or PDE, we can deduce various properties of the measure μ and thereby indirectly recover information about asymptotics of the functions $u(h)$.

5.1 CONSTRUCTION, EXAMPLES

In the first two sections of this chapter, we consider a collection of functions $\{u(h)\}_{0<h\leq h_0}$ that is bounded in $L^2(\mathbb{R}^n)$:

$$\sup_{0<h\leq h_0} \|u(h)\|_{L^2} < \infty. \tag{5.1}$$

For the time being, we do not assume that $u(h)$ solves any PDE.

THEOREM 5.1 (An operator norm bound). *Suppose $a \in S$. Then*

$$\|a^w(x, hD)\|_{L^2\to L^2} \leq C \sup_{\mathbb{R}^{2n}} |a| + O(h) \tag{5.2}$$

as $h \to 0$.

Proof. We showed earlier in Theorem 4.20 that if $a \in S$ and $h = 1$, then

$$\|a^w(x, D)\|_{L^2\to L^2} \leq C \sup_{|\alpha|\leq M} |\partial^\alpha a| \tag{5.3}$$

for some M.

Suppose now $a \in S$ and $u \in \mathcal{S}$. We rescale by taking

$$\tilde{x} := h^{-\frac{1}{2}}x,\ \tilde{y} := h^{-\frac{1}{2}}y,\ \tilde{\xi} := h^{-\frac{1}{2}}\xi$$

and

$$\tilde{u}(\tilde{x}) := h^{\frac{n}{4}}u(x) = h^{\frac{n}{4}}u(h^{\frac{1}{2}}\tilde{x}).$$

This is a different rescaling of u from that discussed earlier in (4.29), the advantage being that $u \mapsto \tilde{u}$ is now a unitary transformation of L^2: $\|u\|_{L^2} = \|\tilde{u}\|_{L^2}$.

Then

$$
\begin{aligned}
&a^w(x,hD)u(x) \\
&\qquad = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a\left(\frac{x+y}{2},\xi\right) e^{\frac{i}{h}\langle x-y,\xi\rangle}u(y)\,dyd\xi \\
&\qquad = \frac{h^{-\frac{n}{4}}}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a_h\left(\frac{\tilde{x}+\tilde{y}}{2},\tilde{\xi}\right) e^{i\langle \tilde{x}-\tilde{y},\tilde{\xi}\rangle}\tilde{u}(\tilde{y})\,d\tilde{y}d\tilde{\xi} \qquad (5.4)\\
&\qquad = h^{-\frac{n}{4}}a_h^w(\tilde{x},D)\tilde{u}(\tilde{x}),
\end{aligned}
$$

for

$$a_h(\tilde{x},\tilde{\xi}) := a(x,\xi) = a(h^{\frac{1}{2}}\tilde{x}, h^{\frac{1}{2}}\tilde{\xi}).$$

Hence, noting that $dx = h^{\frac{n}{2}}d\tilde{x}$, we deduce from (5.4) and (5.3) that

$$
\begin{aligned}
\|a^w(x,hD)u\|_{L^2} &= \|a_h^w(\tilde{x},D)\tilde{u}\|_{L^2} \\
&\leq \|a_h^w\|_{L^2\to L^2}\|\tilde{u}\|_{L^2} \\
&\leq C\sup_{|\alpha|\leq M}|\partial^\alpha a_h|\|u\|_{L^2} \\
&\leq C\sup_{|\alpha|\leq M}h^{\frac{|\alpha|}{2}}|\partial^\alpha a|\|u\|_{L^2}\,.
\end{aligned}
$$

This implies (5.2). □

THEOREM 5.2 (Existence of defect measure). *There exists a Radon measure μ on $\mathbb{R}^n$ and a sequence $h_j \to 0$ such that*

$$\langle a^w(x,h_jD)u(h_j),u(h_j)\rangle \to \int_{\mathbb{R}^{2n}} a(x,\xi)\,d\mu \qquad (5.5)$$

for all symbols $a \in S$.

DEFINITION. We call μ a *microlocal defect measure* associated with the family $\{u(h)\}_{0<h\leq h_0}$.

Proof. 1. Let $\{a_k\} \subset C_c^\infty(\mathbb{R}^{2n})$ be dense in $C_0(\mathbb{R}^{2n})$. Select a sequence $h_j^1 \to 0$ such that

$$\langle a_1^w(x,h_j^1D)u(h_j^1),u(h_j^1)\rangle \to \alpha_1.$$

Select a subsequence $\{h_j^2\} \subseteq \{h_j^1\}$ such that

$$\langle a_2^w(x,h_j^2D)u(h_j^2),u(h_j^2)\rangle \to \alpha_2.$$

Continue, at the k^{th} step extracting a subsequence $\{h_j^k\} \subseteq \{h_j^{k-1}\}$ such that

$$\langle a_k^w(x,h_j^kD)u(h_j^k),u(h_j^k)\rangle \to \alpha_k.$$

By a standard diagonal argument, we see that the sequence $h_j := h_j^j$ converges to 0, with

$$\langle a_k^w(x, h_j D)u(h_j), u(h_j)\rangle \to \alpha_k$$

for all $k = 1, \dots$.

2. Define $\Phi(a_k) := \alpha_k$. Owing to Theorem 5.1, we see for each k that

$$\begin{aligned}|\Phi(a_k)| = |\alpha_k| &= \lim_{h_j \to \infty} |\langle a_k^w u(h_j), u(h_j)\rangle| \\ &\le C \limsup_{h_j \to \infty} \|a_k^w\|_{L^2 \to L^2} \le C \sup_{\mathbb{R}^{2n}} |a_k|.\end{aligned}$$

The mapping Φ is bounded, linear and densely defined, and therefore uniquely extends to a bounded linear functional on S, with the estimate

$$|\Phi(a)| \le C \sup_{\mathbb{R}^{2n}} |a|$$

for all $a \in S$. The Riesz Representation Theorem therefore implies the existence of a (possibly complex-valued) Radon measure on $\mathbb{R}^{2n}$ such that

$$\Phi(a) = \int_{\mathbb{R}^{2n}} a(x, \xi)\, d\mu.$$

□

REMARK. Theorem 5.2 is also valid if we replace the Weyl quantization $a^w = \mathrm{Op}_{1/2}(a)$ by $\mathrm{Op}_t(a)$ for any $0 \le t \le 1$, since the error is then $O(h)$. □

THEOREM 5.3 (Positivity). *The measure μ is real and nonnegative:*

(5.6) $$\mu \ge 0.$$

Proof. We must show that $a \ge 0$ implies

$$\int_{\mathbb{R}^{2n}} a\, d\mu \ge 0.$$

Now since $a \ge 0$, the sharp Gårding inequality, Theorem 4.25, implies

$$a^w(x, hD) \ge -Ch;$$

that is,

$$\langle a^w(x, hD)u(h), u(h)\rangle \ge -Ch\|u(h)\|_{L^2}^2$$

for sufficiently small $h > 0$. Let $h = h_j \to 0$, to deduce

$$\int_{\mathbb{R}^{2n}} a\, d\mu = \lim_{h_j \to \infty} \langle a^w(x, h_j D) u(h_j), u(h_j)\rangle \geq 0.$$

$\square$

EXAMPLE 1: Coherent states. Take the *coherent state*

$$u(h)(x) := (\pi h)^{-\frac{n}{4}} e^{\frac{i}{h}\langle x - x_0, \xi_0\rangle - \frac{1}{2h}|x-x_0|^2},$$

and observe that $\|u(h)\|_{L^2} = 1$. Then there exists precisely one associated semiclassical defect measure, namely

$$\mu := \delta_{(x_0, \xi_0)}.$$

To confirm this statement, take $t = 1$ in the quantization and calculate

$$\begin{aligned}
&\langle a(x, hD) u(h), u(h)\rangle \\
&\quad = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(x,\xi) e^{\frac{i}{h}\langle x-y, \xi\rangle} u(h)(y)\overline{u(h)}(x)\, dy d\xi dx \\
&\quad = \frac{2^{\frac{n}{2}}}{(2\pi h)^{\frac{3n}{2}}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(x,\xi) e^{\frac{i}{h}(\langle x-y,\xi\rangle + \langle y - x_0, \xi_0\rangle - \langle x - x_0, \xi_0\rangle)} \\
&\qquad\qquad e^{-\frac{1}{2h}(|y-x_0|^2 + |x-x_0|^2)}\, dy d\xi dx \\
&\quad = \frac{2^{\frac{n}{2}}}{(2\pi h)^{\frac{3n}{2}}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(x,\xi) e^{\frac{i}{h}\langle x-y, \xi - \xi_0\rangle} \\
&\qquad\qquad e^{-\frac{1}{2h}(|y-x_0|^2 + |x-x_0|^2)}\, dy d\xi dx.
\end{aligned}$$

For each fixed x and ξ, the integral in y is

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y, \xi-\xi_0\rangle} e^{-\frac{1}{2h}|y-x_0|^2}\, dy &= e^{\frac{i}{h}\langle x-x_0, \xi-\xi_0\rangle} \int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle y, \xi-\xi_0\rangle} e^{-\frac{1}{2h}|y|^2}\, dy \\
&= e^{\frac{i}{h}\langle x-x_0, \xi-\xi_0\rangle} \mathcal{F}\left(e^{-\frac{1}{2h}|y|^2}\right)\left(\frac{\xi-\xi_0}{h}\right) \\
&= (2\pi h)^{\frac{n}{2}} e^{\frac{i}{h}\langle x-x_0, \xi-\xi_0\rangle} e^{-\frac{1}{2h}|\xi-\xi_0|^2},
\end{aligned}$$

where we used formula (3.3) for the last equality. Therefore

$$\begin{aligned}
&\langle a(x, hD) u(h), u(h)\rangle \\
&= \frac{2^{\frac{n}{2}}}{(2\pi h)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(x,\xi) e^{\frac{i}{h}\langle x-x_0, \xi-\xi_0\rangle} e^{-\frac{1}{2h}(|x-x_0|^2 + |\xi-\xi_0|^2)}\, dx d\xi \\
&= a(x_0,\xi_0)\frac{2^{\frac{n}{2}}}{(2\pi h)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-x_0, \xi-\xi_0\rangle} e^{-\frac{1}{2h}(|x-x_0|^2 + |\xi-\xi_0|^2)}\, dx d\xi + o(1) \\
&= C a(x_0, \xi_0) + o(1),
\end{aligned}$$

for the constant

$$C := \frac{2^{\frac{n}{2}}}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} e^{-\frac{1}{2}(|x|^2+|\xi|^2)}\, dx d\xi.$$

Taking $a \equiv 1$ and recalling that $\|u(h)\|_{L^2} = 1$, we deduce that $C = 1$. $\square$

EXAMPLE 2: Stationary phase and defect measures. For our next example, take

$$u(h)(x) := e^{\frac{i\phi(x)}{h}} b(x),$$

where $\phi, b \in C^\infty$ and $\|b\|_{L^2} = 1$. Then

$$\begin{aligned}&\langle a(x,hD)u(h), u(h)\rangle \\ &\quad= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x,\xi) e^{\frac{i}{h}(\langle x-y,\xi\rangle + \phi(y)-\phi(x))} b(y)\overline{b(x)}\, dy d\xi dx.\end{aligned}$$

We assume $a \in C_c^\infty(\mathbb{R}^{2n})$ and apply stationary phase. For a given value of x, define

$$\Phi(y,\xi) := \langle x-y, \xi\rangle + \phi(y) - \phi(x).$$

Then

$$\partial_y \Phi = \partial\phi(y) - \xi,\ \ \partial_\xi \Phi = x - y.$$

The Hessian matrix of Φ is

$$\partial^2 \Phi = \begin{pmatrix} \partial^2\phi & -I \\ -I & 0 \end{pmatrix}.$$

Since $\operatorname{sgn}\begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} = 0$, we have $\operatorname{sgn}\begin{pmatrix} t\partial^2\phi & -I \\ -I & 0 \end{pmatrix} = 0$ for $0 \le t \le 1$: this is so since the signature of a matrix is integer–valued, and consequently is invariant if we move along a curve of nonsingular matrices. Consequently

$$\operatorname{sgn}(\partial^2\Phi) = 0.$$

In addition, $|\det \partial^2\Phi| = 1$. Thus as $h \to 0$ the stationary phase asymptotic expression (3.47) implies

$$\langle a^w(x,hD)u(h), u(h)\rangle \to \int_{\mathbb{R}^n} a(x, \partial\phi(x))|b(x)|^2\, dx = \int_{\mathbb{R}^{2n}} a(x,\xi)\, d\mu$$

for the semiclassical defect measure

$$\mu := |b(x)|^2 \delta_{\{\xi = \partial\phi(x)\}}\, \mathcal{L}^n,$$

$\mathcal{L}^n$ denoting n-dimensional Lebesgue measure in the x-variables. $\square$

5.2 DEFECT MEASURES AND PDE

We now assume more about the family $\{u(h)\}_{0<h\leq h_0}$, namely that each function $u(h)$ is an approximate solution of a equation involving the operator $P(h) = p^w(x, hD)$ for some symbol p.

First, let us suppose $P(h)u(h)$ vanishes up to an $o(1)$ error term and see what we can conclude about a corresponding semiclassical defect measure μ.

THEOREM 5.4 (Support of defect measure). *Suppose $p \in S(\langle\xi\rangle^m)$ is real and*

$$|p| \geq \gamma \quad \text{if} \ \ |\xi| \geq C$$

for constants $C, \gamma > 0$. Write $P(h) = p^w$.

Suppose that $u(h)$ satisfies

$$\begin{cases} \|P(h)u(h)\|_{L^2} = o(1) \quad \text{as } h \to 0, \\ \|u(h)\|_{L^2} = 1. \end{cases} \tag{5.7}$$

Then if μ is any microlocal defect measure associated with $\{u(h)\}_{0<h\leq 1}$,

$$\operatorname{spt}\mu \subseteq p^{-1}(0). \tag{5.8}$$

INTERPRETATION. We sometimes call $p^{-1}(0)$ the *characteristic variety* or the *zero energy surface* of the symbol p. We understand (5.8) as saying that in the semiclassical limit $h \to 0$, all of the mass of the solutions $u(h)$ coalesces onto this set. □

Proof. Select $a \in C_c^\infty(\mathbb{R}^{2n})$ such that $\operatorname{spt}(a) \cap p^{-1}(0) = 0$. We must show

$$\int_{\mathbb{R}^{2n}} a \, d\mu = 0.$$

To do so, first select $\chi \in C_c^\infty(\mathbb{R}^{2n})$ such that $\operatorname{spt}(a) \cap \operatorname{spt}(\chi) = \emptyset$ and

$$|p + i\chi| \geq \gamma > 0 \quad \text{on } \mathbb{R}^{2n}.$$

Theorem 4.23 ensures us that $P(h) + i\chi^w$ is invertible on L^2, for small enough h. Observe also that

$$\frac{ap}{p + i\chi} - a = i\frac{a\chi}{p + i\chi} \tag{5.9}$$

Now write $A = a^w$. Since a and χ have disjoint support, (5.9) and Theorems 4.21, 4.22 imply

$$\|A(P(h) + i\chi^w)^{-1}P(h) - A\|_{L^2 \to L^2} = O(h).$$

Therefore (5.7) implies

$$\|Au(h)\|_{L^2} = o(1);$$

and thus

$$\langle Au(h), u(h)\rangle \to 0.$$

But also

$$\langle Au(h_j), u(h_j)\rangle \to \int_{\mathbb{R}^{2n}} a\,d\mu.$$

$\square$

Now we make the stronger assumption that the error term in (5.7) is $o(h)$.

THEOREM 5.5 (Flow invariance). *Suppose that p satisfies the assumptions of Theorem 5.4. Assume also*

$$\begin{cases} \|P(h)u(h)\|_{L^2} = o(h) \quad \text{as } h \to 0, \\ \|u(h)\|_{L^2} = 1. \end{cases} \tag{5.10}$$

Then

$$\int_{\mathbb{R}^{2n}} \{p, a\}\,d\mu = 0 \tag{5.11}$$

for all $a \in C_c^\infty(\mathbb{R}^{2n})$.

INTERPRETATION. Let Φ_t be the flow generated by the Hamiltonian vector field H_p. Then

$$\frac{d}{dt}\int_{\mathbb{R}^{2n}} \Phi_t^* a\,d\mu = \int_{\mathbb{R}^{2n}} (H_p a)(\Phi_t)\,d\mu = \int_{\mathbb{R}^{2n}} \{p, a\}\,d\mu.$$

Conseqently (5.11) asserts that *the semiclassical defect measure μ is flow-invariant.* $\square$

Proof. Since p is real, $P(h) = p^w$ is self-adjoint on L^2. Select a as above and write $A = a^w$, $A = A^*$. Then

$$\begin{aligned} \langle [P(h), A]u(h), u(h)\rangle &= \langle (P(h)A - AP(h))u(h), u(h)\rangle \\ &= \langle Au(h), P(h)u(h)\rangle - \langle P(h)u(h), Au(h)\rangle \\ &= o(h), \text{ as } h \to 0. \end{aligned}$$

On the other hand,

$$[P(h), A] = \frac{h}{i}\{p, a\}^w + O(h^2)_{L^2 \to L^2}.$$

Hence

$$\langle [P(h), A]u(h), u(h)\rangle = \frac{h}{i}\langle \{p, a\}^w u(h), u(h)\rangle + \langle o(h)u(h), u(h)\rangle.$$

Cancel h and let $h = h_j \to 0$:

$$\int_{\mathbb{R}^{2n}} \{p, a\}\, d\mu = 0.$$

Note that even though p may not have compact support, $\{p, a\}$ does. □

This proof illustrates one of the basic principles mentioned in Chapter 1, that an assertion about Hamiltonian dynamics involving the Poisson bracket corresponds to a commutator argument at the quantum level.

REMARK. We have similar statements if we replace $\mathbb{R}^n \times \mathbb{R}^n$ by $\mathbb{T}^n \times \mathbb{R}^n$, where $\mathbb{T}^n$ denotes the flat torus. We will need this observation in the following application. □

5.3 APPLICATION: DAMPED WAVE EQUATION

A damped wave equation. In this section $\mathbb{T}^n$ denotes the flat n-dimensional torus.

Consider now the initial-value problem

$$\begin{cases} (\partial_t^2 + a(x)\partial_t - \Delta)u = 0 & \text{on } \mathbb{T}^n \times \mathbb{R} \\ u = 0,\ u_t = f & \text{on } \mathbb{T}^n \times \{t = 0\}, \end{cases} \tag{5.12}$$

in which the smooth function a is nonnegative, and thus represents a damping mechanism, as we will see.

DEFINITION. The *energy* at time t is

$$E(t) := \frac{1}{2} \int_{\mathbb{T}^n} (\partial_t u)^2 + |\partial_x u|^2\, dx.$$

LEMMA 5.6 (Elementary energy estimates).

(i) *If* $a \equiv 0$, $t \mapsto E(t)$ *is constant.*

(ii) *If* $a \geq 0$, $t \mapsto E(t)$ *is nonincreasing.*

Proof. These assertions follow easily from this calculation:

$$\begin{aligned} E'(t) &= \int_{\mathbb{T}^n} \partial_t u \partial_t^2 u + \langle \partial_x u, \partial_{xt}^2 u \rangle\, dx \\ &= \int_{\mathbb{T}^n} \partial_t u (\partial_t^2 u - \Delta u)\, dx \\ &= -\int_{\mathbb{T}^n} a(x)(\partial_t u)^2\, dx \leq 0. \end{aligned}$$

□

Our eventual goal is showing that if the support of the damping term a is large enough, then we have exponential energy decay for our solution of the wave equation (5.12). Here is the key assumption:

DYNAMICAL HYPOTHESIS.

$$\left\{\begin{array}{l}\text{There exists a time } T>0 \text{ such that any}\\ \text{trajectory of the Hamiltonian vector field of}\\ p(x,\xi)=|\xi|^2\text{, starting at time 0 with } |\xi|=1,\\ \text{intersects the set } \{a>0\} \text{ by the time } T.\end{array}\right. \tag{5.13}$$

Another way to write this is stating that for each initial point $m = (x,\xi) \in \mathbb{T}^n \times \mathbb{R}^n$, with $|\xi| = 1$,

$$\langle a\rangle_T := \frac{1}{T}\int_0^T a(x+t\xi)\,dt > 0.$$

Note that $\langle a\rangle_T$ depends upon $m = (x,\xi)$.

MOTIVATION. Since the damping term a in general depends upon x, we cannot use Fourier transform (or Fourier series) in x to solve (5.12). Instead we define

$$u \equiv 0 \quad \text{ for } t < 0$$

and, at first formally, take the Fourier transform in t:

$$\hat{u}(x,\tau) := \int_0^\infty e^{-it\tau}u(x,t)\,dt \qquad (\tau \in \mathbb{R}).$$

Then

$$\begin{aligned}\Delta\hat{u} &= \int_0^\infty e^{-it\tau}\Delta u\,dt\\ &= \int_0^\infty e^{-it\tau}(\partial_t^2 u + a(x)\partial_t u)\,dt\\ &= \int_0^\infty ((i\tau)^2 + a(x)i\tau)e^{-it\tau}u\;dt - f\\ &= (-\tau^2 + a(x)i\tau)\hat{u} - f.\end{aligned}$$

Consequently,

$$P(\tau)\hat{u} := (-\tau^2 + i\tau a(x) - \Delta)\hat{u} = f. \tag{5.14}$$

Now take τ to be complex, with $\operatorname{Re}\tau \geq 0$, and define

$$P(z,h) := (-h^2\Delta + i\sqrt{z}ha(x) - z) \tag{5.15}$$

for the rescaled variable

$$z = \frac{\tau^2}{h^2}. \tag{5.16}$$

Then (5.14) reads

$$P(z,h)\hat{u} = h^2 f; \tag{5.17}$$

and so, if $P(z,h)$ is invertible,

$$\hat{u} = h^2 P(z,h)^{-1} f. \tag{5.18}$$

We consequently need to study the inverse of $P(z,h)$.

THEOREM 5.7 (Resolvent bounds). *Under the dynamical assumption* (5.13)*, there exist constants $\alpha, C, h_0 > 0$ such that*

$$\|P(z,h)^{-1}\|_{L^2 \to L^2} \leq \frac{C}{h} \tag{5.19}$$

for

$$|\mathrm{Im}\ z| \leq \alpha h,\ |z-1| \leq \alpha,\ 0 < h \leq h_0. \tag{5.20}$$

Proof. 1. We will see in the next chapter that $P(z,h)$ is meromorphic. Consequently it is enough to show that

$$\|u\|_{L^2} \leq \frac{C}{h}\|P(z,h)u\|_{L^2}$$

for all $u \in L^2$ and some constant C, provided z and h satisfy (5.20).

We argue by contradiction. If the assertion were false, then for $m = 1, 2, \dots$ there would exist $z_m \in \mathbb{C}$, $0 < h_m \leq 1/m$ and functions u_m in L^2 such that

$$\|P(z_m,h_m)u_m\|_{L^2} \leq \frac{h_m}{m}\|u_m\|_{L^2}, \quad |\mathrm{Im}\ (z_m)| \leq \frac{h_m}{m}, \quad |z_m - 1| \leq \frac{1}{m}.$$

We may assume $\|u_m\|_{L^2} = 1$. Then

$$P(z_m,h_m)u_m = o(h_m). \tag{5.21}$$

Also,

$$z_m \to 1,\ \mathrm{Im}(z_m) = o(h_m). \tag{5.22}$$

2. Let μ be a microlocal defect measure associated with $\{u_m\}_{m=1}^\infty$. Then Theorem 5.4 implies that

$$\mathrm{spt}(\mu) \subseteq \{|\xi|^2 = 1\}.$$

But $\langle u_m, u_m\rangle = 1$, and so

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} d\mu = 1. \tag{5.23}$$

We will derive a contradiction to this.

3. Hereafter write $P_m := P(z_m, h_m)$. Then

$$\begin{aligned} P_m &= -h_m^2\Delta + i\sqrt{z_m}h_m a(x) - z_m, \\ P_m^* &= -h_m^2\Delta - i\sqrt{\bar{z}_m}h_m a(x) - \bar{z}_m; \end{aligned}$$

and therefore

$$P_m - P_m^* = i(\sqrt{z_m}+\sqrt{\bar{z}_m})h_m a(x) - z_m + \bar{z}_m = 2ih_m a(x) + o(h_m), \tag{5.24}$$

since (5.22) implies that $-z_m + \bar{z}_m = -2i\mathrm{Im}(z_m) = o(h_m)$.

Now select $b \in C_c^\infty(\mathbb{T}^n \times \mathbb{R}^n)$ and set $B = b^w$. Then $B = B^*$. Using (5.21) and (5.24), we calculate that

$$\begin{aligned} o(h_m) = 2i\ \mathrm{Im}\langle BP_m u_m, u_m\rangle &= \langle BP_m u_m, u_m\rangle - \langle u, BP_m u_m\rangle \\ &= \langle (BP_m - P_m^* B)u_m, u_m\rangle \\ &= \langle [B, P_m]u_m, u_m\rangle \\ &\quad + \langle (P_m - P_m^*)Bu_m, u_m\rangle \\ &= \frac{h_m}{i}\langle \{b,p\}^w u_m, u_m\rangle \\ &\quad + 2h_m i\langle (ab)^w u_m, u_m\rangle + o(h_m). \end{aligned}$$

Divide by h_m and let $h_m \to 0$ (through a subsequence, if necessary), to discover that

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \{p,b\} + ab\, d\mu = 0. \tag{5.25}$$

We will select b so that $\{p,b\} + ab > 0$ on $\mathrm{spt}(\mu)$. This will imply

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} d\mu = 0, \tag{5.26}$$

a contradiction to (5.23)

4. Define for $(x,\xi) \in \mathbb{T}^n \times \mathbb{R}^n$, with $|\xi| = 1$,

$$c(x,\xi) := \frac{1}{T}\int_0^T (T-t)a(x+\xi t)\, dt,$$

where T is the time from the dynamical hypothesis (5.13). Hence

$$\begin{aligned}\langle \xi, \partial_x c\rangle &= \frac{1}{T}\int_0^T (T-t)\langle \xi, \partial a(x+\xi t)\rangle \, dt \\ &= \frac{1}{T}\int_0^T (T-t)\frac{d}{dt}a(x+\xi t)\, dt \\ &= \frac{1}{T}\int_0^T a(x+\xi t)\, dt - a(x) \\ &= \langle a\rangle_T - a.\end{aligned}$$

Let

$$b := e^c.$$

Then

$$\langle \xi, \partial_x b\rangle = e^c \langle \xi, \partial_x c\rangle = e^c\langle a\rangle_T - ab.$$

Consequently

$$\{p, b\} + ab = \langle \xi, \partial_x b\rangle + ab = e^c\langle a\rangle_T > 0,$$

as desired. □

THEOREM 5.8 (Exponential decay in time). *Assume the dynamic hypothesis* (5.13) *and suppose u solves the wave equation with damping* (5.12).

Then there exists constants $C, \beta > 0$ such that

$$E(t) \le Ce^{-\beta t}\|f\|_{L^2} \quad \textit{for all times } t > 0. \tag{5.27}$$

REMARK. The following calculations are based upon this idea: to get decay estimates of g on the positive real axis, we estimate $\hat{g}$ in a complex strip $|\mathrm{Im}\, z| \le \alpha$. Indeed if $\beta < \alpha$, then

$$\widehat{e^{\beta t} g} = \int_{-\infty}^{\infty} e^{\beta t} g(t) e^{-it\tau}\, dt = \int_{-\infty}^{\infty} g(t) e^{-it(\tau + i\beta)}\, dt = \hat{g}(\tau + i\beta).$$

Hence an L^2 estimate of $\hat{g}(\cdot + i\beta)$ will imply exponential decay of $g(t)$ for $t \to \infty$. □

Proof. 1. Recall from (5.15), (5.16) that

$$P(\tau) = h^{-2}P(z,h) \quad \text{for } \tau^2 = h^{-2}z.$$

First we assert that if $|\mathrm{Im}\, \tau| \le \alpha$, then

$$\|P(\tau)^{-1}\|_{L^2 \to H^1} = O(1). \tag{5.28}$$

To prove (5.28) we note that

$$\begin{aligned}\|P(z,h)^{-1}u\|_{H^2_h} &\leq C\|h^2\Delta P(z,h)^{-1}u\|_{L^2} + C\|P(z,h)^{-1}u\|_{L^2}\\ &\leq C\|(z+i\sqrt{z}ha)u\|_{L^2} + \frac{C}{h}\|u\|_{L^2},\end{aligned}$$

the last inequality holding according to Theorem 5.7. Thus

$$\|P(z,h)^{-1}u\|_{H^2_h} \leq \frac{C}{h}\|u\|_{L^2}.$$

Rescaling, we find that

$$\|P(\tau)^{-1}u\|_{H^2} \leq C|\tau|\|u\|_{L^2}$$

if $|\mathrm{Im}\ \tau| \leq \alpha$. Also

$$\|P(\tau)^{-1}u\|_{L^2} \leq \frac{C}{|\tau|}\|u\|_{L^2}.$$

Interpolating between the last two inequalities demonstrates that

$$\|P(\tau)^{-1}u\|_{H^1} \leq C\|u\|_{L^2}.$$

This proves (5.28).

2. Next select $\chi : \mathbb{R} \to \mathbb{R}$, $\chi = \chi(t)$, such that

$$0 \leq \chi \leq 1,\ \chi \equiv 1 \text{ on } [1,\infty), \chi \equiv 0 \text{ on } (-\infty, 0).$$

Then if $u_1 := \chi u$, we have

$$(\partial_t^2 + a(x)\partial_t - \Delta)u_1 = g_1, \tag{5.29}$$

for

$$g_1 := \chi'' u + 2\chi'\partial_t u + a(x)\chi' u. \tag{5.30}$$

Note that $u_1(t) = 0$ for $t \leq 0$, and observe also that the support of g_1 lies within $\mathbb{T}^n \times [0,1]$. Furthermore, using energy estimates in Lemma 5.6, we see that

$$\begin{aligned}&\|g_1\|_{L^2((0,\infty);L^2)}\\ &\qquad \leq C\left(\|u\|_{L^2((0,1);L^2)} + \|\partial_t u\|_{L^2((0,1);L^2)}\right) \leq C\|f\|_{L^2}.\end{aligned} \tag{5.31}$$

Now take the Fourier transform of (5.29) in time:

$$P(\tau)\hat{u}_1 = \hat{g}_1.$$

Then

$$\hat{u}_1 = P(\tau)^{-1}\hat{g}_1.$$

Using (5.28) and (5.31), we deduce that

$$\begin{aligned} \|u_1\|_{L^2((0,\infty);H^1)} &= \|\hat{u}_1\|_{L^2((-\infty,\infty);H^1)} \\ &\leq C\|\hat{g}_1\|_{L^2((-\infty,\infty);L^2)} \\ &\leq C\|g_1\|_{L^2((0,\infty);L^2)} \leq C\|f\|_{L^2}. \end{aligned}$$

3. We now modify the foregoing argument to obtain some exponential decay. For this, we recall the Remark above, and compute

$$\begin{aligned} \|e^{\beta t}u_1\|_{L^2((0,\infty);H^1)} &= \|\hat{u}_1(\cdot + i\beta)\|_{L^2((-\infty,\infty);H^1)} \\ &= \|P(\cdot + i\beta)^{-1}\hat{g}(\cdot + i\beta)\|_{L^2((-\infty,\infty);H^1)} \\ &\leq C\|\hat{g}_1\|_{L^2((-\infty,\infty);L^2)} \\ &\leq C\|g_1\|_{L^2((0,\infty);L^2)} \leq C\|f\|_{L^2}. \end{aligned}$$

Since $u_1 = \chi u$, we deduce that

$$\|e^{\beta t}u\|_{L^2((1,\infty),H^1)} \leq C\|f\|_{L^2}. \tag{5.32}$$

4. Finally, fix $T > 2$ and select a new function $\chi : \mathbb{R} \to \mathbb{R}$, such that

$$0 \leq \chi \leq 1,\ \chi \equiv 0 \text{ for } t \leq T-1, \chi \equiv 1 \text{ for } t \geq T.$$

Let $u_2 = \chi u$. Then

$$(\partial_t^2 + a(x)\partial_t - \Delta)u_2 = g_2, \tag{5.33}$$

for

$$g_2 := \chi'' u + 2\chi'\partial_t u + a(x)\chi' u. \tag{5.34}$$

Therefore spt$g \subseteq \mathbb{T}^n \times (T-1, T)$.

Define

$$E_2(t) := \frac{1}{2}\int_{\mathbb{T}^n} (\partial_t u_2)^2 + |\partial_x u_2|^2\, dx.$$

Modifying the calculations in the proof of Lemma 5.6, we use (5.33) and(5.34) to compute

$$\begin{aligned} E_2'(t) &= \int_{\mathbb{T}^n} \partial_t u_2 \partial_t^2 u_2 + \langle \partial_x u_2, \partial_{xt}^2 u_2\rangle\, dx \\ &= \int_{\mathbb{T}^n} \partial_t u_2(\partial_t^2 u_2 - \Delta u_2)\, dx \\ &= -\int_{\mathbb{T}^n} a(x)(\partial_t u_2)^2\, dx + \int_{\mathbb{T}^n} \partial_t u_2 g_2\, dx \\ &\leq C\int_{\mathbb{T}^n} |\partial_t u_2|(|\partial_t u| + |u|)\, dx \\ &\leq CE_2(t) + C\int_{\mathbb{T}^n} u^2 + (\partial_t u)^2\, dx. \end{aligned}$$

Since $E_2(T-1)=0$ and $E_2(T)=E(T)$, Gronwall's inequality implies that

$$E(T)\le C\left(\|u\|^2_{L^2((T-1,T);L^2)}+\|\partial_t u\|^2_{L^2((T-1,T);L^2)}\right). \tag{5.35}$$

5. We need to control the right hand term in (5.35). For this select $\chi:\mathbb{R}\to\mathbb{R}$, such that

$$\begin{cases} 0\le\chi\le 1,\\ \chi\equiv 0 \text{ for } t\le T-2 \text{ and } t\ge T+1,\\ \chi\equiv 1 \text{ for } T-1\le t\le T.\end{cases}$$

We multiply the wave equation (5.12) by $\chi^2 u$ and integrate by parts, to find

$$\begin{aligned} 0 &= \int_{T-2}^{T+1}\int_{\mathbb{T}^n}\chi^2 u(\partial_t^2 u+a(x)\partial_t u-\Delta u)\,dxdt\\ &= \int_{T-2}^{T+1}\int_{\mathbb{T}^n}-\chi^2(\partial_t u)^2-2\chi\chi' u\partial_t u+\chi^2 a(x)u\partial_t u+\chi^2|\partial_x u|^2\,dxdt.\end{aligned}$$

From this identity we derive the estimate

$$\|\partial_t u\|_{L^2((T-1,T);L^2)}\le C\|u\|_{L^2((T-2,T+1);H^1)}.$$

This, (5.35) and (5.32) therefore imply

$$E(T)\le C\|u\|^2_{L^2((T-2,T+1);H^1)}\le Ce^{-\beta T}\|f\|_{L^2},$$

as asserted. □

REMARK. Our methods extend with no difficulty if $\mathbb{T}^n$ is replaced by a general compact Riemannian manifold: see Appendix D. □

6. Eigenvalues and eigenfunctions

In this chapter we are given the potential $V : \mathbb{R}^n \to \mathbb{R}$, and investigate how the symbol

$$p(x, \xi) = |\xi|^2 + V(x)$$

provides interesting information about the corresponding operator

$$P(h) = -h^2 \Delta + V(x).$$

We will focus mostly upon learning how p controls the aysmptotic distribution of the eigenvalues of $P(h)$ in the semiclassical limit $h \to 0$.

6.1 THE HARMONIC OSCILLATOR

Our plan is to consider first the simplest case, when the potential is quadratic; and to simplify even more, we begin in one dimension. So suppose that $n = 1$, $h = 1$ and $V(x) = x^2$. Thus we start with the *one-dimensional quantum harmonic oscillator*, meaning the operator

$$P_0 u := (-\partial^2 + x^2)u.$$

6.1.1 Eigenvalues and eigenfunctions of $\mathbf{P_0}$. We can as follows employ certain auxillary first-order differential operators to compute explicitly the eigenvalues and eigenfunctions for P_0.

NOTATION. Let us write

$$A_+ := D_x + ix, \ A_- := D_x - ix, \tag{6.1}$$

where $D_x = \frac{1}{i}\partial_x$, and call A_+ the *creation operator* and A_- the *annihilation operator*. (This terminology is from particle physics.)

LEMMA 6.1 (Properties of $\mathbf{A_\pm}$). *The creation and annihilation operators satisfy these identities:*

$$A_+^* = A_-, \ A_-^* = A_+, \tag{6.2}$$

$$P_0 = A_+ A_- + 1 = A_- A_+ - 1. \tag{6.3}$$

Proof. 1. It is easy to check that $D_x^* = D_x$ and $(ix)^* = -ix$.

2. Calculate

$$\begin{aligned} A_+A_-u &= (D_x + ix)(D_x - ix)u \\ &= \left(\frac{1}{i}\partial_x + ix\right)\left(\frac{1}{i}u_x - ixu\right) \\ &= -u_{xx} - (xu)_x + xu_x + x^2u \\ &= -u_{xx} - u - xu_x + xu_x + x^2u \\ &= P_0u - u. \end{aligned}$$

Similarly,

$$\begin{aligned} A_-A_+u &= (D_x - ix)(D_x + ix)u \\ &= \left(\frac{1}{i}\partial_x - ix\right)\left(\frac{1}{i}u_x + ixu\right) \\ &= -u_{xx} + (xu)_x - xu_x + x^2u \\ &= P_0u + u. \end{aligned}$$

□

We can now use $A_\pm$ to find all the eigenvalues and eigenfunctions of P_0:

THEOREM 6.2 (Eigenvalues and eigenfunctions).

(i) *We have*

$$\langle P_0u, u\rangle \geq \|u\|_{L^2}^2$$

for all $u \in C_c^\infty(\mathbb{R}^n)$*. That is,* $P_0 \geq 1$.

(ii) *The function*

$$v_0 =: e^{-\frac{x^2}{2}}$$

is an eigenfunction corresponding to the smallest eigenvalue 1.

(iii) *Set*

$$v_n := A_+^n v_0$$

for $n = 1, 2, \ldots$*. Then*

$$P_0v_n = (2n+1)v_n. \tag{6.4}$$

(iv) *Define the normalized eigenfunctions*

$$u_n := \frac{v_n}{\|v_n\|_{L^2}}.$$

Then

$$u_n(x) = H_n(x)e^{-\frac{x^2}{2}} \tag{6.5}$$

where $H_n(x) = c_n x^n + \cdots + c_0$ $(c_n \neq 0)$ *is a polynomial of degree* n.

(v) *We have*

$$\langle u_n, u_m \rangle = \delta_{nm}.$$

(vi) *Furthermore, the collection of eigenfunctions* $\{u_n\}_{n=0}^{\infty}$ *is complete in* $L^2(\mathbb{R}^n)$.

Proof. 1. We note that

$$[D_x, x]u = \frac{1}{i}(xu)_x - \frac{x}{i}u_x = \frac{u}{i},$$

and consequently

$$i[D_x, x] = 1.$$

Therefore

$$\begin{aligned} \|u\|_{L^2}^2 &= \langle i[D_x, x]u, u\rangle \\ &\leq 2\|xu\|_{L^2}\|D_x u\|_{L^2} \\ &\leq \|xu\|_{L^2}^2 + \|D_x u\|_{L^2}^2 = \langle P_0 u, u\rangle. \end{aligned}$$

2. Next, observe

$$A_- v_0 = \frac{1}{i}\left(e^{-\frac{x^2}{2}}\right)_x - ixe^{-\frac{x^2}{2}} = 0;$$

and so $P_0 v_0 = (A_+A_- + 1)v_0 = v_0$.

3. We can further calculate that

$$\begin{aligned} P_0 v_n &= (A_+A_- + 1)A_+ v_{n-1} \\ &= A_+(A_-A_+ - 1)v_{n-1} + 2A_+v_{n-1} \\ &= A_+P_0 v_{n-1} + 2A_+v_{n-1} \\ &= (2n-1)A_+v_{n-1} + 2A_+v_{n-1} \quad \text{(by induction)} \\ &= (2n+1)v_n. \end{aligned}$$

4. The form (6.5) of v_n, u_n follows by induction.

5. Note also that

$$\begin{aligned} [A_-, A_+] &= A_-A_+ - A_+A_- \\ &= (P_0 + 1) - (P_0 - 1) = 2. \end{aligned}$$

Hence if $m > n$,

$$\begin{aligned}\langle v_n, v_m\rangle &= \langle A_+^n v_0, A_+^m v_0\rangle\\ &= \langle A_-^m A_+^n v_0, v_0\rangle \quad (\text{since } A_- = A_+^*)\\ &= \langle A_-^{m-1}(A_+A_- + 2)A_+^{n-1}v_0, v_0\rangle.\end{aligned}$$

After finitely many steps, the foregoing equals

$$\langle (\dots)A_- v_0, v_0\rangle = 0,$$

since $A_- v_0 = 0$.

6. Suppose $\langle u_n, g\rangle = 0$ for $n = 0, 1, 2, \dots$; we must show $g \equiv 0$.
Now since $H_n(x) = c_n x^n + \dots$, with $c_n \neq 0$, we have

$$\int_{-\infty}^{\infty} g(x)e^{-\frac{x^2}{2}}p(x)\,dx = 0$$

for each polynomial p. Hence

$$\int_{-\infty}^{\infty} g(x)e^{-\frac{x^2}{2}}e^{-ix\xi}\,dx = \int_{-\infty}^{\infty} g(x)e^{-\frac{x^2}{2}}\sum_{k=0}^{\infty}\frac{(-ix\xi)^k}{k!}\,dx;$$

and so $\mathcal{F}\left(ge^{-\frac{x^2}{2}}\right) \equiv 0$. This implies $ge^{-\frac{x^2}{2}} \equiv 0$ and consequently $g \equiv 0$. □

6.1.2 Higher dimensions, rescaling. Suppose now $n > 1$, and write

$$P_0 := -\Delta + |x|^2;$$

this is the *n-dimensional quantum harmonic oscillator*. We define also

$$u_\alpha(x) := \prod_{j=1}^{n} u_{\alpha_j}(x_j) = \prod_{j=1}^{n} H_{\alpha_j}(x_j)e^{-\frac{|x|^2}{2}}$$

for each multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$. Then

$$P_0 u_\alpha = (-\Delta + |x|^2)u_\alpha = (2|\alpha| + n)u_\alpha,$$

for $|\alpha| = \alpha_1 + \dots + \alpha_n$. Hence u_α is an eigenfunction of P_0 corresponding to the eigenvalue $2|\alpha| + n$.

We next restore the parameter $h > 0$ by setting

$$P_0(h) := -h^2\Delta + |x|^2, \tag{6.6}$$

$$u_\alpha(h)(x) := h^{-\frac{n}{4}}\prod_{j=1}^{n} H_{\alpha_j}\left(\frac{x_j}{\sqrt{h}}\right)e^{-\frac{|x|^2}{2h}} \tag{6.7}$$

and

$$E_\alpha(h) := (2|\alpha| + n)h. \tag{6.8}$$

Then

$$P_0(h)u_\alpha(h) = E_\alpha(h)u_\alpha(h).$$

Upon reindexing, we can write these eigenfunction equations as

$$P_0(h)u_j(h) = E_j(h)u_j(h) \qquad (j = 1, \dots). \tag{6.9}$$

6.1.3 Asymptotic distribution of eigenvalues. With these explicit formulas in hand, we can study the behavior of the eigenvalues $E(h)$ in the semiclassical limit:

THEOREM 6.3 (Weyl's law for harmonic oscillator). *Assume that* $0 \le a < b < \infty$. *Then*

(6.10)

$$\begin{aligned} &\#\{E(h) \mid a \le E(h) \le b\} \\ &\qquad\qquad = \frac{1}{(2\pi h)^n}(\mathrm{Vol}\{a \le |\xi|^2 + |x|^2 \le b\} + o(1)). \end{aligned}$$

as $h \to 0$.

In this formula and hereafter, "Vol" means volume, that is, Lebesgue measure.

Proof. We may assume that $a = 0$. Since $E(h) = (2|\alpha| + n)h$ for some multiindex α, we have

$$\begin{aligned} \#\{E(h) \mid 0 \le E(h) \le b\} &= \#\left\{\alpha \mid 0 \le 2|\alpha| + n \le \frac{b}{h}\right\} \\ &= \#\left\{\alpha \mid \alpha_1 + \dots + \alpha_n \le R\right\}, \end{aligned}$$

for $R := \frac{b-nh}{2h}$. Therefore

$$\begin{aligned} &\#\{E(h) \mid 0 \le E(h) \le b\} \\ &\qquad = \mathrm{Vol}\ \{x \mid x_i \ge 0,\ x_1 + \dots + x_n \le R\} + o(R^n) \\ &\qquad = \frac{1}{n!}R^n + o(R^n) \quad \text{as } R \to \infty \\ &\qquad = \frac{1}{n!}\left(\frac{b}{2h}\right)^n + o(\frac{1}{h^n}) \quad \text{as } h \to 0. \end{aligned}$$

Recall that the volume of the simplex $\{x \mid x_i \ge 0,\ x_1 + \dots + x_n \le 1\}$ is $(n!)^{-1}$.

Next we note that $\mathrm{Vol}\{|\xi|^2 + |x|^2 \le b\} = \alpha(2n)b^n$, where $\alpha(k) := \pi^{\frac{k}{2}}(\Gamma(\frac{k}{2}+1))^{-1}$ is the volume of the unit ball in $\mathbb{R}^k$. Setting $k = 2n$, we compute that $\alpha(2n) = \pi^n(n!)^{-1}$. Hence

$$\begin{aligned}\#\{E(h) \mid 0 \le E(h) \le b\} &= \frac{1}{n!}\left(\frac{b}{2h}\right)^n + o(\frac{1}{h^n}) \\ &= \frac{1}{(2\pi h)^n}(\mathrm{Vol}\{|\xi|^2 + |x|^2 \le b\}) + o(\frac{1}{h^n}).\end{aligned}$$

□

6.2 SYMBOLS AND EIGENFUNCTIONS

For this section, we take the symbol

(6.11) $$p(x,\xi) = |\xi|^2 + V(x),$$

corresponding to the operator

(6.12) $$P(h) = -h^2\Delta + V(x).$$

We assume that the potential $V : \mathbb{R}^n \to \mathbb{R}$ is smooth, and satisfies the growth conditions:

(6.13) $$|\partial^\alpha V(x)| \le C_\alpha \langle x\rangle^k, \quad V(x) \ge C\langle x\rangle^k \quad \text{for } |x| \ge R,$$

for appropriate constants $k, C, C_\alpha, R > 0$.

Our plan in the next section is to employ our detailed knowledge about the eigenvalues of the harmonic oscillator

$$P_0(h) = -h^2\Delta + |x|^2$$

to estimate the asymptotics of the eigenvalues of $P(h)$. This section develops some useful techniques that will aid us in this task.

6.2.1 Concentration in phase space. First, we make the important observation that in the semiclassical limit eigenfunctions $u(h)$ "are concentrated in phase space" on appropriate energy surface $\{|\xi|^2+V(x) = E\}$ of the symbol. The proof of this assertion illustrates well the power of the pseudodifferential operator techniques.

THEOREM 6.4 ($\mathbf{h^\infty}$-estimates). *Suppose that $u(h) \in L^2(\mathbb{R}^n)$ solves*

(6.14) $$P(h)u(h) = E(h)u(h).$$

Assume as well that $a \in S$ is a symbol satisfying

$$\{|\xi|^2 + V(x) = E\} \cap \mathrm{spt}(a) = \emptyset.$$

Then if

$$|E(h) - E| < \delta$$

for some sufficiently small $\delta > 0$, *we have the estimate*

$$\|a^w(x, hD)u(h)\|_{L^2} = O(h^\infty)\|u(h)\|_{L^2}. \tag{6.15}$$

Proof. 1. The set $K := \{|\xi|^2 + V(x) = E\} \subset \mathbb{R}^{2n}$ is compact. Hence there exists a compactly supported C^∞ function χ on $\mathbb{R}^{2n}$ such that

$$0 \leq \chi \leq 1, \ \chi \equiv 1 \text{ on } K, \ \chi \equiv 0 \text{ on } \operatorname{spt}(a).$$

Define the symbol

$$b := |\xi|^2 + V(x) - E(h) + i\chi = p(x, \xi) - E(h) + i\chi$$

and the order function

$$m := \langle \xi \rangle^2 + \langle x \rangle^k.$$

2. Then if $|E(h) - E|$ is small enough,

$$|b| \geq \frac{1}{C} m \quad \text{ on } \mathbb{R}^{2n}$$

for some constant $C > 0$. Consequently $b \in S(m)$, with $b^{-1} \in S(m^{-1})$. Thus there exist $c \in S(m^{-1})$, $r_1, r_2 \in S$ such that

$$\begin{cases} b^w \circ c^w = I + r_1^w \\ c^w \circ b^w = I + r_2^w. \end{cases}$$

where r_1^w, r_2^w are $O(h^\infty)$.

Then

$$a^w(x, hD) \circ c^w(x, hD) \circ b^w(x, hD) = a^w(x, hD) + O(h^\infty), \tag{6.16}$$

and

$$b^w(x, hD) = P(h) - E(h) + i\chi^w \tag{6.17}$$

Furthermore

$$a^w(x, hD) \circ c^w(x, hD) \circ \chi^w(x, hD) = O(h^\infty), \tag{6.18}$$

since $\operatorname{spt}(a) \cap \operatorname{spt}(\chi) = \emptyset$. Since $P(h)u(h) = E(h)u(h)$, (6.16) and (6.17) imply that

$$a^w(x, hD)u(h) = a^w \circ c^w \circ (P(h) - E(h) + i\chi^w)u(h) + O(h^\infty) = O(h^\infty).$$

$\square$

For the next result, we temporarily return to the case of the quantum harmonic oscillator, developing some sharper estimates:

THEOREM 6.5 (Improved estimates for the harmonic oscillator). *Suppose that $u(h) \in L^2(\mathbb{R}^n)$ is an eigenfuction of the harmonic oscillator:*

$$P_0(h)u(h) = E(h)u(h). \tag{6.19}$$

If the symbol a belongs to S, then

$$\|a^w(x, hD)u(h)\|_{L^2} = O\left(\left(\frac{h}{E(h)}\right)^{\infty}\right)\|u(h)\|_{L^2}. \tag{6.20}$$

REMARK. The precise form of the right hand side of (6.20) will later let us handle eigenvalues $E(h) \to \infty$. □

Proof. 1. We as follows rescale the harmonic oscillator, in order that we may work near a fixed energy level E.

Set

$$y := \frac{x}{\sqrt{E}}, \quad \tilde{h} := \frac{h}{E}, \quad E(\tilde{h}) := \frac{E(h)}{E};$$

so that $|E(h) - E| \leq \delta E$. Then put

$$P_0(h) := -h^2\Delta_x + |x|^2, \quad P_0(\tilde{h}) := -\tilde{h}^2\Delta_y + |y|^2;$$

whence

$$P_0(h) - E(h) = E(P(\tilde{h}) - \widetilde{E}(\tilde{h})).$$

We next introduce the unitary transformation

$$U_E u(y) := E^{\frac{n}{2}} u(E^{\frac{1}{2}} y).$$

Then

$$U_E P_0(h) U_E^{-1} = E P_0(\tilde{h});$$

and more generally

$$U_E b^w(x, hD) U_E^{-1} = \widetilde{b}^w(y, \tilde{h}D), \quad \widetilde{b}(y, \eta) := b(E^{\frac{1}{2}} y, E^{\frac{1}{2}} \eta).$$

We will denote the symbol classes defined using $\tilde{h}$ by the symbol $\widetilde{S}_\delta$.

2. Theorem 6.4, applied now to eigenfuctions of $P_0(\tilde{h})$, shows that if $|E(\tilde{h}) - 1| < \delta$ and $(P_0(\tilde{h}) - E(\tilde{h}))\widetilde{u}(\tilde{h}) = 0$, then

$$\|\widetilde{b}^w(y, \tilde{h}D)\widetilde{u}(\tilde{h})\|_{L^2} = O(\tilde{h}^{\infty})\|\widetilde{u}(\tilde{h})\|_{L^2},$$

where $\widetilde{b}(y, \eta) \in \widetilde{S}$ is assumed to have support contained, say, in $|y|^2 + |\eta|^2 \leq 1/2$.

Translated to the original h and x as above, this assertion provides us with the bound

$$\|b^w(x, hD)u(h)\|_{L^2} = O((h/E)^{\infty})\|u(h)\|_{L^2}, \tag{6.21}$$

for

$$b(x,\xi) = \widetilde{b}(E^{-1/2}x, E^{-1/2}\xi) \in S.$$

Note that $\mathrm{spt}(b) \subset \{|x|^2 + |\xi|^2 \leq E/2\}$.

3. In view of (6.21), we only need to show that for a in the statement of the theorem, we have

$$\|(a^w(x,hD)(1 - b^w(x,hD))\|_{L^2\to L^2} = O((h/E)^\infty),$$

where b is as in (6.21). That is the same as showing

$$\|\widetilde{a}^w(y,\tilde{D})(1 - \widetilde{b}^w(y,\tilde{h}D))\|_{L^2\to L^2} = O(\tilde{h}^\infty), \tag{6.22}$$

for

$$\widetilde{a}(y,\eta) = a(E^{\frac{1}{2}}y, E^{\frac{1}{2}}\eta) \in \widetilde{S}_{\frac{1}{2}},$$

with

$$\mathrm{dist}(\mathrm{spt}(\widetilde{a}), \mathrm{spt}(1-\widetilde{b})) \geq 1/C > 0,$$

uniformly in $\tilde{h}$. And estimate (6.22) is a consequence of Theorem 4.22. $\square$

6.2.2 Projections. We next study how projections onto the span of various eigenfunctions of the harmonic oscillator $P_0(h)$ are related to our symbol calculus.

THEOREM 6.6 (Projections and symbols). *Suppose that a is a symbol such that*

$$\mathrm{spt}(a) \subset \{|\xi|^2 + |x|^2 < R\}.$$

Let

$\Pi :=$ *projection in L^2 onto*

$$\mathit{span}\{u(h) \mid P_0(h)u(h) = E(h)u(h) \ \mathit{for}\ E(h) \leq R+1\},$$

Then

$$\|a^w \circ (I - \Pi)\|_{L^2\to L^2} = O(h^\infty) \tag{6.23}$$

and

$$\|(I - \Pi)\circ a^w\|_{L^2\to L^2} = O(h^\infty). \tag{6.24}$$

Proof. First of all observe that we can write

$$(I - \Pi) = \sum_{E_j(h) > R+1} u_j(h) \otimes u_j(h),$$

meaning that

$$(I-\Pi)u = \sum_{E_j(h)>R+1} \langle u_j(h), u\rangle u_j(h).$$

Therefore

$$a^w \circ (I - \Pi) = \sum_{E_j(h) > R+1} (a^w u_j(h)) \otimes u_j(h);$$

and so

$$\|a^w \circ (I - \Pi)\|_{L^2 \to L^2} \leq \left(\sum_{E_j(h) > R+1} \|a^w u_j(h)\|_{L^2}^2 \right)^{\frac{1}{2}}. \tag{6.25}$$

Next, observe that Weyl's Law for the harmonic oscillator, Theorem 6.3, implies that

$$E_j(h) \geq c j^{\frac{1}{n}} h$$

for some constant $c > 0$. According then to Theorem 6.5, for each $M < N$ we have

$$\begin{aligned} \|a^w u_j(h)\|_{L^2} &\leq C_N \left(\frac{h}{E_j(h)} \right)^N \\ &\leq C h^M \left(\frac{h}{E_j(h)} \right)^{N-M} \\ &\leq C h^M j^{-\frac{N-M}{n}}. \end{aligned}$$

Consequently, if we fix $N - M > n$, the sum on the right hand side of (6.25) is less than or equal to Ch^M. This proves (6.23) $\square$

6.3 WEYL'S LAW

6.3.1 Spectrum and resolvents. We next show that the spectrum of $P(h)$ consists entirely of eigenvalues.

THEOREM 6.7 (Resolvents and spectrum). *There exists a constant $h_0 > 0$ such that if $0 < h \leq h_0$, then the resolvent*

$$(P(h) - z)^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is a meromorphic function with only simple, real poles.

In particular, the spectrum of $P(h)$ is discrete.

Proof. 1. Let $|z| \leq E$, where E is fixed; and as before let $P_0(h) = -h^2 \Delta + |x|^2$ be the harmonic oscillator. As in Theorem 6.6 define

$$\begin{aligned} \Pi := &\text{ projection in } L^2 \text{ onto} \\ &\operatorname{span}\{u \mid P_0(h) u = E(h) u \text{ for } E(h) \leq R + 1\}. \end{aligned}$$

Suppose now $\mathrm{spt}(a) \subset \{|x|^2 + |\xi|^2 \leq R\}$. Owing to Theorem 6.6, we have

$$\|a^w - a^w \Pi\|_{L^2 \to L^2} = \|a^w - \Pi a^w\|_{L^2 \to L^2} = O(h^\infty). \tag{6.26}$$

2. Fix $R > 0$ so large that

$$\{|\xi|^2 + V(x) \leq E\} \subset \{|x|^2 + |\xi|^2 < R\}.$$

Select $\chi \in C^\infty(\mathbb{R}^{2n})$ with $\mathrm{spt}(\chi) \subset \{|x|^2 + |\xi|^2 \leq R\}$ so that

$$|\xi|^2 + V(x) - z + \chi \geq \frac{1}{C} m$$

for $m = \langle\xi\rangle^2 + \langle x\rangle^k$ and all $|z| \leq E$. From (6.26) we see that $\chi = \Pi\chi\Pi + O(h^\infty)$. Recall that the symbolic calculus guarantees that $P - z + \chi$ is invertible, if h is small enough. Consequently, so is $P - z + \Pi\chi\Pi$, since the two operators differ by an $O(h^\infty)$ term.

3. Now write

$$P - z = P - z + \Pi\chi\Pi - \Pi\chi\Pi$$

Consequently

$$P - z \;\; = \;\; (P - z + \Pi\chi\Pi)(I - (P - z + \Pi\chi\Pi)^{-1}\chi)$$

Note that $\Pi\chi\Pi$ is an operator of finite rank. So Lemma 6.8, stated and proved below, asserts that the family of operators

$$(I - (P - z + \Pi\chi\Pi)^{-1}\Pi\chi\Pi)^{-1}$$

is meromorphic in z. It follows that $(P - z)^{-1}$ is meromorphic on L^2. The poles are the eigenvalues, and the self-adjointness of P implies that they are real and simple. □

LEMMA 6.8 (Inverses). *Suppose that $z \mapsto M(z)$ is an analytic mapping of a connected open set $\Omega \subset \mathbb{C}$ into the space of finite rank operators on a Hilbert space H.*

If $(I + M(z_0))^{-1}$ exists at some point $z_0 \in \Omega$, then $(I + M(z))^{-1}$ is a meromorphic family of operators on Ω.

Proof.[1] 1. If H is finite dimensional, we observe that the function $\det(I + M(z))$ is analytic and not identically zero in Ω, in view of our assumption that $(I + M(z_0))^{-1}$ exists. Thus

$$(I + M(z))^{-1} = \frac{\mathrm{cof}\,(I + M(z))}{\det(I + M(z))} \quad \text{(Cramer's rule)}$$

[1]Omit this proof on first reading.

is meromorphic. Here "cof A" means the cofactor matrix of A.

2. For the general case that H is infinite dimensional, let z_1 be an arbitrary point in Ω. We can choose two finite rank operators

$$R_- : \mathbb{C}^{n_-} \to \mathcal{H}, \quad R_+ : \mathcal{H} \to \mathbb{C}^{n_+}$$

such that

$$\begin{aligned} R_-(\mathbb{C}^{n_-}) \cap \operatorname{Range}(I + M(z_1)) &= \{0\}, \\ \operatorname{Ker}\, (R_+|_{\operatorname{Ker}})(I + M(z_1)) &= \{0\}. \end{aligned}$$

We can choose $R_\pm$ to be of maximal rank, in which case

$$\begin{aligned} n_+ &= \dim \operatorname{Ker}\, (I + M(z)) \\ n_- &= \dim \operatorname{Ker}\, (I + M(z)^*). \end{aligned}$$

An argument from linear algebra shows that

$$\mathcal{P}(z) := \begin{pmatrix} I + M(z) & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{H} \otimes \mathbb{C}^{n_-} \longrightarrow \mathcal{H} \otimes \mathbb{C}^{n_+} ,$$

is invertible at $z = z_1$ and hence for z in a neighbourhood of z_1. We can write the inverse as

$$\mathcal{P}(z)^{-1} = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix} : \mathcal{H} \otimes \mathbb{C}^{n_+} \longrightarrow \mathcal{H} \otimes \mathbb{C}^{n_-}.$$

Next, the simple and celebrated *Schur complement formulas*

$$\begin{aligned} (I + M(z))^{-1} &= E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z) , \\ E_{-+}(z)^{-1} &= -R_-(I + M(z))^{-1}R_+ \end{aligned} \tag{6.27}$$

show that $(I + M(z))^{-1}$ is invertible if and only if $n_+ = n_-$ and $E_{-+}(z)$ is invertible.

3. The foregoing argument shows that there exists a locally finite covering $\{\Omega_j\}$ of Ω, such that for $z \in \Omega_j$, $I + M(z)$ is invertible precisely when $E_{-+}(z)$, defined for $z \in \Omega_j$, is invertible.

Since Ω is connected and since $I + M(z_0)$ is invertible, we deduce that $n_- = n_+$ for all points in Ω and that $\det E_{-+}(z)$ is not identically zero in Ω_j. The finite dimensional argument shows that $E_{-+}(z)^{-1}$ is meromorphic and then (6.27) gives the meromorphy of $(I + M(z))^{-1}$. $\square$

REMARK: Theorem 6.7 can be obtained more directly by using the Spectral Theorem and compactness of $(P + i)^{-1}$. We will demonstrate this in our later discussion Schrödinger operators on manifolds in Appendix D. The approach using the Schur formulas is *Grushin's method*: see [S-Z2]. $\square$

6.3.2 Spectral asymptotics. We are now ready for the main result of this section:

THEOREM 6.9 (Weyl's Law). *Suppose that V satisfies the conditions* (6.13) *and that $E(h)$ are the eigenvalues of $P(h) = -h^2\Delta + V(x)$.*

Then for each $a < b$, we have

$$\begin{aligned} &\#\{E(h) \mid a \leq E(h) \leq b\} \\ &\qquad = \frac{1}{(2\pi h)^n}(\mathrm{Vol}\{a \leq |\xi|^2 + V(x) \leq b\} + o(1)). \end{aligned} \tag{6.28}$$

as $h \to 0$.

Proof. 1. Let

$$N(\lambda) = \#\{E(h) \mid E(h) \leq \lambda\}.$$

Select $\chi \in C_c^\infty(\mathbb{R}^{2n})$ so that

$$\chi \equiv 1 \text{ on } \{p \leq \lambda + \epsilon\},\ \chi \equiv 0 \text{ on } \{p \geq \lambda + 2\epsilon\}.$$

Then

$$a := p + (\lambda + \epsilon)\chi - \lambda \geq \frac{1}{C_\epsilon} m,$$

for $m = \langle \xi \rangle^2 + \langle x \rangle^m$, is elliptic. Consequently for small h, a^w is invertible.

2. *Claim #1*: We have

$$\langle (P(h) + (\lambda + \epsilon)\chi^w - \lambda)u, u \rangle \geq \gamma \|u\|_{L^2}^2 \tag{6.29}$$

for some $\gamma > 0$.

To see this, take $b \in S(m^{1/2})$ so that $b^2 = a$. Then $b^2 = b\#b + r$, where $r = O(h)$; and thus

$$\mathrm{Op}(a) = \mathrm{Op}(b^2) = \mathrm{Op}(b)\mathrm{Op}(b) + \mathrm{Op}(r) = BB + O(h)_{L^2 \to L^2}$$

for $B = \mathrm{Op}(b)$. Hence for sufficiently small $h > 0$,

$$\begin{aligned} \langle (P(h) + (\lambda + \epsilon)\chi^w - \lambda)u, u \rangle &= \langle \mathrm{Op}(a)u, u \rangle \\ &= \|Bu\|_{L^2} + \langle O(h)u, u \rangle \\ &\geq \|Bu\|^2 - O(h)\|u\|_{L^2}^2 \geq \gamma \|u\|_{L^2}^2, \end{aligned}$$

for some $\gamma > 0$, since B^{-1} exists. This proves (6.29).

3. *Claim #2*: For each $\delta > 0$, there exists a bounded linear operator Q such that

$$\chi^w = Q + O(h^\infty)_{L^2 \to L^2} \tag{6.30}$$

and

$$\text{rank}(Q) \leq \frac{1}{(2\pi h)^n}(\text{Vol}\{p \leq \lambda + 2\epsilon\} + \delta). \tag{6.31}$$

To prove this, cover the set $\{p \leq \lambda + 2\epsilon\}$ with balls $B_j = B((x_j, \xi_j), r_j)$ $(j = 1, \ldots, N)$ such that

$$\sum_{j=1}^{N} \text{Vol } B_j \leq \text{Vol}\{p \leq \lambda + 2\epsilon\} + \frac{\delta}{2}.$$

Define the "shifted" harmonic oscillator

$$P_j(h) := (hD_x - \xi_j)^2 + (x - x_j)^2;$$

and set

$$\begin{aligned} \Pi \;\; := \;\; & \text{projection in } L^2 \text{ onto } V\text{, the span of} \\ & \{u \mid P_j(h)u = E_j(h)u,\ E_j(h) \leq r_j,\ j = 1, \ldots, N\}. \end{aligned}$$

Then

$$\begin{aligned} \chi^w &= \Pi\chi^w + (I - \Pi)\chi^w \\ &= \Pi\chi^w + O(h^\infty) \quad \text{ by Theorem 6.6} \\ &= Q + O(h^\infty) \end{aligned}$$

for

$$Q := \Pi\chi^w.$$

Clearly Q has finite rank, since

$$\begin{aligned} \text{rank}\, Q &= \text{dim(image of } Q) \\ &\leq \text{dim(image of } \Pi) \\ &= \sum_{j=1}^{N} \#\{E_j(h) \mid E_j(h) \leq r_j\} \\ &= \frac{1}{(2\pi h)^n}\left(\sum_{j=1}^{N} \text{Vol } B_j + o(1)\right), \end{aligned}$$

according to Weyl's law for the harmonic oscillator, Theorem 6.3. Consequently

$$\text{rank}\, Q \leq \frac{1}{(2\pi h)^n}\left(\text{Vol}\{p \leq \lambda + 2\epsilon\} + \frac{\delta}{2} + o(1)\right).$$

This proves Claim #2.

4. We next employ Claims #1,2 and Theorem B.4. We have

$$\begin{aligned}\langle P(h)u,u\rangle &\geq (\lambda+\gamma)\|u\|_{L^2}^2-(\lambda+\epsilon)\langle Qu,u\rangle+\langle O(h^\infty)u,u\rangle\\ &\geq \lambda\|u\|_{L^2}^2-(\lambda+\epsilon)\langle Qu,u\rangle,\end{aligned}$$

where rank $Q \leq \frac{1}{(2\pi h)^n}(\mathrm{Vol}\{p\leq\lambda+2\epsilon\}+\delta)$. Theorem B.4,(i) implies then that

$$N(\lambda)\leq\frac{1}{(2\pi h)^n}(\mathrm{Vol}\{p\leq\lambda+2\epsilon\}+\delta+o(1)).$$

This holds for all $\epsilon,\delta>0$, and so

$$N(\lambda)\leq\frac{1}{(2\pi h)^n}(\mathrm{Vol}\{p\leq\lambda\}+o(1)) \tag{6.32}$$

as $h\to 0$.

5. We must prove the opposite inequality.

Claim #3: Suppose $B_j=B((x_j,\xi_j),r_j)\subset\{p<\lambda\}$. Then if

$$P_j(h)u=E_j(h)u$$

and $E(h)\leq r_j$, we have

$$\langle P_j(h)u,u\rangle\leq(\lambda+\epsilon+O(h^\infty))\|u\|_{L^2}^2 \tag{6.33}$$

To prove this claim, select a symbol $a\in C_{\mathrm{c}}^\infty(\mathbb{R}^{2n})$, with

$$a\equiv 1 \text{ on } \{p\leq\lambda\},\ \mathrm{spt}(a)\subset\{p\leq\lambda+\epsilon\}.$$

Let $c:=1-a$. Then $u-a^wu=c^wu=O(h^\infty)$ according to Theorem 6.6, since $\mathrm{spt}(1-a)\cap B_j=\emptyset$.

Define $b^w:=P(h)a^w$. Now $p\in S(m)$ and $a\in S(m^{-1})$. Thus b^w is bounded in L^2, since $b=pa+O(h)\in S$. Observe also that $b\leq\lambda+\frac{\epsilon}{2}$, and so

$$b^w=\mathrm{Op}(h)\leq\lambda+\frac{3\epsilon}{4}.$$

Therefore

$$\langle P(h)a^wu,u\rangle=\langle b^wu,u\rangle\leq\left(\lambda+\frac{3\epsilon}{4}\right)\|u\|_{L^2}^2.$$

Since $a^wu=u+O(h^\infty)$, we deduce

$$\langle P(h)u,u\rangle\leq(\lambda+\epsilon+O(h^\infty))\|u\|_{L^2}^2.$$

This proves Claim #3.

6. Now find disjoint balls $B_j\subset\{p<\lambda\}$ such that

$$\mathrm{Vol}\{p<\lambda\}\leq\sum_{j=1}^N\mathrm{Vol}\ B_j+\delta.$$

Let $V := \text{span}\{u \mid P_j(h)u = E_j(h)u,\ E_j(h) \leq r_j,\ j = 1, \ldots, N\}$. Owing to Claim #3,

$$\langle Pu, u \rangle \leq (\lambda + \delta)\|u\|_{L^2}^2$$

for all $u \in V$. Also, Theorem 6.3 implies

$$\begin{aligned}
\dim V &\geq \sum_{j=1}^{N} \#\{E_j(h) \leq r_j\} \\
&= \frac{1}{(2\pi h)^n} \left(\sum_{j=1}^{N} \text{Vol } B_j + o(1) \right) \\
&\geq \frac{1}{(2\pi h)^n} (\text{Vol}\{p < \lambda\} - \delta + o(1)).
\end{aligned}$$

According then to Theorem B.4,(ii),

$$N(\lambda) \geq \frac{1}{(2\pi h)^n} (\text{Vol}(p < \lambda\} - \delta + o(1)).$$

$\square$

7. Exponential estimates for eigenfunctions

This chapter continues our study of semiclassical behavior of eigenfunctions:

$$P(h)u(h) = E(h)u(h)$$

for

$$P(h) = -h^2\Delta + V(x).$$

We first demonstrate that if $E(h)$ is close to the energy level E, then $u(h)$ exponentially small in the physically forbidden region $V^{-1}((E,\infty))$. Then we show, conversely, that in any open set the L^2 norm of $u(h)$ is bounded from below by a quantity exponentially small in h.

We conclude with a discussion of the order of vanishing of eigenfunctions in the semiclassical limit.

7.1 ESTIMATES IN CLASSICALLY FORBIDDEN REGIONS

For the classical Hamiltonian

$$p = |\xi|^2 + V(x)$$

the *classically forbidden region* at energy level E is the open set

$$V^{-1}((E,\infty)).$$

We will show in this section that an eigenfunction of $P(h) = -h^2\Delta + V(x)$, for V satisfying the assumptions of §6.3, is exponentially small within the classically forbidden region.

We begin with some general facts and definitions.

DEFINITION. Let $V \subset \mathbb{R}^n$ be an open set. The *semiclassical Sobolev norms* are defined as

$$\|u\|_{H_h^k(V)} := \left(\sum_{|\alpha|\le k} \int_V |(hD)^\alpha u|^2 dx \right)^{1/2}$$

for $u \in C^\infty(V)$.

LEMMA 7.1 (Semiclassical elliptic estimates). *Let $V \subset\subset U$ be open sets. Then there exists a constant C such that*

$$\|u\|_{H_h^2(V)} \le C(\|P(h)u\|_{L^2(U)} + \|u\|_{L^2(U)}) \tag{7.1}$$

for all $u \in C^\infty(U)$.

Proof. 1. Let $\chi \in C_c^\infty(U)$, $\chi \equiv 1$ on V. We multiply $P(h)u$ by $\chi^2\bar{u}$ and integrate by parts:

$$\int_U h^2\langle \partial(\chi^2\bar{u}), \partial u\rangle + (V-E)|u|^2\chi^2\,dx = \int_U P(h)u\bar{u}\chi^2\,dx.$$

Therefore

$$h^2\int_U \chi^2|\partial u|^2\,dx \le C\int_U |P(h)u|^2 + |u|^2\,dx;$$

and so

$$h^2\int_V |\partial u|^2\,dx \le C\int_U |P(h)u|^2 + |u|^2\,dx$$

2. Similarly, multiply $P(h)u$ by $\chi^2\Delta\bar{u}$ and integrate by parts, to show

$$h^4\int_V |\partial^2 u|^2\,dx \le C\int_U |P(h)u|^2 + |u|^2\,dx.$$

□

Before turning again to eigenfunctions, we present the following general estimates for the operator $P(h) = -h^2\Delta + V(x)$ where $V \in C^\infty(\mathbb{R}^n)$.

THEOREM 7.2 (Exponential estimate from above). *Suppose that U is an open set such that*

$$\overline{U} \subset\subset V^{-1}(E,\infty).$$

Then for each open set $W \supset\supset \overline{U}$ and for each λ in a small neighborhood of E, there exist constants $\delta, C > 0$, such that

$$\|u\|_{L^2(U)} \le Ce^{-\delta/h}\|u\|_{L^2(W)} + C\|(P(h)-\lambda)u\|_{L^2(W)} \tag{7.2}$$

for all $u \in C_c^\infty(\mathbb{R}^n)$.

We call (7.2) an *Agmon-type estimate.*

Proof. 1. Select $\psi, \phi \in C_0^\infty(W)$ such that $0 \le \psi, \phi \le 1$, $\psi \equiv 1$ on U, and $\phi \equiv 1$ on spt ψ.

We may as well assume that $W \subset\subset V^{-1}((E,\infty))$ and require also that

$$\text{spt}\,\phi \subset W \subset\subset V^{-1}((E,\infty)).$$

2. Observe that the symbol of

$$e^{\delta\psi/h}(P(h)-\lambda)e^{-\delta\psi/h} = P(h) - \lambda - \delta^2|\partial\psi|^2 - h\delta\Delta\psi + i\delta\langle\partial\psi, hD\rangle$$

is

$$(\xi + i\delta\partial\psi)^2 + V(x) - \lambda + O(h).$$

Now for λ close to E, $x \in W$ and δ sufficiently small, we have

$$|(\xi + i\delta\partial\psi)^2 + V(x) - \lambda)|^2 \geq c_0 > 0$$

for some positive constant c_0. Then according to the easy Gårding inequality, Theorem 4.24, we see that provided $\delta > 0$ is sufficiently small, then

$$A(h)^* A(h) \geq c^2 \phi^2 \tag{7.3}$$

for some constant $c > 0$ in the sense of operators, for

$$A(h) := e^{\delta\psi/h}(P(h) - \lambda)e^{-\delta\psi/h}\phi.$$

3. Estimate (7.3) implies that

$$\begin{aligned}\|\phi e^{\delta\psi/h} u\|_{L^2} &\leq c^{-1}\|A(h)u\|_{L^2} \\ &\leq c^{-1}\|e^{\delta\psi/h}\phi(P(h) - \lambda)u\|_{L^2} + c^{-1}\|e^{\delta\psi/h}[P(h), \phi]u\|_{L^2},\end{aligned}$$

for $u \in C_c^\infty(\mathbb{R}^n)$. Since $\psi \equiv 0$ on spt $[P(h), \phi]u$, Lemma 7.1 gives

$$\begin{aligned}\|e^{\delta\psi/h}[P(h), \phi]u\|_{L^2} &\leq C\big(\|hD_x u\|_{L^2(V)} + \|u\|_{L^2(V)}\big) \\ &\leq C\|u\|_{L^2(V)} + C\|(P(h) - \lambda)u\|_{L^2(V)}.\end{aligned}$$

Combining these estimates, we conclude that

$$e^{\delta/h}\|u\|_{L^2(\Omega)} \leq C\|u\|_{L^2(V)} + C(e^{\delta/h} + C)\|(P(h) - z)u\|_{L^2(V)};$$

and the theorem follows. $\square$

Specializing to eigenfunctions, we obtain

THEOREM 7.3 (Exponential decay estimates). *Suppose that $U \subset\subset V^{-1}((E, \infty))$, and that $u(h) \in L^2(\mathbb{R}^n)$ solves*

$$P(h)u(h) = E(h)u(h),$$

where

$$E(h) \to E \quad \textit{as } h \to 0.$$

Then there exists a constant $\delta > 0$ such that

$$\|u(h)\|_{L^2(U)} \leq e^{-\delta/h}\|u(h)\|_{L^2(\mathbb{R}^n)}. \tag{7.4}$$

7.2 TUNNELING

In the previous section we showed that an eigenfunction is exponentially small in the physically forbidden region. In this section we will show that it can never be "smaller" than that: in any open set the L^2 norm of an eigenfuction is bounded *from below* by a quantity exponentially small in h. This is a mathematical version of quantum mechanical "tunneling into the physically forbidden region".

We will assume in this that section $u = u(h)$ solves

$$P(h)u = E(h)u \quad \text{ in } \mathbb{R}^n,$$

where

$$P = P(h) = -h^2\Delta + V(x) = \mathrm{Op}(p)$$

for the symbol

$$p(x,\xi) = |\xi|^2 + V(x).$$

Our goal is deriving for small $h > 0$ the lower bound

$$\|u\|_{L^2(U)} \geq e^{-\frac{C}{h}} \|u\|_{L^2(\mathbb{R}^n)}$$

where U is a bounded, open subset of $\mathbb{R}^n$.

NOTATION. We will also use conjugation by an exponential and to make it more systematic we now introduce some notation. Assume that $\phi : \mathbb{R}^n \to \mathbb{R}$ and define

$$P_\phi = P_\phi(h) := e^{\phi/h} P e^{-\phi/h} = \mathrm{Op}(p_\phi + O(h)) \tag{7.5}$$

for

$$p_\phi(x,\xi) := (\xi + i\partial\phi(x))^2 + V(x).$$

We call P_ϕ the *conjugation* of P by the term $e^{\phi/h}$. By carefully selecting the weight ϕ we can ensure that P_ϕ has some desirable properties.

DEFINITION. *Hörmander's hypoellipticity condition* is the requirement that:

$$\text{if } \ p_\phi = 0, \quad \text{then} \quad i\{p_\phi, \overline{p_\phi}\} > 0. \tag{7.6}$$

REMARK. Observe that for any complex function $q = q(x,\xi)$,

$$i\{q,\bar{q}\} = i\{\mathrm{Re}\ q + i\ \mathrm{Im}\ q,\ \mathrm{Re}\ q - i\ \mathrm{Im}\ q\} = 2\{\mathrm{Re}\ q, \mathrm{Im}\ q\}.$$

Hence the expression $i\{p_\phi, \overline{p_\phi}\}$ is real. □

LEMMA 7.4 ($\mathbf{L^2}$-estimate). *If Hörmander's condition* (7.6) *is valid for all x in $\bar{V} \subset\subset \mathbb{R}^n$, then*

$$h^{1/2}\|u\|_{L^2(V)} \leq C\|P_\phi u\|_{L^2(V)} \tag{7.7}$$

for all $u \in C^\infty_{\mathrm{c}}(V)$, provided $h_0 > 0$ is sufficiently small and $0 < h \leq h_0$.

Proof. We calculate that

$$\begin{aligned}\|P_\phi u\|_{L^2}^2 &= \langle P_\phi u, P_\phi u\rangle = \langle P_\phi^* P_\phi u, u\rangle \\ &= \langle P_\phi P_\phi^* u, u\rangle + \langle [P_\phi^*, P_\phi]u, u\rangle \\ &= \|P_\phi^* u\|_{L^2} + \langle [P_\phi^*, P_\phi]u, u\rangle.\end{aligned}$$

The symbol of $P_\phi P_\phi^* + [P_\phi^*, P_\phi]$ is

$$a = p_\phi \bar{p}_\phi + \frac{h}{i}\{\overline{p_\phi}, p_\phi\} + O(h^2) = |p_\phi|^2 + ih\{p_\phi, \overline{p_\phi}\} + O(h^2).$$

Owing to Hörmander's condition (7.6), we have

$$a \geq Ch + O(h^2) \geq \frac{C}{2}h$$

if $0 < h \leq h_0$. We apply the easy Gårding inequality, Theorem 4.24, to deduce for small h that

$$\|P_\phi u\|_{L^2}^2 = \langle P_\phi P_\phi^* + [P_\phi^*, P_\phi]u, u\rangle \geq Ch\|u\|_{L^2}^2. \tag{7.8}$$

□

LEMMA 7.5 (Constructing a weight). *Let $0 < r < R$. There exists a C^∞, bounded, radial function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that Hörmander's condition* (7.6) *holds on $B(0,R) - B(0,r)$.*

Proof. 1. We will take $p = |\xi|^2 + V(x) - E$, and then compute

$$\begin{aligned} p_\phi &= (\xi + i\partial\phi(x))^2 + V(x) - E \\ &= |\xi|^2 - |\partial\phi(x)|^2 + V(x) - E + 2i\langle \xi, \partial\phi(x)\rangle. \end{aligned} \tag{7.9}$$

So $p_\phi = 0$ implies

$$|\xi|^2 - |\partial\phi(x)|^2 + V(x) - E = 0 \tag{7.10}$$

and

$$\langle \xi, \partial\phi(x)\rangle = 0. \tag{7.11}$$

Furthermore,

$$\begin{aligned} \frac{i}{2}\{p_\phi, \overline{p_\phi}\} &= \{\mathrm{Re}\ p_\phi, \mathrm{Im}\ p_\phi\} \\ &= \langle \partial_\xi(|\xi|^2 - |\partial\phi|^2 + V - E), \partial_x\langle 2\xi, \partial\phi\rangle\rangle \\ &\quad -\langle \partial_x(|\xi|^2 - |\partial\phi|^2 + V - E), \partial_\xi\langle 2\xi, \partial\phi\rangle\rangle \\ &= 4\langle \partial^2\phi\xi, \xi\rangle - \langle -2\partial\phi\partial^2\phi + \partial V, \partial\phi\rangle \\ &= 4\langle \partial^2\phi\xi, \xi\rangle + 2\langle \partial^2\phi\partial\phi, \partial\phi\rangle - \langle \partial V, \partial\phi\rangle. \end{aligned} \tag{7.12}$$

2. Assume now

$$\phi = e^{\lambda\psi},$$

where $\lambda > 0$ will be selected and $\psi : \mathbb{R}^n \to \mathbb{R}$ is positive and radial; that is, $\psi = \psi(|x|)$. Then

$$\partial\phi = \lambda e^{\lambda\psi}\partial\psi$$

and

$$\partial^2\phi = e^{\lambda\psi}(\lambda^2\partial\psi \otimes \partial\psi + \lambda\partial^2\psi).$$

Hence

$$\langle\partial^2\phi\xi,\xi\rangle = e^{\lambda\psi}(\lambda^2\langle\partial\psi,\xi\rangle^2 + \lambda\langle\partial^2\psi\xi,\xi\rangle) = e^{\lambda\psi}\lambda\langle\partial^2\psi\xi,\xi\rangle, \tag{7.13}$$

since $p_\phi = 0$ implies $\langle\partial\phi,\xi\rangle = 0$ and so $\langle\partial\psi,\xi\rangle = 0$. Also

$$\langle\partial^2\phi\partial\phi,\partial\phi\rangle = \lambda^4|\partial\psi|^4e^{4\lambda\psi} + \lambda^3\langle\partial^2\psi\partial\psi,\partial\psi\rangle e^{3\lambda\psi},$$

and

$$\langle\partial V,\partial\phi\rangle = \lambda e^{\lambda\psi}\langle\partial V,\partial\psi\rangle.$$

According to (7.12), we have

$$\begin{aligned}\frac{i}{2}\{p_\phi,\overline{p_\phi}\} = 4\lambda e^{\lambda\psi}\langle\partial^2\psi\xi,\xi\rangle + 2\lambda^4|\partial\psi|^4e^{4\lambda\psi}&\\ + 2\lambda^3\langle\partial^2\psi\partial\psi,\partial\psi\rangle e^{3\lambda\psi} - \lambda\langle\partial V,\partial\psi\rangle e^{\lambda\psi}&.\end{aligned} \tag{7.14}$$

Now take

$$\psi := C - |x|,$$

for a constant C so large that $\psi \geq 1$ on $B(0,R)$. Then

$$|\partial\psi| = 1,\ |\partial^2\psi| \leq C \quad \text{ on } B(0,R) - B(0,r).$$

Furthermore according to (7.10) we have

$$|\xi|^2 \leq C + |\partial\phi|^2 \leq C + C\lambda^2e^{2\lambda\psi} \quad \text{ on } B(0,R) - B(0,r). \tag{7.15}$$

Plugging these estimates into (7.14), we compute

$$\frac{i}{2}\{p_\phi,\overline{p_\phi}\} = 2\lambda^4e^{4\lambda\psi} - C\lambda^3e^{3\lambda\psi} - C \geq 1,$$

if λ is selected large enough. We can now modify ψ in $B(0,r)$ to obtain a smooth fuction on $B(0,R)$. $\square$

THEOREM 7.6 (Exponential estimate from below). *Let $U \subset \mathbb{R}^n$ be an open set. There exist constants C, $h_0 > 0$ such that if $u(h)$ solves*

$$P(h)u = E(h)u(h) \quad \text{ in } \mathbb{R}^n$$

for $E(h) \in [a,b]$, then

$$\|u(h)\|_{L^2(U)} \geq e^{-\frac{C}{h}}\|u(h)\|_{L^2(\mathbb{R}^n)} \tag{7.16}$$

for all $0 < h \leq h_0$.

We call (7.16) a *Carleman-type estimate.*

Proof. 1. Without loss $U = B(0, 2r)$ for some (possibly small) $r > 0$. Select $R > 0$ so large that

$$p(x,\xi) = |\xi|^2 + V(x) \geq |\xi|^2 + \langle x \rangle^k$$

for $|x| \geq R$. That is, the symbol p is elliptic in $\mathbb{R}^n - B(0,R)$.

Select two radial functions $\chi_1, \chi_2 : \mathbb{R}^{2n} \to \mathbb{R}$ such that $0 \leq \chi_1 \leq 1$,

$$\begin{cases} \chi_1 \equiv 0 & \text{on } B(0,r), \\ \chi_1 \equiv 1 & \text{on } B(0,R+2) - B(0,2r), \\ \chi_1 \equiv 0 & \text{on } \mathbb{R}^{2n} - B(0,R+3) \end{cases}$$

and $0 \leq \chi_2 \leq 1$,

$$\begin{cases} \chi_2 \equiv 0 & \text{on } B(0,R) \\ \chi_2 \equiv 1 & \text{on } \mathbb{R}^{2n} - B(0,R+1). \end{cases}$$

2. Since p is semiclassically elliptic on $\mathbb{R}^n - B(0,R)$, we have the estimate

$$\|u\|_{L^2(\mathbb{R}^n - B(0,R))} \leq C\|P(h)u\|_{L^2(\mathbb{R}^n - B(0,R))}$$

for all $u \in C_c^\infty(\mathbb{R}^n - B(0,R))$.

Hence

$$\|\chi_2 u\|_{L^2} \leq C\|P(h)(\chi_2 u)\|_{L^2} = C\|[P(h),\chi_2]u\|_{L^2}.$$

Now $[P(h),\chi_2]u = -h^2 u\Delta\chi_2 - 2h^2\langle\partial\chi_2, \partial u\rangle$. Thus $[P(h),\chi_2]u$ is supported in $B(0,R+1) - B(0,R)$. Hence, according to Lemma 7.1,

$$\|\chi_2 u\|_{L^2} \leq Ch\|\chi_1 u\|_{L^2(B(0,R+1)-B(0,R))}. \tag{7.17}$$

3. Next apply Lemma 7.4:

$$h^{1/2}\left\|e^{\frac{\phi}{h}}\chi_1 u\right\|_{L^2} \leq C\left\|e^{\frac{\phi}{h}}P(h)(\chi_1 u)\right\|_{L^2} = C\left\|e^{\frac{\phi}{h}}[P(h),\chi_1]u\right\|_{L^2}.$$

Now $[P(h),\chi_1]$ is supported in the union of $B(0,2r) - B(0,r)$ and $B(0,R+3) - B(0,R+2)$. Thus

$$h^{1/2}\left\|e^{\frac{\phi}{h}}\chi_1 u\right\|_{L^2} \leq Ch\left\|e^{\frac{\phi}{h}}\chi_2 u\right\|_{L^2(B(0,R+3)-B(0,R+2))} + C\|u\|_{L^2(U)}.$$

4. Select a positive constant ϕ_0 so that

$$\begin{cases} \phi > \phi_0 & \text{on } B(0,R+1) - B(0,R) \\ \phi < \phi_0 & \text{on } B(0,R+3) - B(0,R+2). \end{cases}$$

Multiply (7.17) by $e^{\phi_0/h}$ and add to (7.17):

$$\begin{aligned}
&\left\|e^{\frac{\phi_0}{h}}\chi_2 u\right\|_{L^2} + \left\|e^{\frac{\phi}{h}}\chi_1 u\right\|_{L^2} \\
&\qquad \leq Ch\left\|e^{\frac{\phi_1}{h}}\chi_1 u\right\|_{L^2(B(0,R+1)-B(0,R))} \\
&\qquad\qquad + Ch^{1/2}\left\|e^{\frac{\phi}{h}}\chi_2 u\right\|_{L^2(B(0,R+3)-B(0,R+2))} + Ch^{-1/2}\|u\|_{L^2(U)} \\
&\qquad \leq Ch\left\|e^{\frac{\phi}{h}}\chi_1 u\right\|_{L^2} + Ch^{1/2}\left\|e^{\frac{\phi_0}{h}}\chi_2 u\right\|_{L^2} + Ch^{-1/2}\|u\|_{L^2(U)}.
\end{aligned}$$

Take $0 < h \leq h_0$, for h_0 sufficiently small, to deduce

$$\left\|e^{\frac{\phi_0}{h}}\chi_2 u\right\|_{L^2} + \left\|e^{\frac{\phi}{h}}\chi_1 u\right\|_{L^2} \leq Ch^{-1/2}\|u\|_{L^2(U)}.$$

Since $\chi_1 + \chi_2 \geq 1$ on $\mathbb{R}^n - B(0, 2r) = \mathbb{R}^n - U$, the theorem follows. □

7.3 ORDER OF VANISHING

Assume, as usual, that

$$P(h)u(h) = E(h)u(h), \tag{7.18}$$

where $E(h) \in [a, b]$. To simplify notation, we will in this subsection write u for $u(h)$.

We propose now to give an estimate for the order of vanishing of u, following a suggestion of N. Burq.

DEFINITION. We say u *vanishes to order* N at the point x_0 if

$$u(x) = O(|x - x_0|^N) \quad \text{as } x \to x_0.$$

We will consider potentials which are analytic in x and, to avoid technical difficulties, make a strong assumption on the growth of derivatives:

$$V(x) \geq \langle x\rangle^m/C_0 - C_0, \quad |\partial^\alpha V(x)| \leq C_0^{1+|\alpha|}|\alpha|^{|\alpha|}\langle x\rangle^m \tag{7.19}$$

for some $m > 0$ and all multiindices α.

We note that the second condition holds when V has a holomorphic extension bounded by $|z|^m$ into a conic neighbourhood of $\mathbb{R}^n$ in $\mathbb{C}^n$.

THEOREM 7.7 (Semiclassical estimate on vanishing order). *Suppose that $u \in L^2$ solves* (7.18) *for $a \leq E(h) \leq b$ and that V a real analytic potential satisfying* (7.19). *Let K be compact subset of $\mathbb{R}^n$.*

Then there exists a constant C such that if u vanishes to order N at a point $x_0 \in K$, we have the estimate

$$N \leq Ch^{-1}. \tag{7.20}$$

We need the following lemma to establish analyticity of the solution in a semiclassically quantitative way:

LEMMA 7.8 (Semiclassical derivative estimates). *If u satisfies the assumptions of Theorem 7.7, then there exists a constant C_1 such that for any positive integer k:*

$$\|u\|_{H^k_h(\mathbb{R}^n)} \le C_1^k (1+kh)^k \|u\|_{L^2(\mathbb{R}^n)}. \tag{7.21}$$

Proof. 1. By adding C_0 to V we can assume without loss that $V(x) \ge \langle x\rangle^m / C_0$. The Lemma will follow from the following stronger estimate, which we will prove by induction:

$$\begin{aligned} &\|\langle x\rangle^{m/2}(hD)^\alpha u\|_{L^2} + \|(h\partial)(hD)^\alpha u\|_{L^2} \\ &\qquad\qquad \le C_2^{k+2}(1+kh)^{k+1}\|u\|_{L^2}. \end{aligned} \tag{7.22}$$

for $|\alpha| = k$.

2. To prove this inequality, we observe first that our multiplying (7.18) by $\bar{u}$ and integrating by parts shows that estimate (7.22) holds for $|\alpha| = 0$

Next, note that

$$\begin{aligned} &\|V^{\frac12}(hD)^\alpha u\|_{L^2}^2 + \|(h\partial)(hD)^\alpha u\|_{L^2}^2 \\ &\qquad = \langle(-h^2\Delta + V - E(h))(hD)^\alpha u, (hD)^\alpha u\rangle + E(h)\|(hD)^\alpha u\|_{L^2}^2 \\ &\qquad = \langle V^{-\frac12}[V,(hD)^\alpha]u, V^{\frac12}(hD)^\alpha u\rangle + E(h)\|(hD)^\alpha u\|_{L^2}^2 \\ &\qquad \le 2\|V^{-\frac12}[V,(hD)^\alpha]u\|_{L^2}^2 + \frac12\|V^{\frac12}(hD)^\alpha u\|_{L^2}^2 + E(h)\|(hD)^\alpha u\|_{L^2}^2. \end{aligned}$$

Hence

$$\begin{aligned} &\frac12\|V^{\frac12}(hD)^\alpha u\|_{L^2}^2 + \|(h\partial)(hD)^\alpha u\|_{L^2}^2 \\ &\qquad\qquad \le 2\|V^{-\frac12}[V,(hD)^\alpha]u\|_{L^2}^2 + E(h)\|(hD)^\alpha u\|_{L^2}^2. \end{aligned} \tag{7.23}$$

3. We can now expand the commutator, to deduce from (7.19) (with V replaced by $V + C_0$) the inequality:

$$\begin{aligned} &\|V^{-\frac12}[V,(hD)^\alpha]u\|_{L^2} \le \\ &\qquad \sum_{l=0}^{k-1}\binom{k}{l} C_0^{k-l+1}(h(k-l))^{k-l}\sup_{|\beta|=l}\|\langle x\rangle^{m/2}(hD)^\beta u\|_{L^2}. \end{aligned} \tag{7.24}$$

We proceed by induction, and so now assume that (7.22) is valid for $|\alpha| < k$. Now Stirling's formula implies

$$\binom{k}{l} \leq C \frac{k^k}{l^l (k-l)^{k-l}}.$$

Hence, in view of (7.23) and (7.24), it is enough to show that there exists a constant C_2 such that

$$\sum_{l=0}^{k-1} h^{k-l} \frac{k^k}{l^l} C_0^{k-l+1} C_2^{l+2} (1+lh)^{l+1} + C_2^{k+1}(1+hk)^k \leq C_2^{k+2}(1+hk)^{k+1}.$$

This estimate we can rewrite as

$$C_0 \sum_{l=1}^{k-1} \left(\frac{C_0}{C_2}\right)^{k-l} (hl)^{-l}(1+hl)^l(1+hl) + C_2^{-1}(hk)^{-k}(1+hk)^k$$
$$\leq (hk)^{-k}(1+hk)^k(1+hk).$$

Since we can choose C_2 to be large and since we can estimate the $(1+hl)$ factor in the sum by $(1+hk)$, this will follow once we show that for ϵ small enough,

$$\sum_{l=0}^{k-1} \epsilon^{k-l} a_l \leq a_k \quad \text{for} \quad a_l := (1+(hl)^{-1})^l.$$

This is true by induction if a_{k-1}/a_k is bounded:

$$\sum_{l=0}^{k-1} \epsilon^{k-l} a_l \leq 2\epsilon a_{k-1}.$$

In our case,

$$\begin{aligned}
\frac{a_{k-1}}{a_k} &= \left(\frac{1+(h(k-1))^{-1}}{1+(hk)^{-1}}\right)^{k-1} (1+(hk)^{-1})^{-1} \\
&= \left(1 + \frac{1}{(k-1)(1+hk)}\right)^{k-1} (1+(hk)^{-1})^{-1} \\
&\leq \exp\left(\frac{1}{1+hk}\right) \frac{hk}{1+hk} < 1.
\end{aligned}$$

□

Proof of Theorem 7.7: Assume now that $\|u\|_{L^2} = 1$ and that u vanishes to order N at a point $x_0 \in K$.

Then $D^\alpha u(x_0) = 0$ for $|\alpha| < N$ and Taylor's formula shows that

$$|u(x)| \le \frac{\epsilon^N}{N!} \sup_{|\alpha|=N} \sup_{y\in\mathbb{R}^n} |D^\alpha u(y)| \quad \text{for } |x - x_0| < \epsilon. \tag{7.25}$$

The Sobolev inequality (Lemma 3.5) and Lemma 7.8 allow us to estimate the derivatives. If say $M = N + n$ and $|\alpha| = N$, then

$$\sup_{y\in\mathbb{R}^n} |D^\alpha u(y)| \le \|u\|_{H^M} \le h^{-M}\|u\|_{H_h^M} \le h^{-M} C_1^M (1 + hM)^M.$$

Inserting this into (7.25) and using Stirling's formula, we deduce that if for $|x - x_0| < \epsilon$, then

$$|u(x)| \le \left(\frac{e\epsilon}{N}\right)^N \left(\frac{C}{h}\right)^M (1 + hM)^M \le \left(\frac{N}{e\epsilon}\right)^n \left(\frac{e\epsilon C}{Nh}\right)^M (1 + hM)^M.$$

If we put $A := Mh$, then for ϵ small enough we have, with $K = C\epsilon^{-1}$ large,

$$\begin{aligned} |u(x)| &\le (KAh^{-1})^n \left(\frac{1}{KA}\right)^{Ah^{-1}} (1 + A)^{Ah^{-1}} \\ &= (KAh^{-1})^n (1 + 1/A)^{Ah^{-1}} \exp(-Ah^{-1}\log K). \end{aligned}$$

We can assume that A is large, as otherwise there is nothing to prove. Hence

$$|u(x)| \le \exp(-\alpha A h^{-1}),$$

for $\alpha > 0$ and $|x - x_0| < \epsilon$. It follows that

$$\int_{\{|x-x_0|<\epsilon\}} |u(x)|^2\, dx \le C_1 e^{-2\alpha A/h},$$

uniformly in h. But according to Theorem 7.6,

$$\int_{\{|x-x_0|<\epsilon\}} |u(x)|^2\, dx > e^{-C_2/h}.$$

Consequently $A = Mh = (N + n)h$ is bounded, and this means that $N \le Ch^{-1}$, as claimed. □

EXAMPLE : Optimal order of vanishing. Theorem 7.7 is optimal in the semiclassical limit, meaning as regards the dependence on h in estimate (7.20).

We can see this by considering the harmonic oscillator in dimension $n = 2$. In polar coordinates (r, θ) the harmonic oscillator for $h = 1$ takes the form

$$P_0 = r^{-2}((rD_r)^2 + D_\theta^2 + r^4).$$

The eigenspace corresponding to the eigenvalue $2k+2$ has dimension $k+1$, corresponding to the number of multiindicies $\alpha = (\alpha_1, \alpha_2)$, with $|\alpha| = \alpha_1 + \alpha_2 = k$. Separating variables, we look for eigenfunctions of the form

$$u = u_{km}(r)e^{im\theta}.$$

Then

$$r^{-2}((rD_r)^2 + m^2 + r^4 - (2n+1)r^2)u_{km}(r) = 0.$$

Since the number of linearly independent eigenfunctions is $k+1$, there must be solution for some integer $m > k/2$. Near $r = 0$, we have the asymptotics $u_{km} \simeq r^{\pm m}$, and the case $u_{km} \simeq r^{-m}$ is impossible since $u \in L^2$. Therefore $u \simeq r^m$ has to vanish to order m.

Rescaling to the semiclassical case, we see that for the eigenvalue $E(h) = (2k+1)h \simeq 1$ we have an eigenfunction vanishing to order $\simeq 1/h$. $\square$

8. More on the symbol calculus

This chapter collects together various more advanced topics concerning the symbol calculus, discussing in particular invariance properties under changes of variable, a semiclassical version of Beals's characterization of pseudodifferential operators, extensions to manifolds, etc. Chapters 9 and 10 will provide applications.

8.1 INVARIANCE, HALF-DENSITIES

Invariance. We begin with a general discussion concerning the invariance of various quantities under the change of variables

$$\tilde{x} = \boldsymbol{\kappa}(x), \tag{8.1}$$

where $\boldsymbol{\kappa} : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism.

• **Functions.** We note first that functions transform under (8.1) by pull-back. This means that we transform u into a function of the new variables $\tilde{x}$ by the rule

$$\tilde{u}(\tilde{x}) = \tilde{u}(\boldsymbol{\kappa}(x)) := u(x), \tag{8.2}$$

for $x \in \mathbb{R}^n$. Observe however that in general the integral of u over a set E is not then invariant:

$$\int_{\boldsymbol{\kappa}(E)} \tilde{u}(\tilde{x})\, d\tilde{x} \neq \int_E u(x)\, dx.$$

• **Densities.** One way to repair this defect is to change our definition (8.2) to include the Jacobian of the transformation $\boldsymbol{\kappa}$. We elegantly accomplish this by turning our attention to *densities*, which we denote symbolically as

$$u(x)|dx|.$$

We therefore modify our earlier definition (8.2), now to read

$$\tilde{u}(\tilde{x}) = \tilde{u}(\boldsymbol{\kappa}(x)) := u(x)|\det(\partial\boldsymbol{\kappa}(x))|^{-1}. \tag{8.3}$$

Then we have the invariance assertion

$$\text{“}\,\tilde{u}(\tilde{x})|d\tilde{x}| = u(x)|dx|,\text{”}$$

meaning that

$$\int_{\boldsymbol{\kappa}(E)} \tilde{u}(\tilde{x})\, d\tilde{x} = \int_E u(x)\, dx$$

for all Borel sets $E \subseteq \mathbb{R}^n$.

• **Half-densities.** Next recall our general motivation coming from quantum mechanics. The eigenfunctions u we study are then interpreted as *wave functions* and the squares of their moduli are the probability densities in the position representation: the probability of "finding our state in the set E" is given by

$$\int_E |u(x)|^2 dx.$$

This probability should be invariantly defined, and so should not depend on the choice of coordinates x. As above, this means that it is *not* the function $u(x)$ which should be defined invariantly but rather the density $|u(x)|^2 dx$, or equivalently the *half-density*

$$u(x)|dx|^{\frac{1}{2}}.$$

For half-densities we therefore demand that

$$\text{“}\,\tilde{u}(\tilde{x})|d\tilde{x}|^{\frac{1}{2}} = u(x)|dx|^{\frac{1}{2}}\text{”},$$

which means that integrals of the squares should be invariantly defined. To accomplish this, we once again modify our original definition (8.2), this time to become

$$\tilde{u}(\tilde{x}) = \tilde{u}(\boldsymbol{\kappa}(x)) := u(x)|\det(\partial \boldsymbol{\kappa}(x))|^{-\frac{1}{2}}. \tag{8.4}$$

Then

$$\int_{\boldsymbol{\kappa}(E)} |\tilde{u}(\tilde{x})|^2\, d\tilde{x} = \int_E |u(x)|^2\, dx.$$

for all Borel subsets $E \subseteq \mathbb{R}^n$.

DISCUSSION. The foregoing formalism is at first rather unintuitive, but turns out later to play a crucial role in the rigorous semiclassical calculus, in particular in the theory of Fourier integral operators, which we will touch upon later. Section 8.2 below will demonstrate how the half-density viewpoint fits naturally within the Weyl calculus, and Section 10.2 will explain how half-densities simplify some related calculations for a propagator.

Our Appendix D provides a more careful foundation of these concepts in terms of the *s-density line bundles* over $\mathbb{R}^n$, denoted $\Omega^s(\mathbb{R}^n)$. In this notation, a density is a smooth section of $\Omega^1(\mathbb{R}^n)$ and a half-density is a smooth section of $\Omega^{\frac{1}{2}}(\mathbb{R}^n)$. $\square$

NOTATION. We write

$$u|dx| \in C^\infty(\mathbb{R}^n, \Omega^1(\mathbb{R}^n))$$

for densities, and

$$u|dx|^{\frac{1}{2}} \in C^\infty(\mathbb{R}^n, \Omega^{\frac{1}{2}}(\mathbb{R}^n))$$

for half-densities.

REMARK: Operator kernels. Another nice aspect of half-densities appears when we use operator kernels. Suppose that

$$K \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n; \Omega^{\frac{1}{2}}(\mathbb{R}^n \times \mathbb{R}^n)).$$

Then K, acting as an integral kernel, defines a map

$$K \ : \ C^\infty_{\mathrm{c}}(\mathbb{R}^n, \Omega^{\frac{1}{2}}(\mathbb{R}^n)) \ \to C^\infty(\mathbb{R}^n, \Omega^{\frac{1}{2}}(\mathbb{R}^n)),$$

in an invariant way, independently of the choice of densities:

$$\begin{aligned} Ku(x)|dx|^{\frac{1}{2}} &= \int_{\mathbb{R}^n} K(x,y)|dx|^{\frac{1}{2}}|dy|^{\frac{1}{2}}u(y)|dy|^{\frac{1}{2}} \\ &:= \left(\int_{\mathbb{R}^n} K(x,y)u(y)dy\right)|dx|^{\frac{1}{2}}. \end{aligned} \tag{8.5}$$

$\square$

8.2 CHANGING VARIABLES

In this section we illustrate the usefulness of half-densities in charterizing invariance properties of quantization under changes of variables.

We observe that the half-density sections over $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ are identified with functions if we consider symplectic changes of variables, in particular

$$(x,\xi) \longmapsto (\tilde{x}, \tilde{\xi}) = (\boldsymbol{\kappa}(x), (\partial\boldsymbol{\kappa}(x)^T)^{-1}\xi). \tag{8.6}$$

(Recall the derivation of this formula in Example 1 of Section 2.3.)

We will consider the Weyl quantization of $a \in C^\infty_{\mathrm{c}}(\mathbb{R}^{2n})$ *as an operator acting on half-densities*. That is done as in (8.5) by defining

$$\begin{aligned} &K_a(x,y)|dx|^{\frac{1}{2}}|dy|^{\frac{1}{2}} \\ &\qquad := \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} a\left(\frac{x+y}{2},\xi\right)e^{i\frac{\langle x-y,\xi\rangle}{h}}d\xi|dx|^{\frac{1}{2}}|dy|^{\frac{1}{2}}. \end{aligned} \tag{8.7}$$

Then

$$\mathrm{Op}(a)(u(y)|dy|^{\frac{1}{2}})|dx|^{\frac{1}{2}} := \int_{\mathbb{R}^n} K_a(x,y)u(y)\,dy|dx|^{\frac{1}{2}}. \tag{8.8}$$

The arguments of Chapter 4 show that for $a \in S$ we obtain a bounded operator that quantizes a:

$$\mathrm{Op}(a) \ : \ L^2(\mathbb{R}^n, \Omega^{\frac{1}{2}}(\mathbb{R}^n)) \ \to \ L^2(\mathbb{R}^n, \Omega^{\frac{1}{2}}(\mathbb{R}^n)).$$

Next, let $\boldsymbol{\kappa} : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth diffeomorphism and $a \in S$. We write $A = \mathrm{Op}(a)$ for the operator acting on *half-densities.* As above, if we write $\tilde{x} = \boldsymbol{\kappa}(x)$, we define $\tilde{u}$ by

$$\tilde{u}(\tilde{x})|d\tilde{x}|^{\frac{1}{2}} = u(x)|dx|^{\frac{1}{2}}. \tag{8.9}$$

Then $\tilde{A} = (\boldsymbol{\kappa}^{-1})^* A \boldsymbol{\kappa}^*$, acting on half-densities, is given by the rule

$$\tilde{A}\tilde{u}(\tilde{x}) = Au(x), \tag{8.10}$$

when acting on functions.

THEOREM 8.1 (Operators and half-densities).

(i) *Consider A acting on half-densities. Then*

$$(\boldsymbol{\kappa}^{-1})^* A \boldsymbol{\kappa}^* = \mathrm{Op}(\tilde{a}) \tag{8.11}$$

for

$$\tilde{a}(x,\xi) := a(\boldsymbol{\kappa}^{-1}(x), \partial\boldsymbol{\kappa}(x)^T \xi) + O(h^2). \tag{8.12}$$

That is,

$$a(x,\xi) = \tilde{a}(\boldsymbol{\kappa}(x), (\partial\boldsymbol{\kappa}(x)^T)^{-1}\xi) + O(h^2). \tag{8.13}$$

(ii) *When we consider A acting on functions and define*

$$A_1 = (\boldsymbol{\kappa}^{-1})^* A \boldsymbol{\kappa}^*,$$

then

$$A_1 = \mathrm{Op}(a_1),$$

for

$$a_1(x,\xi) := a(\boldsymbol{\kappa}^{-1}(x), \partial\boldsymbol{\kappa}(x)^T \xi) + O(h). \tag{8.14}$$

That is,

$$a(x,\xi) = a_1(\boldsymbol{\kappa}(x), (\partial\boldsymbol{\kappa}(x)^T)^{-1}\xi) + O(h). \tag{8.15}$$

INTERPRETATION. The point is that assertion (i) for half-densities (with error term of order $O(h^2)$) is more precise than the assertion (ii) for functions (with error term $O(h)$). □

Proof. 1. Remember that

$$Au(x)|dx|^{\frac{1}{2}} = \int_{\mathbb{R}^n} K_a(x,y)|dx|^{\frac{1}{2}}|dy|^{\frac{1}{2}}u(y)|dy|^{\frac{1}{2}}$$

for

$$K_a(x,y) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} a\left(\frac{x+y}{2}, \xi\right) e^{\frac{i\langle x-y,\xi\rangle}{h}}\, d\xi.$$

Likewise

$$\tilde{A}\tilde{u}(\tilde{x}) = \int_{\mathbb{R}^n} K_{\tilde{a}}(\tilde{x},\tilde{y}) |d\tilde{x}|^{\frac{1}{2}} |d\tilde{y}|^{\frac{1}{2}} \tilde{u}(\tilde{y})\, |d\tilde{y}|^{\frac{1}{2}}$$

for

$$K_{\tilde{a}}(\tilde{x},\tilde{y}) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \tilde{a}\left(\frac{\tilde{x}+\tilde{y}}{2}, \tilde{\xi}\right) e^{\frac{i\langle \tilde{x}-\tilde{y},\tilde{\xi}\rangle}{h}}\, d\tilde{\xi}.$$

Since $\tilde{u}(\tilde{y})|d\tilde{y}|^{\frac{1}{2}} = u(y)|dy|^{\frac{1}{2}}$ and $d\tilde{y} = |\det \partial\boldsymbol{\kappa}(y)|dy$, it follows that

$$\tilde{A}\tilde{u}(x) = \int_{\mathbb{R}^n} K_{\tilde{a}}(\tilde{x},\tilde{y}) |\det \partial\boldsymbol{\kappa}(y)|^{\frac{1}{2}} |\det \partial\boldsymbol{\kappa}(x)|^{\frac{1}{2}} u(y)\, dy.$$

Hence we must show

$$K_a(x,y) = K_{\tilde{a}}(\tilde{x},\tilde{y}) |\det \partial\boldsymbol{\kappa}(y)|^{\frac{1}{2}} |\det \partial\boldsymbol{\kappa}(x)|^{\frac{1}{2}} + O(h^2). \tag{8.16}$$

2. Now

$$\begin{aligned} K_{\tilde{a}}(\tilde{x},\tilde{y}) &= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \tilde{a}\left(\frac{\tilde{x}+\tilde{y}}{2}, \tilde{\xi}\right) e^{\frac{i}{h}\langle \tilde{x}-\tilde{y},\tilde{\xi}\rangle}\, d\tilde{\xi} \\ &= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \tilde{a}\left(\frac{\boldsymbol{\kappa}(x)+\boldsymbol{\kappa}(y)}{2}, \tilde{\xi}\right) e^{\frac{i}{h}\langle \boldsymbol{\kappa}(x)-\boldsymbol{\kappa}(y),\tilde{\xi}\rangle}\, d\tilde{\xi}. \end{aligned}$$

We have

$$\boldsymbol{\kappa}(x) - \boldsymbol{\kappa}(y) = \langle k(x,y), x-y\rangle, \tag{8.17}$$

where

$$k(x,y) = \partial\boldsymbol{\kappa}\left(\frac{x+y}{2}\right) + O(|x-y|^2). \tag{8.18}$$

Also

$$\boldsymbol{\kappa}(x) + \boldsymbol{\kappa}(y) = 2\boldsymbol{\kappa}\left(\frac{x+y}{2}\right) + O(|x-y|^2). \tag{8.19}$$

Let us also write

$$\tilde{\xi} = (k(x,y)^T)^{-1}\xi. \tag{8.20}$$

Substituting above, we deduce that

$$\begin{aligned} &K_{\tilde{a}}(\tilde{x},\tilde{y}) \\ &= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \left[\tilde{a}\left(\boldsymbol{\kappa}\left(\frac{x+y}{2}\right), (k(x,y)^T)^{-1}\xi\right) + O(|x-y|^2)\right] \\ &\qquad\qquad e^{\frac{i}{h}\langle \boldsymbol{\kappa}(x)-\boldsymbol{\kappa}(y),(k(x,y)^T)^{-1}\xi\rangle}\, d\tilde{\xi}. \end{aligned}$$

Now, we apply the "Kuranishi trick" to rewrite this expression as a pseudodifferential operator. First,

$\langle \boldsymbol{\kappa}(x) - \boldsymbol{\kappa}(y), (k(x,y)^T)^{-1}\xi\rangle = \langle k(x,y)^{-1}(\boldsymbol{\kappa}(x) - \boldsymbol{\kappa}(y)), \xi\rangle = \langle x-y, \xi\rangle,$

according to (8.17). Remembering also (8.18), we compute

$$\begin{aligned}
&K_{\tilde a}(\tilde x, \tilde y) = \\
&\tfrac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\left[\tilde a\left(\boldsymbol{\kappa}\left(\tfrac{x+y}{2}\right), \left(\partial\boldsymbol{\kappa}\left(\tfrac{x+y}{2}\right)^T\right)^{-1}\xi\right) + O(|x-y|^2)\right] e^{\frac{i}{h}\langle x-y,\xi\rangle} d\tilde\xi \\
&\qquad = \tfrac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\left[a\left(\tfrac{x+y}{2}, \xi\right) + O(|x-y|^2)\right] e^{\frac{i}{h}\langle x-y,h\rangle} d\tilde\xi.
\end{aligned}$$

Furthermore $d\tilde\xi = |\det\, k(x,y)|^{-1} d\xi$ and

$$\det\, k(x,y) = \det\, \partial\boldsymbol{\kappa}\left(\frac{x+y}{2}\right) + O(|x-y|^2).$$

Also

$$\left|\det\, \partial\boldsymbol{\kappa}\left(\frac{x+y}{2}\right)\right|^2 = |\det\, \partial\boldsymbol{\kappa}(x)||\det\, \partial\boldsymbol{\kappa}(y)| + O(|x-y|^2).$$

3. Finally we observe that

$$(x-y)^\alpha e^{\frac{i}{h}\langle x-y,\xi\rangle} = (hD_\xi)^\alpha e^{\frac{i}{h}\langle x-y,\xi\rangle}. \tag{8.21}$$

Hence integrating by parts in the terms with $O(|x-y|^2)$ gives us terms of order $O(h^2)$. So

$$\begin{aligned}
&K_{\tilde a}(\tilde x, \tilde y) = \\
&\frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\left(a\left(\frac{x+y}{2}, \xi\right) + O(h^2)\right) e^{\frac{i}{h}\langle x-y,\xi\rangle} d\xi \\
&\qquad\qquad\qquad |\det\, \partial\boldsymbol{\kappa}(x)|^{-1/2}|\det\, \partial\boldsymbol{\kappa}(y)|^{-1/2}.
\end{aligned}$$

This proves (8.16).

4. When A acts on functions, then K_a has to transform as a density. In other words, we need to show

$$K_a(x,y) = K_{a_1}(\tilde x, \tilde y)|\det\, \partial\boldsymbol{\kappa}(y)| + O(h), \tag{8.22}$$

instead of (8.16). Since

$$|\det\, \partial\boldsymbol{\kappa}(y)| = |\det\, \partial\boldsymbol{\kappa}(y)|^{1/2}|\det\, \partial\boldsymbol{\kappa}(x)|^{1/2} + O(|x-y|),$$

we see from (8.21) that (8.22) follows from (8.16) with $a_1 = \tilde a + O(h)$. $\square$

8.3 ESSENTIAL SUPPORT, WAVEFRONT SETS

We devote this section to a few definitions built around the notion of the essential support of a symbol.

Let m denote some order function.

DEFINITION. Let $a = a(x, \xi, h)$ be a symbol in $S(m)$. We define the *essential support* of a, denoted

$$\text{ess-spt}(a),$$

to be the *complement* of the set of points with neighbourhoods in which

$$|\partial^\alpha a| \leq C_{\alpha,N} m h^N, \tag{8.23}$$

for each N.

REMARK. The proof of Theorem 8.1 shows that this notion does not depend on the choice of coordinates, in the sense that if κ is a diffeomorphism and $\mathrm{Op}(a_{\boldsymbol{\kappa}}) = \mathrm{Op}((\boldsymbol{\kappa}^{-1})^* \mathrm{Op}(a) \boldsymbol{\kappa}^*)$, then

$$\text{ess-spt}(a_{\boldsymbol{\kappa}}) = \{(\boldsymbol{\kappa}(x), (\partial \boldsymbol{\kappa}(x)^T)^{-1} \xi) \mid (x, \xi) \in \text{ess-spt}(a)\}. \tag{8.24}$$

$\square$

DEFINITION. A family of smooth functions $\{u(h)\}_{0<h\leq h_0}$ is called *h-tempered* on $\mathbb{R}^n$ if for each multiindex α there exist $N = N_\alpha$ and $C = C_\alpha$ such that

$$\|\langle x \rangle^{-N} \partial^\alpha u(h)\|_{L^2(\mathbb{R}^n)} \leq C h^{-N}$$

for $0 < h \leq h_0$.

DEFINITION. For a tempered family we define the *semiclassical wavefront set*

$$WF_h(u)$$

to the *complement* of the set of points $(x, \xi) \in \mathbb{R}^{2n}$ for which there exists a symbol $a \in S$ such that:

$$a(x, \xi) \neq 0$$

and

$$\|a^w(x, hD) u(h)\|_{L^2} \leq C_\alpha h^k \tag{8.25}$$

for all integers k. We note that this a *local* property of the family $u(h)$ in *phase space* – see Theorem 8.3 below.

Thus

(8.26)

$$\begin{aligned} WF_h(u) = \{(x,\xi) \ \mid \ & \text{there exists } a \in S \text{ such that} \\ & a(x,\xi) \neq 0 \text{ and } \ a^w(x,hD)u(h) = O_{L^2}(h^\infty)\}^c . \end{aligned}$$

The superscript "c" means the complement.

The meaning of the wavefront set is illucidated by the following

LEMMA 8.2 (Localization and wavefront sets). *Suppose that* $(x_0,\xi_0) \notin WF_h(u)$ *then for any* $b \in C_c^\infty(\mathbb{R}^n)$ *with support sufficiently close to* (x_0,ξ_0) *we have*

$$b^w(x,hD)u = O_{\mathcal{S}}(h^\infty) ,$$

where the notation means that we have decay in any seminorm in the Schwartz class $\mathcal{S}$*.*

Proof: 1. Suppose $a \in S$ has the property that $a(x_0,\xi_0) \neq 0$. Then there exists $\chi \in C^\infty(\mathbb{R}^{2n})$ supported near (x_0,ξ_0) such that

$$|\chi(x,\xi)(a(x,\xi) - a(x_0,\xi_0)) + 1| > 1/C , \quad (x,\xi) \in \mathbb{R}^{2n} .$$

By Theorem 4.23 there exists $c \in S$ such that

$$c^w(x,hD)(\chi^w(x,hD)a^w(x,hD) + a(x_0,\xi_0)(I - \chi^w(x,hD))) = I .$$

2. Now consider

$$\begin{aligned} b^w(x,hD)u(h) =& b^w(x,hD)c^w(x,hD)\chi^w(x,hD)a^w(x,hD)u(h) \\ & + b^w(x,hD)a(x_0,\xi_0)(I - \chi^w(x,hD)))u(h) . \end{aligned}$$

If we choose a to be the symbol appearing in (8.25) then the first term on the right hand side is bounded by $O(h^\infty)$ in $b(x,hD)L^2 \subset \mathcal{S}$. If support of b is sufficiently close to (x_0,ξ_0) then $\mathrm{spt}(b) \cap \mathrm{spt}(1-\chi) = \emptyset$ and the second term has the same property. This proves the lemma. $\square$

Much can be said about the properties of semiclassical wave front sets and we refer to [A] for a recent discussion. Here we will only present the following general result:

THEOREM 8.3 (Wavefront sets and pseudodifferential operators). *Suppose that* $a \in S(m)$ *for some order function* m*, and that* $u(h)$ *is an* h*-tempered family of functions. Then*

$$WF_h(a^w(x,hD)u) \subset WF_h(u) \cap \text{ess-spt}(a) .$$

Proof: 1. We need to show that if $(x,\xi) \notin WF_h(u)$ *or* $(x,\xi) \notin \text{ess-spt}(a)$ then $(x,\xi) \notin WF_h(a^w(x,hD)u)$.

2. Suppose that (x,ξ) is not in $WF_h(u)$. Choose $b \in C_0^\infty(\mathbb{R}^{2n})$, with $b(x,\xi) \neq 0$ a and such that $b^w(x,hD)u = O_{L^2}(h^\infty)$. The existence of such b's is clear from Lemma 8.2. Now the pseudodifferential calculus shows that

$$b^w(x,hD)a^w(x,hD) = c^w(x,hD) + r^w(x,hD)\,,$$
$$\operatorname{spt} c \subset \operatorname{spt} b\,, \quad r \in S^{-\infty}(1)\,.$$

Lemma 8.2 implies that

$$\|b^w(x,hD)a^w(x,hD)u(h)\|_{L^2} = O(h^\infty)\,,$$

showing $(x,\xi) \notin WF_h(a^w(x,hD))$.

3. Now suppose that $(x,\xi) \notin \text{ess-spt}(a)$ and use the same b as above. Then, if the support of b is sufficiently close to (x,ξ), (8.23) implies that $b^w(x,hD)a^w(x,hD) = c^w(x,hD)$ where $c = O_{\mathcal{S}}(h^\infty)$, and that shows that for any h-tempered family

$$\|b^w(x,hD)a^w(x,hD)u(h)\|_{L^2} = O(h^\infty)\,.$$

$\square$

Motivated by Theorem 8.3 we give the following

DEFINITION. Let A be an h-dependent family of operators. We define the *wavefront set* of A to be

$$WF_h(A) := \bigcup WF_h(u) \cap WF_h(Au), \tag{8.27}$$

the union taken over all h-tempered families $\{u(h)\}$.

It is not hard to check that if $A = \operatorname{Op}(a)$, then

$$WF_h(A) = \text{ess-spt}(a)$$

and hence is a closed set. $\square$

REMARK. The definitions of wavefront sets do not depend on the choice of coordinates. We can use it to give an alternative definition of ess-spt(a) which can then be adapted to the case of manifolds:

$$\begin{cases} \text{if } A = \operatorname{Op}(a), \text{ then } (x,\xi) \notin \text{ess-spt}(a) \ \text{ if and only if} \\ \text{for any tempered family } \{u(h)\}_{0<h\le h_0}, \\ \text{we have } (x,\xi) \notin WF_h(Au). \end{cases} \tag{8.28}$$

$\square$

8.4 WAVEFRONT SETS AND L^∞ BOUNDS

Here we will show how a natural frequency localization condition on approximate solutions to (pseudo)-differential equations implies h-dependent L^∞ bounds. As an application we will give well known bounds on eigenfuction clusters for compact Riemannian manifolds.

We start with the following semiclassical version of the Sobolev inequality (Lemma 3.5):

LEMMA 8.4 (Basic L^∞ bounds). *Suppose that $u(h) \in L^2(\mathbb{R}^k)$ is an h-tempered family of functions, and that there exists $\psi \in C_c^\infty(\mathbb{R}^n)$ such that*

$$\|(1-\psi(hD))u(h)\|_{L^2} = \mathcal{O}(h^\infty)\|u\|_{L^2}\,. \tag{8.29}$$

Then

$$\|u(h)\|_{L^\infty} \leq Ch^{-k/2}\|u(h)\|_{L^2}\,.$$

Proof: 1. We can assume that $\|u(h)\|_{L^2} = 1$. We can also assume that $u(h)$ is compactly supported since for $\phi \in C_c^\infty(\mathbb{R}^n)$, $(1-\psi(hD))\phi = \psi(1-\psi(hD)) + r(x, hD)$ with $r \in S$ and ess-spt(r) compact. We can choose $\psi_1 \in C_c^\infty$ for which

$$(1-\psi_1)(1-\psi) = (1-\psi_1)\,, \quad (1-\psi_1)|_{\text{ess-spt}(r)} = 0\,.$$

Then

$$\begin{aligned}(1-\psi_1(hD))\phi u(h) &= (1-\psi_1(hD))\psi(1-\psi(hD))u(h)\\ &\quad + (1-\psi_1(hD))r(x,hD)u(h)\\ &= O_{L^2}(h^\infty)\,.\end{aligned}$$

2. For $u(h)$ compactly supported (uniformly in h) the h-temperance assumption implies that $\|\langle hD\rangle^k u(h)\| \leq h^{-N_k}$ for every k. Hence, by the Cauchy-Schwartz inequality,

$$\begin{aligned}\|\langle hD\rangle^k(1-\psi(hD))u(h)\| &\leq \|\langle hD\rangle^{2k}u(h)\|^{\frac{1}{2}}\|(1-\psi(hD))^2u(h)\|^{\frac{1}{2}}\\ &= \mathcal{O}(h^\infty)\,,\end{aligned}$$

and by Lemma 3.5

$$\|(1-\psi(hD))u(h)\|_{L^\infty} = O(h^\infty)\,.$$

3. It remains to estimate $\|\psi(hD)u\|_{L^\infty}$ and we simply use the semiclassical inverse Fourier transform (3.25):

$$\begin{aligned}\|\psi(hD)u\|_{L^\infty} &\leq \frac{1}{(2\pi h)^k}\|\psi\|_{L^2}\|\mathcal{F}_h u\|_{L^2}\\ &= \frac{1}{(2\pi h)^{k/2}}\|\psi\|_{L^2}\|u\|_{L^2}\,,\end{aligned}$$

proving the lemma. ☐

THEOREM 8.5 (L^∞ bounds for approximate solutions). *Let $m = m(x, \xi)$ an order function. Suppose that $p \in S(T^*\mathbb{R}^n, m)$ is real valued, and for some fixed $K \subset T^*\mathbb{R}^n$,*

$$p(x,\xi) = 0\,, \quad (x,\xi) \in K \implies \partial_\xi p(x,\xi) \neq 0\,. \tag{8.30}$$

If an h-tempered family $u(h)$ satisfies the frequency localization condition (8.29)*, and*

$$\|p^w(x, hD)u(h)\|_{L^2} = O(h)\|u(h)\|_{L^2}\,, \quad WF_h(u) \subset K\,, \tag{8.31}$$

then

$$\|u(h)\|_{L^\infty} \leq Ch^{-(n-1)/2}\|u(h)\|_{L^2}\,. \tag{8.32}$$

REMARK. The bound given in Theorem 8.5 is already optimal in the simplest case in which the assumptions are satisfied: $p(x, \xi) = \xi_1$. Indeed, write $x = (x_1, x')$ and let $\chi_1 \in C_c^\infty(\mathbb{R})$, and $\chi \in C_c^\infty(\mathbb{R}^{n-1})$. Then

$$u(h) := h^{-(n-1)/2}\chi_1(x_1)\chi(x'/h)$$

satisfies

$$P(h)u(h) = hD_{x_1}u(h) = O_{L^2}(h)\,, \quad \|u(h)\|_{L^2} = O(1)\,,$$

and for any non-trivial choices of χ_1 and χ,

$$\|u(h)\|_{L^\infty} \simeq h^{-(n-1)/2}\,.$$

The condition (8.30) is in general necessary as shown by another simple example. Let $p(x, \xi) = x_1$, and

$$u(h) = h^{-n/2}\chi_1(x_1/h)\chi(x'/h)\,.$$

Then

$$P(h)u(h) = hh^{-n/2}(t\chi_1(t))|_{t=x_1/h}\chi(x'/h) = O_{L^2}(h)\,, \ \|u(h)\|_{L^2} = O(1)\,,$$

and

$$\|u(h)\|_{L^\infty} \simeq h^{-n/2}\,,$$

which is the general bound of Lemma 8.4.

Proof of Theorem 8.5: 1. First we observe that, as in Lemma 8.4 we can assume that $u(h)$ is compactly supported. We also note that the hypothesis on $u(h)$ is local in phase space: if $\chi \in C_c^\infty(T^*\mathbb{R}^n)$ then, normalizing to $\|u(h)\|_{L^2} = 1$,

$$\begin{aligned} P(h)\chi^w(x, hD)u(h) &= \chi^w(x, hD)P(h)u(h) + [P(h), \chi^w(x, hD)]u(h) \\ &= O_{L^2}(h)\,, \end{aligned}$$

and, by Theorem 8.3, $WF_h(\chi^w u) \subset K$.

2. Hence it is enough to prove the theorem for $u(h)$ replaced by $\chi^w u(h)$, where χ is supported near a given point in K as a partition of unity argument will then give the bound on $u(h)$. A partition of unity, in this case, means a set of functions,,

$$\{\chi_j\}_{j=0}^N \subset C_c^\infty(T^*\mathbb{R}^n)\,,$$

such that

$$\begin{gathered}\sum_{j=1}^N \chi_j(x,\xi) = \chi_0(x,\xi)\,, \\ \operatorname{spt}\chi_j \subset U_j\,, \quad \operatorname{spt}\chi_0 \subset U_0 := \bigcup_{j=1}^N U_j\,,\end{gathered} \tag{8.33}$$

where U_0 is a neighbourhood of $WF_h(u)$, a compact set, in which (8.30) holds.

3. Suppose that $p \neq 0$ on the support of χ. Then we can use Theorem 4.23 as in part 1 of the proof of Lemma 8.2 to see that $P(h)\chi^w u(h) = \mathcal{O}_{L^2}(h)$ implies that $\chi^w u(h) = \mathcal{O}_{L^2}(h)$. Lemma 8.4 then shows that

$$\|\chi^w u(h)\|_{L^\infty} \leq Chh^{-n/2} \leq Ch^{-(n-1)/2}\,.$$

4. Now suppose that p vanishes in the support of χ. By applying a linear change of variables we can assume that $p_{\xi_1} \neq 0$ there. The implicit function theorem shows that

$$p(x,\xi) = e(x,\xi)(\xi_1 - a(x,\xi'))\,, \quad \xi = (\xi_1,\xi')\,, \quad e(x,\xi) > 0\,, \tag{8.34}$$

holds in a neighbourhood of $\operatorname{spt}\chi$. We extend e arbitrarily to $e \in S$, $e \geq 1/C$, and $a(x,\xi')$ to a real valued $a(x,\xi') \in S$. The pseudodifferential calculus shows that

$$\begin{aligned} e^w(x,hD)(hD_{x_1} - a(x,hD_{x'}))(\chi^w u(h)) &= P(h)(\chi^w u(h)) + O_{L^2}(h) \\ &= O_{L^2}(h)\,, \end{aligned}$$

and since e^w is elliptic,

$$(hD_{x_1} - a(x,hD_{x'}))(\chi^w u(h)) = O_{L^2}(h)\,. \tag{8.35}$$

5. The proof will be completed if we show that

$$\|(\chi^w u)(x_1,\bullet)\|_{L^2(\mathbb{R}^{n-1})} = O(1)\,, \tag{8.36}$$

and for that we need another elementary lemma:

LEMMA 8.6 (A simple energy estimate). *Suppose that $a \in S(\mathbb{R}\times T^*\mathbb{R}^k)$ is real valued, and that*

$$(hD_t + a^w(t,x,hD_x))u(t,x) = f(t,x)\,, \quad u(0,x) = u_0(x)\,,$$
$$f \in L^2(\mathbb{R}\times\mathbb{R}^k)\,, \quad u_0 \in L^2(\mathbb{R}^k)\,.$$

Then

$$\|u(t,\bullet)\|_{L^2(\mathbb{R}^k)} \leq \frac{\sqrt{t}}{h}\|f\|_{L^2(\mathbb{R}\times\mathbb{R}^k)} + \|u_0\|_{L^2(\mathbb{R}^k)}\,. \tag{8.37}$$

Proof: Since $a^w(t,x,hD)$ is family of bounded operators on $L^2(\mathbb{R}^k)$ existence of solutions follows from existence theory for (linear) ordinary differential equations in t. Suppose first that $f \equiv 0$. Then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\mathbb{R}^k)} &= \mathrm{Re}\langle \partial_t u(t), u(t)\rangle_{L^2(\mathbb{R}^k)} \\ &= \frac{1}{h}\mathrm{Re}\langle i a^w(x,hD)u(t), u(t)\rangle = 0\,.\end{aligned}$$

Thus, if we put $E(t)u_0 := u(t)$,

$$\|E(t)u_0\|_{L^2(\mathbb{R}^k)} = \|u_0\|_{L^2(\mathbb{R}^k)}\,.$$

If $f \neq 0$, Duhamel's formula gives

$$u(t) = E(t)u_0 + \frac{i}{h}\int_0^t E(t-s)f(s)ds\,,$$

and hence

$$\|u(t)\|_{L^2(\mathbb{R}^k)} \leq \|u_0\|_{L^2(\mathbb{R}^k)} + \int_0^t \|f(s)\|_{L^2(\mathbb{R}^k)}\,.$$

The estimate (8.37) is an immediate consequence. □

The estimate (8.36) is immediate from the lemma and (8.35). We now apply Lemma 8.4 in x' variables only, that is with $k = n-1$. That is allowed since we clearly have

$$\|(1-\psi(hD'))\chi^w u(h)(x_1,\bullet)\|_{L^2(\mathbb{R}^{n-1})} = O(h^\infty)\,,$$

uniformly in x_1. □

As an application we give a well known L^∞ bound on spectral clusters. The statement requires the material presented in Appendix D.3.

THEOREM 8.7 (Bounds on eigenfuctions). *Suppose that M is an n-dimensional compact Riemannian manifold and let Δ_g be its Laplace-Beltrami operator. If*

$$0 = \lambda_0 < \lambda_1 \leq \cdots \lambda_j \to \infty$$

is the complete set of eigenvalues of $-\Delta_g$, *and*

$$-\Delta_g \phi_j = \lambda_j \phi_j$$

are the corresponding eigenfunctions, then for any $c_j \in \mathbb{C}$, $j = 0, 1, \cdots$,

$$\| \sum_{\mu \leq \sqrt{\lambda_j} \leq \mu+1} c_j \phi_j \|_{L^\infty} \leq C \mu^{(n-1)/2} \| \sum_{\mu \leq \sqrt{\lambda_j} \leq \mu+1} c_j \phi_j \|_{L^2} \,. \tag{8.38}$$

In particular

$$\|\phi_j\|_{L^\infty} \leq C \lambda_j^{(n-1)/4} \|\phi_j\|_{L^2} \,. \tag{8.39}$$

Proof: We put $P(h) := -h^2 \Delta_g - 1$. Then the assumption (8.30) holds everywhere. If

$$u(h) := \sum_{\mu \leq \sqrt{\lambda_j} \leq \mu+1} c_j \phi_j \,, \quad \mu = 1/h \,.$$

then

$$\begin{aligned} \|P(h)u(h)\|_{L^2} &= \| \sum_{\mu \leq \sqrt{\lambda_j} \leq \mu+1} c_j (h^2 \lambda_j - 1) \phi_j \|_{L^2} \\ &= \left(\sum_{\mu \leq \sqrt{\lambda_j} \leq \mu+1} |c_j|^2 (h^2 \lambda_j - 1)^2 \|\phi_j\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\leq 2h \| \sum_{\mu \leq \sqrt{\lambda_j} \leq \mu+1} c_j \phi_j \|_{L^2} \,, \end{aligned}$$

which means that the assumption (8.31) holds. On a compact manifold the frequency localization (8.29) follows from

$$\begin{gathered} \|(1 - \varphi(-h^2 \Delta_g)) u(h)\| = \mathcal{O}(h^\infty) \|u(h)\|_L^2 \,, \\ \varphi \in C_c^\infty(\mathbb{R}) \,, \quad \varphi(t) \equiv 1 \,, \ |t| \leq 2 \,, \end{gathered}$$

and that follows from the spectral theorem. $\square$

The estimate (8.38) is essentially optimal. On the other hand the optimality of (8.39) is very rare – see [S-Z] for a recent discussion.

8.5 BEALS'S THEOREM

We next present a semiclassical version of Beals's Theorem, a characterization of pseudodifferential operators in terms of h-dependent bounds on commutators.

EXAMPLE: Resolvents. As motivation, take the following simple example. Suppose $a \in S$ is real-valued. Consequently $A = \mathrm{Op}(a)$ is a self-adjoint operator on $L^2(\mathbb{R}^n)$, and so the resolvent $(A+i)^{-1}$ is a bounded operator on $L^2(\mathbb{R}^n)$.

But we are then confronted with a basic question:

$$\text{Is } \ (A+i)^{-1} = \mathrm{Op}(b) \ \text{ for some symbol } b \in S?$$

Theorem 8.9 below provides a simple criterion from which this conclusion easily follows. We will return to this example later. □

We start with

THEOREM 8.8 (Estimating a symbol by operator norms). *Take $h = 1$. Then there exist constants $C, M > 0$ such that*

$$\|b\|_{L^\infty} \leq C \sum_{|\alpha| \leq M} \|\mathrm{Op}(\partial^\alpha b)\|_{L^2 \to L^2}, \tag{8.40}$$

for all $b \in \mathcal{S}'(\mathbb{R}^{2n})$.

Proof. 1. We will first consider the classical quantization

$$b(x,D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} b(x,\xi) e^{i\langle x,\xi\rangle} \hat{u}(\xi)\, d\xi,$$

where by the integration we mean the Fourier transform in $\mathcal{S}'$.

Then if $\phi = \phi(x), \psi = \psi(\xi)$ are functions in the Schwartz space $\mathcal{S}$, we can regard $\mathcal{F}(b\bar{\phi}\hat{\psi}e^{i\langle x,\xi\rangle})$ as a function of the dual variables $(x^*, \xi^*) \in \mathbb{R}^{2n}$. We have

$$\begin{aligned}
\frac{1}{(2\pi)^n} |\mathcal{F}(b\bar{\phi}\hat{\psi}e^{i\langle x,\xi\rangle})(0,0)| &= \frac{1}{(2\pi)^n} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} b(x,\xi)\bar{\phi}(x)\hat{\psi}(\xi) e^{i\langle x,\xi\rangle} dx d\xi \right| \\
&= |\langle b(x,D)\psi, \phi\rangle| \leq \|b\|_{L^2 \to L^2} \|\phi\|_{L^2} \|\psi\|_{L^2}.
\end{aligned}$$

Fix $(x^*, \xi^*) \in \mathbb{R}^{2n}$ and rewrite the foregoing inequality with $\phi(x)e^{i\langle x^*,x\rangle}$ replacing $\phi(x)$ and $\psi(\xi)e^{-i\langle \xi^*,\xi\rangle}$ replacing $\psi(\xi)$, a procedure which does not change the L^2 norms. It follows that

$$\frac{1}{(2\pi)^n} |\mathcal{F}(b\bar{\phi}\hat{\psi}e^{i\langle x,\xi\rangle})(x^*,\xi^*)| \leq \|b\|_{L^2 \to L^2} \|\phi\|_{L^2} \|\psi\|_{L^2}\,. \tag{8.41}$$

2. Now take $\chi \in C_c^\infty(\mathbb{R}^{2n})$. Select $\phi, \psi \in \mathcal{S}$ so that

$$\begin{cases} \phi(x) = 1 & \text{if } (x,\xi) \in \mathrm{spt}\chi \\ \hat{\psi}(\xi) = 1 & \text{if } (x,\xi) \in \mathrm{spt}\chi. \end{cases}$$

Write

$$\Phi = \chi e^{-i\langle x,\xi\rangle}. \tag{8.42}$$

Then

$$\chi(x,\xi) = \Phi(x,\xi)\phi(x)\hat{\psi}(\xi)e^{i\langle x,\xi\rangle}.$$

According to (3.20),

$$\|\mathcal{F}\chi\|_{L^1} \le C \sum_{|\alpha|\le 2n+1} \|\partial^\alpha \chi\|_{L^1};$$

and so (8.42) implies

$$\|\mathcal{F}\Phi\|_{L^1} \le C \sum_{|\alpha|\le 2n+1} \|\partial^\alpha \chi\|_{L^1}. \tag{8.43}$$

Thus (8.41) shows that for any $(x^*,\xi^*) \in \mathbb{R}^{2n}$ we have

$$\begin{aligned}
|\mathcal{F}(\chi b)(x^*,\xi^*)| &\le \|\mathcal{F}(\Phi b\bar{\phi}\hat{\psi}e^{i\langle x,\xi\rangle})\|_{L^\infty} \\
&= \frac{1}{(2\pi)^n}\|\mathcal{F}(\Phi) * \mathcal{F}(b\bar{\phi}\hat{\psi}e^{i\langle x,\xi\rangle})\|_{L^\infty} \\
&\le \frac{1}{(2\pi)^n}\|\mathcal{F}(b\bar{\phi}\hat{\psi}e^{i\langle x,\xi\rangle})\|_{L^\infty}\|\mathcal{F}(\Phi)\|_{L^1} \\
&\le C\|b\|_{L^2\to L^2},
\end{aligned}$$

the constant C depending on ϕ, ψ and χ, but not (x^*,ξ^*). Hence

$$\|\mathcal{F}(\chi b)\|_{L^\infty} \le C\|b\|_{L^2\to L^2} \tag{8.44}$$

with the same constant for any translate of χ.

3. Next, we assert that

$$|\mathcal{F}(\chi b)(x^*,\xi^*)| \le C\langle(x^*,\xi^*)\rangle^{-2n-1} \sum_{|\alpha|\le 2n+1} \|\mathrm{Op}(\partial^\alpha b)\|_{L^2\to L^2}. \tag{8.45}$$

To see this, compute

$$\begin{aligned}
(x^*)^\alpha(\xi^*)^\beta\mathcal{F}(\chi b)(x^*,\xi^*) &= \int_{\mathbb{R}^{2n}} (x^*)^\alpha(\xi^*)^\beta e^{-i(\langle x^*,\,x\rangle+\langle\xi^*,\,\xi\rangle)}\chi b(x,\xi)\,dxd\xi \\
&= \int_{\mathbb{R}^{2n}} e^{-i(\langle x^*,\,x\rangle+\langle\xi^*,\,\xi\rangle)} D_x^\alpha D_\xi^\beta(\chi b)\,dxd\xi.
\end{aligned}$$

Summing absolute values of the left hand side over all (α,β) with $|\alpha|+|\beta| \le 2n+1$ and using the estimate (8.44), we obtain the bound

$$\begin{aligned}
\|\langle(x^*,\xi^*)\rangle^{2n+1}\mathcal{F}(\chi b)\|_{L^\infty} &\le C_1 \sum_{|\alpha|+|\beta|\le 2n+1} \|\mathcal{F}(D_x^\alpha D_\xi^\beta(\chi b))\|_{L^\infty} \\
&\le C_2 \sum_{|\alpha|\le 2n+1} \|\mathrm{Op}(\partial^\alpha b)\|_{L^2\to L^2}.
\end{aligned}$$

This gives (8.45).

Consequently,

$$\|\chi b\|_{L^\infty} \leq C\|\mathcal{F}(\chi b)\|_{L^1} \leq C \sum_{|\alpha|\leq 2n+1} \|\mathrm{Op}(\partial^\alpha b)\|_{L^2\to L^2}.$$

4. This implies the desired inequality (8.40), except that we used the classical ($t = 1$), and not the Weyl ($t = 1/2$) quantization. To remedy this, recall from Theorem 4.13 that if

$$b = e^{\frac{i}{2}\langle D_x, D_\xi\rangle}\tilde{b},$$

then

$$\begin{cases} b^w(x, D) = \tilde{b}(x, D) \\ (\partial^\alpha b)^w(x, D) = (\partial^\alpha \tilde{b})(x, D). \end{cases}$$

The continuity statement in Theorem 4.16 shows that

$$\|b\|_{L^\infty} \leq C \sum_{|\alpha|\leq K} \|\partial^\alpha \tilde{b}\|_{L^\infty},$$

and reduces the argument to the classical quantization.

$\square$

NOTATION. We henceforth write

$$\mathrm{ad}_B A := [B, A];$$

"ad" is called the *adjoint action.*

Recall also that we identify a pair $(x^*, \xi^*) \in \mathbb{R}^{2n}$ with the linear operator $l(x, \xi) = \langle x^*, x\rangle + \langle \xi^*, \xi\rangle$.

THEOREM 8.9 (Semiclassical form of Beals's Theorem). *Let $A : \mathcal{S} \to \mathcal{S}'$ be a continuous linear operator. Then*

(i) $A = \mathrm{Op}(a)$ for a symbol $a \in S$

if and only if

(ii) *for all $N = 0, 1, 2, \ldots$ and all linear functions $l_1, \ldots, l_N$, we have*

$$\|\mathrm{ad}_{l_1(x,hD)} \circ \cdots \circ \mathrm{ad}_{l_N(x,hD)} A\|_{L^2\to L^2} = O(h^N). \tag{8.46}$$

Proof. 1. That (i) implies (ii) follows from the symbol calculus developed in Chapter 4. Indeed, $\|A\|_{L^2\to L^2} = O(1)$ and each commutator with an operator $l_j(x, hD)$ yields a term of order h.

2. That (ii) implies (i) is harder. First of all, the Schwartz Kernel Theorem (Theorem B.8) asserts that we can write

$$Au(x) = \int_{\mathbb{R}^n} K_A(x,y)u(y)\,dy \tag{8.47}$$

for $K_A \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$. We call K_A the *kernel* of A.

3. We now claim that if we define $a \in \mathcal{S}'(\mathbb{R}^{2n})$ by

$$a(x,\xi) := \int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle w,\xi\rangle} K_A\left(x + \frac{w}{2}, x - \frac{w}{2}\right)\,dw, \tag{8.48}$$

then

$$K_A(x,y) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} a\left(\frac{x+y}{2},\xi\right) e^{\frac{i}{h}\langle x-y,\xi\rangle}\,d\xi, \tag{8.49}$$

where the integrals are a shorthand for the Fourier transforms defined on $\mathcal{S}'$.

To confirm this, we calculate that

$$\begin{aligned}
&\frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} a\left(\frac{x+y}{2},\xi\right) e^{\frac{i}{h}\langle x-y,\xi\rangle} d\xi \\
&\qquad = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y-w,\xi\rangle} K_A\left(\frac{x+y}{2} + \frac{w}{2}, \frac{x+y}{2} - \frac{w}{2}\right) dw d\xi \\
&\qquad = K_A(x,y),
\end{aligned}$$

since

$$\frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y-w,\xi\rangle} d\xi = \delta_{x-y}(w) \quad \text{in } \mathcal{S}'.$$

In view of (8.47) and (8.49), we see that $A = \mathrm{Op}(a)$, for a defined by (8.48).

4. Now we must show that a belongs to the symbol class S; that is,

$$\sup_{\mathbb{R}^{2n}} |\partial^\alpha a| \le C_\alpha \tag{8.50}$$

for each multiindex α.

To do so we will make use of our hypothesis (8.46) with $l = x_j, \xi_j$, that is, with $l(x, hD) = x_j, hD_j$. We compute

$$
\begin{aligned}
&\mathrm{Op}(hD_{\xi_j} a)u(x) \\
&\qquad = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} hD_{\xi_j} \left(a\left(\frac{x+y}{2}, \xi \right) \right) e^{i\frac{\langle x-y,\xi\rangle}{h}} u(y)\, d\xi dy \\
&\qquad = -\frac{1}{(2\pi h)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a\left(\frac{x+y}{2}, \xi \right) hD_{\xi_j} \left(e^{\frac{i\langle x-y,\xi\rangle}{h}} \right) u(y)\, d\xi dy \\
&\qquad = -\frac{1}{(2\pi h)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a\left(\frac{x+y}{2}, \xi \right) e^{\frac{i\langle x-y,\xi\rangle}{h}} (x_j - y_j) u(y)\, d\xi dy \\
&\qquad = -[x_j, A]u = -\mathrm{ad}_{x_j} A u(x).
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
&\mathrm{Op}(hD_{x_j} a)u(x) \\
&\qquad = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a_{x_j} \left(\frac{x+y}{2}, \xi \right) e^{\frac{i\langle x-y,\xi\rangle}{h}} u(y)\, d\xi dy \\
&\qquad = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} h(D_{x_j} + D_{y_j}) \left(a\left(\frac{x+y}{2}, \xi \right) \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad e^{\frac{i\langle x-y,\xi\rangle}{h}} u(y)\, d\xi dy \\
&\qquad = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} hD_{x_j} \left(a\left(\frac{x+y}{2}, \xi \right) \right) e^{\frac{i\langle x-y,\xi\rangle}{h}} u(y)\, d\xi dy \\
&\qquad + \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a\left(\frac{x+y}{2}, \xi \right) e^{\frac{i\langle x-y,\xi\rangle}{h}} (\xi_j - D_{y_j} u(y))\, d\xi dy \\
&\qquad = hD_{x_j}(Au) - A(hD_{x_j} u) \\
&\qquad = [hD_{x_j}, A]u = \mathrm{ad}_{hD_{x_j}} A u(x).
\end{aligned}
$$

In summary, for $j = 1, \dots, n$,

$$
\begin{cases} \mathrm{ad}_{x_j} A = -\mathrm{Op}(hD_{\xi_j} a) \\ \mathrm{ad}_{hD_{x_j}} A = \mathrm{Op}(hD_{x_j} a). \end{cases} \tag{8.51}
$$

5. Next we convert to the case $h = 1$ by rescaling. For this, define

$$
U_h u(x) := h^{n/4} u(h^{1/2} x)
$$

and check that $U_h : L^2 \to L^2$ is unitary. Then

$$
U_h a^w(x, hD) U_h^{-1} = a^w(h^{1/2} x, h^{1/2} D) = \mathrm{Op}(a_h)
$$

for

$$
a_h(x, \xi) := a(h^{1/2} x, h^{1/2} \xi). \tag{8.52}
$$

Our hypothesis (8.46) is invariant under conjugation by U_h, and is consequently equivalent to

$$\mathrm{ad}_{l_1(h^{1/2}x,h^{1/2}D)} \circ \cdots \circ a_{l_N(h^{1/2}x,h^{1/2}D)} \mathrm{Op}(a_h) = O(h^N). \tag{8.53}$$

But since l_j is linear, $l_j(h^{1/2}x, h^{1/2}D) = h^{1/2}l(x,D)$. Thus (8.53) is equivalent to

$$\mathrm{ad}_{l_1(x,D)} \circ \cdots \circ a_{l_N(x,D)} \circ \mathrm{Op}(a_h) = O(h^{N/2}). \tag{8.54}$$

Taking $l_k(x,\xi) = x_j$ or ξ_j, it follows from (8.54) that

$$\|\mathrm{Op}(\partial^\beta a_h)\| \leq Ch^{\frac{|\beta|}{2}} \tag{8.55}$$

for all multiindices β.

6. Finally, we claim that

$$|\partial^\alpha a_h| \leq C_\alpha h^{|\alpha|/2} \text{ for each multiindex } \alpha. \tag{8.56}$$

But this follows from Theorem 8.8, owing to estimate (8.55):

$$\|a_h^\alpha\|_{L^\infty} \leq C \sum_{|\beta|\leq n+1} \|\mathrm{Op}(\partial^{\alpha+\beta} a_h)\|_{L^2\to L^2} \leq C_\alpha h^{|\alpha|}.$$

Recalling (8.52), we rescale to derive the desired inequality (8.50). □

REMARK. In Beals's Theorem we can replace linear symbols by symbols in the class S, since for any linear l, we can find $a \in S$ so that $H_l = H_a$ locally. We will need this observation in Chapter 10. □

EXAMPLE. We can now go back to the example in the beginning of this section, in which $A = \mathrm{Op}(a)$ for a real-valued symbol $a \in S(1)$, and $B = (A+i)^{-1}$. Since

$$\mathrm{ad}_l B = -B(\mathrm{ad}_l A)B,$$

we see that the assumptions of Beals's Theorem are satisfied and hence

$$B = \mathrm{Op}(b) \quad \text{for} \quad b \in S(1).$$

□

8.5 APPLICATION: EXPONENTIATION OF OPERATORS

As we have see in Theorem 4.6. quantization of exponetiation commutes with quantization for linear symbols. That is of course not true for non-linear symbols – see Section 10.2 below for an example for the subtleties involved in exponention of skew-adjoint pseudodifferential operators, that is in the study of propagation.

In this section we will consider one parameter families of operators which give exponentials of self-adjoint pseudodifferential operators. We will show that on the level of *order functions* exponentiation commutes with quantization. This is a special of a general result [B-C, Théoreme 6.4].

THEOREM 8.10. *Let $m(x,\xi)$ be an order function and suppose that $G = G(x,\xi,h)$ satisfies*

$$\begin{gathered} G(x,\xi) - \log m(x,\xi) = O(1)\,, \\ \partial^\alpha G \in S_\delta(1)\,, \quad |\alpha| = 1\,, \quad 0 \leq \delta \leq \frac{1}{2}\,. \end{gathered} \tag{8.57}$$

Then the equation

$$\frac{d}{dt}\mathcal{B}(t) = G^w(x,hD)\mathcal{B}(t)\,, \quad \mathcal{B}(0) = Id\,,$$

has a unique solution $\mathcal{B}(t) : \mathcal{S} \to \mathcal{S}$, and

$$\mathcal{B}(t) = B_t^w(x,hD)\,, \quad B_t \in S(m^t)\,. \tag{8.58}$$

Theorem 8.10 gives a construction of $\exp(tG^w(x,D))$ and describes it as a quantization of an element of $S_\delta(m^t)$. Since m may grow in some directions and decay in other directions that is far from obvious. In Chapter 7 we saw advantages of conjugation by $\exp(\phi(x))$ where ϕ was real valued. In many problems one may want to do it in phase space and then operators of the form

$$P_t := e^{-tG^w(x,hD)}Pe^{tG^w(x,hD)}$$

are useful – see [D-S-Z], [S-Z3] for examples of such techniques, and [M] for a slightly different perspective. Here we only note one application of Theorem 8.10: if G is as above and P is bounded on L^2, then P_t is bounded on L^2. In fact, we simply apply Theorems 4.17 and 4.20. It would be difficult to obtain this basic result without Theorem 8.10.

To prove Theorem 8.10 we start with

LEMMA 8.11. *Let $U(t) \stackrel{\text{def}}{=} (\exp tG)^w(x,D) \;:\; \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$. For $|t| < \epsilon_0(G)$, the operator $U(t)$ is invertible, and*

$$U(t)^{-1} = B_t^w(x,D)\,, \quad B_t \in S(m^{-t})\,.$$

Proof: 1. We apply the composition formula given in Theorem 4.17 to obtain

$$U(-t)U(t) = Id + E_t^w(x,D)\,, \quad E_t \in S(1)\,.$$

2. More explicitely we write

$$
\begin{aligned}
E_t(x_1,\xi) &= \int_0^s e^{sA(D)}A(D)(e^{-tG(x_1,\xi_1)+tG(x_2,\xi_2)})|_{x_2=x_1=x,\xi_2=\xi_1=\xi}ds \\
&= \int_0^s ite^{sA(D)}Fe^{-tG(x_1,\xi_1)+tG(x_2,\xi_2)}|_{x_2=x_1=x,\xi_2=\xi_1=\xi}ds/2\,,
\end{aligned}
$$

where

$$
\begin{gathered}
A(D) = i\sigma(D_{x_1}, D_{\xi_1}; D_{x_2}, D_{\xi_2})/2\,, \\
F = \partial_{x_1}G(x_1,\xi_1)\cdot\partial_{\xi_2}G(x_2,\xi_2) - \partial_{\xi_1}G(x_1,\xi_1)\cdot\partial_{x_2}G(x_2,\xi_2)\,.
\end{gathered}
$$

3. We conclude that $E_t = t\widetilde{E}_t$ where $\widetilde{E}_t \in S(1)$, and thus

$$
E_t^w(x,D) = O(t) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\,.
$$

This shows that for $|t|$ small enough $Id + E_t^w(x,D)$ is invertible, and Theorem 8.9 gives

$$
(Id + E_t^w(x,D))^{-1} = C_t^w(x,D)\,, \quad C_t \in S(1)\,.
$$

Hence $B_t = C_t \# \exp(-tG(x,\xi)) \in S(m^{-t})$. $\square$

Proof of Therem 8.10: 1. We first note that we only need to prove the result in the case $h = 1$ by using the rescaling given in (4.28). Also, $G^w(x,hD) : \mathcal{S} \to \mathcal{S}$ shows that $\mathcal{B}_t : \mathcal{S} \to \mathcal{S}$ is unique.

2. The hypotheses on G in (8.57) are equivalent to the statement that $\exp(tG) \in S(m^t)$, for all $t \in \mathbb{R}$. We now observe that

$$
\begin{gathered}
\frac{d}{dt}\left(U(-t)\exp(tG^w(x,D))\right) = V(t)\exp(tG^w(x,D))\,, \\
V(t) = A_t^w(x,D)\,, \quad A_t \in S(m^{-t})\,.
\end{gathered}
\tag{8.59}
$$

In fact, we see that

$$
\begin{gathered}
\frac{d}{dt}U(-t) = -(G\exp(-tG))^w(x,D)\,, \\
U(-t)G^w(x,D) = (\exp(tG)\#G)^w(x,D)\,.
\end{gathered}
$$

As before, the composition formula (4.22) gives

$$
\begin{gathered}
\exp(-tG)\#G - G\exp(-tG) = \\
\int_0^1 \exp(sA(D))A(D)\exp(-tG(x^1,\xi^1)G(x^2,\xi^2)|_{x^1=x^2=x,\xi^1=\xi^2=\xi}\,, \\
A(D) = i\sigma(D_{x^1}, D_{\xi^1}; D_{x^2}, D_{\xi^2})/2\,.
\end{gathered}
$$

From the hypothesis on G we see that $A(D)\exp(tG(x^1,\xi^1))G(x^2,\xi^2)$ is a sum of terms of the form $a(x^1,\xi^1)b(x^2,\xi^2)$ where $a \in S(m^{-t})$ and

$b \in S(1)$. The continuity of $\exp(A(D))$ on the spaces of symbols in Theorem 4.16 gives (8.59).

3. If we put

$$C(t) \stackrel{\text{def}}{=} -V(t)U(-t)^{-1},$$

then by Lemma 8.11, $C(t) = c_t^w$ where $c_t \in S(1)$. Symbolic calculus shows that c_t depends smoothly on t and

$$(\partial_t + C(t))(U(-t)\exp(tG^w(x,D))) = 0.$$

4. The proof of Theorem 8.10 is now reduced to showing

LEMMA 8.12. *Suppose that $C(t) = c_t^w(x,D)$, where $c_t \in S(1)$, depends continuously on $t \in (-\epsilon_0, \epsilon_0)$. Then the solution of*

$$(\partial_t + C(t))Q(t) = 0, \quad Q(0) = q^w(x,D), \quad q \in S(1), \tag{8.60}$$

is given by $Q(t) = q_t(x,D)$, where $q_t \in S(1)$ depends continuously on $t \in (-\epsilon_0, \epsilon_0)$.

Proof: The Picard existence theorem for ODEs shows that $Q(t)$ is bounded on L^2. If $\ell_j(x,\xi)$ are linear functions on $T^*\mathbb{R}^n$ then

$$\begin{gathered}\frac{d}{dt}\mathrm{ad}_{\ell_1(x,D)} \circ \cdots \circ \mathrm{ad}_{\ell_N(x,D)} Q(t) + \\ \mathrm{ad}_{\ell_1(x,D)} \circ \cdots \circ \mathrm{ad}_{\ell_N(x,D)}(C(t)Q(t)) = 0, \\ \mathrm{ad}_{\ell_1(x,D)} \circ \cdots \circ \mathrm{ad}_{\ell_N(x,D)} Q(0) \ : \ L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n).\end{gathered}$$

If we show that for any choice of $\ell_j's$ and any N

$$\mathrm{ad}_{\ell_1(x,D)} \circ \cdots \circ \mathrm{ad}_{\ell_N(x,D)} Q(t) \ : \ L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n), \tag{8.61}$$

then Beals's Theorem concludes the proof. We proceed by induction on N:

$$\begin{gathered}\mathrm{ad}_{\ell_1(x,D)} \circ \cdots \circ \mathrm{ad}_{\ell_N(x,D)}(C(t)Q(t)) = \\ C(t)\mathrm{ad}_{\ell_1(x,D)} \circ \cdots \circ \mathrm{ad}_{\ell_N(x,D)} Q(t) + R(t),\end{gathered}$$

where $R(t)$ is the sum of terms of the form

$$A_k(t)\mathrm{ad}_{\ell_1(x,D)} \circ \mathrm{ad}_{\ell_k(x,D)} Q(t), \quad k < N, \quad A_k(t) = a_k(t)^w,$$

where $a_k(t) \in S(1)$ depend continuously on t. This also follows by an inductive based on the derivation property of ad_ℓ:

$$\mathrm{ad}_\ell(CD) = (\mathrm{ad}_\ell C)D + C(\mathrm{ad}_\ell D).$$

Hence by the induction hypothesis $R(t)$ is bounded on L^2, and depends continuously on t. Thus

$$(\partial_t + C(t))\,\mathrm{ad}_{\ell_1(x,D)} \circ \cdots \circ \mathrm{ad}_{\ell_N(x,D)} Q(t) = R(t)$$

is bounded on L^2. Since (8.61) is valid at $t = 0$ we obtain it for all $t \in (-\epsilon_0, \epsilon_0)$. $\square$

9. Quantum ergodicity

In this chapter we are given a smooth potential V on a compact Riemannian manifold (M, g) and write

$$p(x,\xi) = |\xi|_g^2 + V(x) \tag{9.1}$$

for $(x,\xi) \in T^*M$, the cotangent space of M. As explained in Appendix D, the associated quantum operator is

$$P(h) = -h^2\Delta_g + V, \tag{9.2}$$

and the Hamiltonian flow generated by p is denoted

$$\Phi_t = \exp(tH_p) \qquad (t \in \mathbb{R}).$$

We address in this chapter quantum implications of ergodicity for the classical evolution $\{\Phi_t\}_{t\in\mathbb{R}}$. The proofs will rely on various advanced material presented in Appendix D.

9.1 CLASSICAL ERGODICITY

We hereafter select $a < b$, and assume that

$$|\partial p| \geq \gamma > 0 \text{ on } \{a \leq p \leq b\}. \tag{9.3}$$

According then to the Implicit Function Theorem, for each $a \leq c \leq b$, the set

$$\Sigma_c := p^{-1}(c)$$

is a smooth, $2n-1$ dimensional hypersurface in the cotangent space T^*M. We can interpret Σ_c as an energy surface.

NOTATION. For each $c \in [a,b]$, we denote by μ *Liouville measure* on the hypersurface $\Sigma_c = p^{-1}(c)$ corresponding to p. This measure is characterized by the formula

$$\iint_{p^{-1}[a,b]} f\,dxd\xi = \int_a^b \int_{\Sigma_c} f\,d\mu\,dc$$

for all $a < b$ and each continuous function $f : T^*M \to \mathbb{R}^n$.

DEFINITION. Let $m \in \Sigma_c$ and $f : T^*M \to \mathbb{C}$. For $T > 0$ we define the *time average*

$$\langle f\rangle_T := \frac{1}{T}\int_0^T f\circ\Phi_t(m)\,dt = ⨍_0^T f\circ\Phi_t(m)\,dt, \tag{9.4}$$

the slash through the second integral denoting an average. Note carefully that $\langle f\rangle_T = \langle f\rangle_T(m)$ depends upon the starting point m.

DEFINITION. We say the flow Φ_t is *ergodic* on $p^{-1}[a,b]$ if for each $c \in [a,b]$,

$$\left\{\begin{array}{l}\text{if } E \subseteq \Sigma_c \text{ is flow invariant, then}\\ \text{either } \mu(E)=0 \text{ or else } \mu(E)=\mu(\Sigma_c).\end{array}\right. \tag{9.5}$$

In other words, we are requiring that each flow invariant subset of the energy level Σ_c have either zero measure or full measure.

THEOREM 9.1 (Mean Ergodic Theorem). *Suppose the flow is ergodic on $\Sigma_c := p^{-1}(c)$. Then for each $f \in L^2(\Sigma_c,\mu)$ we have*

$$\lim_{T\to\infty}\int_{\Sigma_c}\left(\langle f\rangle_T - \int_{\Sigma_c} f\,d\mu\right)^2 d\mu = 0. \tag{9.6}$$

REMARK. According to Birkhoff's Ergodic Theorem, for μ–a.e. point m belonging to Σ_c,

$$\langle f\rangle_T \to \int_{\Sigma_c} f\,d\mu \quad \text{ as } T\to\infty.$$

But we will only need the weaker statement of Theorem 8.1. □

Proof. 1. Define

$$A := \{f \in L^2(\Sigma_c,\mu) \mid \ \Phi_t^* f = f \text{ for all times } t\},$$
$$B_0 := \{H_p g \mid g \in C^\infty(\Sigma_c)\}, \quad B := \bar{B}_0.$$

We claim that

$$h \in B_0^\perp \ \text{ if and only if } \ h \in A. \tag{9.7}$$

To see this, first let $h \in A$ and $f = H_p g \in B_0$. Then

$$\begin{aligned}\int_{\Sigma_c} h\bar{f}\,d\mu = \int_{\Sigma_c} h\overline{H_p g}\,d\mu &= \frac{d}{dt}\int_{\Sigma_c} h\overline{\Phi_t^* g}\,d\mu|_{t=0}\\ &= \frac{d}{dt}\int_{\Sigma_c}\Phi_{-t}^* h\overline{g}\,d\mu|_{t=0} = \frac{d}{dt}\int_{\Sigma_c} h\overline{g}\,d\mu|_{t=0} = 0;\end{aligned}$$

and consequently $h \in B_0^\perp$.

Conversely, suppose $h \in B_0^\perp$. Then for any $g \in C^\infty$, we have

$$0 = \int_{\Sigma_c} h\,\overline{H_p\Phi_{-t}^* g}\,d\mu = \frac{d}{dt}\int_{\Sigma_c} h\,\overline{\Phi_{-t}^* g}\,d\mu = \frac{d}{dt}\int_{\Sigma_c}\Phi_t^* h\,\overline{g}\,d\mu.$$

Therefore for all times t and all functions g,

$$\int_{\Sigma_c} \Phi_t^* h \, \overline{g} \, d\mu = \int_{\Sigma_c} h \, \overline{g} \, d\mu.$$

Hence $\Phi_t^* h \equiv h$, and so $h \in A$.

2. It follows from (9.7) that we have the orthogonal decomposition

$$L^2(\Sigma_c, \mu) = A \oplus B.$$

Thus if we write $f = f_A + f_B$, for $f_A \in A$, $f_B \in B$, then

$$\langle f_A \rangle_T \equiv f_A$$

for all T.

Now suppose $f_B = H_p g \in B_0$. We can then compute

$$\begin{aligned}
\int_{\Sigma_c} |\langle f_B \rangle_T|^2 \, d\mu &= \frac{1}{T^2} \int_{\Sigma_c} \left| \int_0^T (d/dt) \Phi_t^* g dt \right|^2 d\mu \\
&= \frac{1}{T^2} \int_{\Sigma_c} |\Phi_T^* g - g|^2 \, d\mu \\
&\leq \frac{4}{T^2} \int_{\Sigma_c} |g|^2 d\mu \longrightarrow 0,
\end{aligned}$$

as $T \to \infty$. Since $f_B \in B := \overline{B}_0$, we have $\langle f_B \rangle_T \to 0$ in $L^2(\Sigma_c, d\mu)$.

3. The ergodicity hypothesis is equivalent to saying that A consists of constant functions. Indeed, for any $h \in A$, the set $h^{-1}([\alpha, \infty))$ is invariant under the flow, and hence has either full measure or measure zero. Since the functions in $L^2(\Sigma_c, d\mu)$ are defined up to sets of measure zero, h is equivalent to a constant function.

Lastly, observe that the orthogonal projection $f \mapsto f_A$ is just the space average with respect to μ. This proves (9.6). □

9.2 EGOROV'S THEOREM

We next estimate the difference between the classical and quantum evolutions governed by our symbol $p(x, \xi) = |\xi|^2 + V(x)$.

NOTATION. (i) We write

$$e^{-\frac{itP(h)}{h}} \qquad (t \in \mathbb{R}) \tag{9.8}$$

for the unitary group on $L^2(M)$ generated by the self-adjoint operator $P(h)$.

Note that since $P(h) u_j(h) = E_j(h) u_j(h)$, we have

$$e^{-\frac{itP(h)}{h}} u_j(h) = e^{-\frac{itE_j}{h}} u_j(h) \qquad (t \in \mathbb{R}). \tag{9.9}$$

(ii) If A is a symbol in $\Psi^{-\infty}$, we also write

$$A_t := e^{\frac{itP(h)}{h}} A e^{-\frac{itP(h)}{h}} \qquad (t \in \mathbb{R}). \tag{9.10}$$

THEOREM 9.2 (Weak form of Egorov's Theorem). *Fix a time $T > 0$ and define for $0 \leq t \leq T$*

$$\tilde{A}_t := \mathrm{Op}(a_t), \tag{9.11}$$

where

$$a_t(x, \xi) := a(\Phi_t(x, \xi)). \tag{9.12}$$

Then

$$\|A_t - \tilde{A}_t\|_{L^2 \to L^2} = O(h) \quad \textit{uniformly for } 0 \leq t \leq T. \tag{9.13}$$

Proof. We have

$$\frac{d}{dt} a_t = \{p, a_t\}.$$

Recall from Appendix D that σ denotes the symbol of an operator. Then, since $\sigma\left(\frac{i}{h}[P(h), B]\right) = \{p, \sigma(B)\}$, it follows that

$$\frac{d}{dt} \tilde{A}_t = \frac{i}{h}[P(h), \tilde{A}_t] + E_t, \tag{9.14}$$

with an error term $\|E_t\|_{L^2 \to L^2} = O(h)$. Hence

$$\begin{aligned}
\frac{d}{dt} & \left(e^{-\frac{itP(h)}{h}} \tilde{A}_t e^{\frac{itP(h)}{h}} \right) \\
&= e^{-\frac{itP(h)}{h}} \left(\frac{d}{dt} \tilde{A}_t - \frac{i}{h}[P(h), \tilde{A}_t] \right) e^{\frac{itP(h)}{h}} \\
&= e^{-\frac{itP(h)}{h}} \left(\frac{i}{h}[P(h), \tilde{A}_t] + E_t - \frac{i}{h}[P(h), \tilde{A}_t] \right) e^{\frac{itP(h)}{h}} \\
&= e^{-\frac{itP(h)}{h}} E_t e^{\frac{itP(h)}{h}} = O(h).
\end{aligned}$$

Integrating, we deduce

$$\|e^{-\frac{itP(h)}{h}} \tilde{A}_t e^{\frac{itP(h)}{h}} - A\|_{L^2 \to L^2} = O(h);$$

and so

$$\|\tilde{A}_t - A_t\|_{L^2 \to L^2} = \|\tilde{A}_t - e^{\frac{itP(h)}{h}} A e^{-\frac{itP(h)}{h}}\|_{L^2 \to L^2} = O(h),$$

uniformly for $0 \leq t \leq T$. $\square$

9.3 WEYL'S THEOREM GENERALIZED

NOTATION. We hereafter consider the eigenvalue problems

$$P(h)u_j(h) = E_j(h)u_j(h) \qquad (j = 1, \dots).$$

To simplify notation a bit, we write $u_j = u_j(h)$ and $E_j = E_j(h)$. We assume as well the normalization

$$\|u_j\|_{L^2(M)} = 1. \tag{9.15}$$

The following result generalizes Theorem 6.9, showing that we can localize the asymptotics using a quantum observable:

THEOREM 9.3 (Weyl's Theorem generalized). *Let $B \in \Psi(M)$. Then*

$$(2\pi h)^n \sum_{a \le E_j \le b} \langle Bu_j, u_j \rangle \to \iint_{\{a \le p \le b\}} \sigma(B)\, dx d\xi. \tag{9.16}$$

REMARK. If $B = I$, whence $\sigma(B) \equiv 1$, (9.16) reads

$$(2\pi h)^n \#\{a \le E_j \le b\} \to \mathrm{Vol}(\{a \le p \le b\}).$$

This is the usual form of Weyl's Law, Theorem D.7. $\square$

Proof. 1. We first assume that $B \in \Psi^{-\infty}$; so that the operator $B : L^2(M) \to L^2(M)$ is of trace class. According to Lidskii's Theorem B.9 from Appendix B, we have

$$\mathrm{tr}(B) = \frac{1}{(2\pi h)^n} \left(\int_M \int_{\mathbb{R}^n} \sigma(B)\, dx d\xi + O(h) \right). \tag{9.17}$$

2. Fix a small munber $\epsilon > 0$ and write $\Omega_\epsilon := p^{-1}(a - \epsilon, a + \epsilon) \cup p^{-1}(b - \epsilon, b + \epsilon)$. Select $\psi_\epsilon \in C_c^\infty$, $\phi_\epsilon \in C^\infty$ so that

$$\begin{cases} WF_h(\psi_\epsilon) \subset \{a < p < b\} \\ WF_h(\phi_\epsilon) \subset \{p < a\} \cup \{p > b\} \\ WF_h(I - \phi_\epsilon + \psi_\epsilon) \subset \Omega_\epsilon. \end{cases}$$

Define

$$\Pi := \text{projection onto the span of } \{u_j \mid a \le E_j \le b\}.$$

We claim that

$$\begin{cases} \psi_\epsilon \Pi = \psi_\epsilon + O(h^\infty) \\ \phi_\epsilon \Pi = O(h^\infty). \end{cases} \tag{9.18}$$

The second assertion follows by an adaptation of the proof of Theorem D.7.

To establish the first part, we need to show that $\psi_\epsilon(I-\Pi) = O(h^\infty)$. We can find f satisfying the assumptions of Theorem 7.6 and such that $\psi_\epsilon(x,\xi)/f(p(x,\xi))$ is smooth. Using a symbolic construction we can find $T_\epsilon \in \Psi^{-\infty}$ with $WF_h(T_\epsilon) = WF_h(\psi_\epsilon)$, for which

$$\psi_\epsilon(I-\Pi) = T_\epsilon f(P)(I-\Pi) + O(h).$$

The first term on the right hand side can be rewritten as

$$\sum_{E_j(h)<a, E_j(h)>b} f(E_j(h)) T_\epsilon u_j \otimes \bar{u}_j.$$

The rough estimate (D.31) and the rapid decay of f show that for all M we have the bound

$$f(E_j(h)) \leq C_M (hj)^{-M}.$$

The proof of Theorem 6.4 shows also that $T_\epsilon u_j = O(h^\infty)$, uniformly in j. Hence

$$\|T_\epsilon f(P)(I-\Pi)\|_{L^2 \to L^2} = O(h^\infty).$$

3. We now write

$$\begin{aligned}\sum_{a \leq E_j \leq b} \langle B u_j, u_j \rangle &= \operatorname{tr}(\Pi B \Pi) \\ &= \operatorname{tr}(\Pi B(\psi_\epsilon + \phi_\epsilon + (1-\phi_\epsilon - \psi_\epsilon)\Pi).\end{aligned}$$

Using (9.18) we see that

$$(2\pi h)^n \operatorname{tr}(\Pi B \phi_\epsilon \Pi) = O(h).$$

The Weyl Law given in Theorem D.7 implies

$$(2\pi h)^n \operatorname{tr}(\Pi B(1-\phi_\epsilon - \psi_\epsilon)\Pi) = O(\epsilon),$$

since $1-\phi_\epsilon-\psi_\epsilon \neq 0$ only on Ω_ϵ. Furthermore,

$$\begin{aligned}&(2\pi h)^n \operatorname{tr}(\Pi B \psi_\epsilon \Pi) \\ &\qquad = (2\pi h)^n \operatorname{tr}(\Pi B \psi_\epsilon) + O(h^\infty) \\ &\qquad = (2\pi h)^n \operatorname{tr}(((\psi_\epsilon + \phi_\epsilon \\ &\qquad\qquad + (1-\phi_\epsilon-\psi_\epsilon)\Pi B \psi_\epsilon) + O(h^\infty) \\ &\qquad = (2\pi h)^n \operatorname{tr}(\psi_\epsilon B \psi_\epsilon) + O(h^\infty) + O(\epsilon).\end{aligned}$$

Combining these calculations gives

$$\begin{aligned}(2\pi h)^n \sum_{a\le E_j\le b} \langle Bu_j, u_j\rangle &= (2\pi h)^n \mathrm{tr}(\psi_\epsilon B \psi_\epsilon) + O(h) + O(\epsilon)\\ &= \iint \sigma(\psi_\epsilon)^2 \sigma(B)\, dx d\xi + O(h) + O(\epsilon)\\ &\to \iint_{\{a\le p\le b\}} \sigma(B)\, dx d\xi\end{aligned}$$

as $h \to 0, \epsilon \to 0$.

4. Finally, to pass from $B \in \Psi^{-\infty}$ to an arbitrary $B \in \Psi$, we decompose the latter as

$$B = B_0 + B_1,$$

with $B_0 \in \Psi^{-\infty}$ and

$$\begin{aligned}&WF_h(B_0) \subset \{a-2 < p < b+2\},\\ &WF_h(B_1) \cap \{a-1 < p < b+1\} = \emptyset.\end{aligned}$$

We have $B_1 u_j = O(h^\infty)$ for $a \le E_j(h) \le b$; and hence only the B_0 part contributes to the limit. □

9.4 A QUANTUM ERGODIC THEOREM

Assume now that $A \in \Psi(M)$ has the symbol $\sigma(A)$ with the property that

$$\alpha := ⨍_{\Sigma_c} \sigma(A)\, d\mu \ \text{ is the same for all } c \in [a,b], \tag{9.19}$$

where the slash through the integral denotes the average. In other words, we are requiring that the averages of the symbol of A over each level surface $p^{-1}(c)$ are equal.

THEOREM 9.4 (Quantum ergodicity). *Assume the ergodic condition* (9.5) *and that $A \in \Psi(M)$ satisfies the condition* (9.19).

(i) *Then*

$$(2\pi h)^n \sum_{a\le E_j\le b} \left| \langle Au_j, u_j\rangle - ⨍_{\{a\le p\le b\}} \sigma(A)\, dx d\xi \right|^2 \longrightarrow 0. \tag{9.20}$$

(ii) *In addition, there exists a family of subsets $\Lambda(h) \subseteq \{a \le E_j \le b\}$ such that*

$$\lim_{h\to 0} \frac{\#\Lambda(h)}{\#\{a \le E_j \le b\}} = 1; \tag{9.21}$$

and for each $A \in \Psi(M)$ satisfying (9.19), *we have*

$$\langle Au_j, u_j \rangle \to \fint_{\{a \le p \le b\}} \sigma(A)\, dx d\xi \quad \textit{as } h \to 0 \tag{9.22}$$

for $E_j \in \Lambda(h)$.

Proof. 1. We first show that assertion (i) implies (ii). For this, let

$$B := A - \alpha I, \tag{9.23}$$

α defined by (9.19). Then $\int_{\{a \le p \le b\}} \sigma(B)\, dx d\xi = 0$. According to (9.20),

$$(2\pi h)^n \sum_{a \le E_j \le b} \langle Bu_j, u_j \rangle^2 =: \epsilon(h) \to 0.$$

Define

$$\Gamma(h) := \{a \le E_j \le b \mid \langle Bu_j, u_j \rangle^2 \ge \epsilon^{1/2}(h)\};$$

so that

$$(2\pi h)^n \#\Gamma(h) \le \epsilon(h)^{1/2}.$$

Next, write

$$\Lambda(h) := \{a \le E_j \le b\} - \Gamma(h).$$

Then if $E_j \in \Lambda(h)$,

$$|\langle Bu_j, u_j \rangle| \le \epsilon^{1/4}(h);$$

and so

$$|\langle Au_j, u_j \rangle - \alpha| \le \epsilon^{1/4}(h).$$

Also,

$$\frac{\#\Lambda(h)}{\#\{a \le E_j \le b\}} = 1 - \frac{\#\Gamma(h)}{\#\{a \le E_j < b\}}.$$

But according to Weyl's law,

$$\frac{\#\Gamma(h)}{\#\{a \le E_j \le b\}} = \frac{(2\pi h)^n \#\Gamma(h)}{\mathrm{Vol}(\{a \le p \le b\}) + o(1)} \le C\epsilon(h)^{1/2} \to 0.$$

This proves (ii).

2. Next we establish assertion (i). Let B be again given by (9.23); so that in view of our hypothesis (9.19)

$$\int_{\Sigma_c} \sigma(B)\, d\mu = 0 \quad \text{for each } c \in [a, b]. \tag{9.24}$$

Define

$$\epsilon(h) := (2\pi h)^n \sum_{a \le E_j \le b} \langle Bu_j, u_j \rangle^2;$$

we must show $\epsilon(h) \to 0$.

Now

$$\langle Bu_j, u_j\rangle = \langle Be^{-\frac{itE_j}{h}}u_j, e^{-\frac{itE_j}{h}}u_j\rangle = \langle Be^{-\frac{itP(h)}{h}}u_j, e^{-\frac{itP(h)}{h}}u_j\rangle$$

according to (9.9). Consequently

$$\langle Bu_j, u_j\rangle = \langle e^{\frac{itP(h)}{h}}Be^{-\frac{itP(h)}{h}}u_j, u_j\rangle = \langle B_t u_j, u_j\rangle \tag{9.25}$$

in the notation of (9.10). This identity is valid for each time $t \in \mathbb{R}$. We can therefore average:

$$\langle Bu_j, u_j\rangle = \langle \fint_0^T B_t\, dt\, u_j, u_j\rangle = \langle\langle B\rangle_T u_j, u_j\rangle, \tag{9.26}$$

for

$$\langle B\rangle_T := \frac{1}{T}\int_0^T B_t\, dt = \fint_0^T B_t\, dt.$$

Now since $\|u_j\|^2 = 1$, (9.26) implies

$$\langle Bu_j, u_j\rangle^2 = \langle\langle B\rangle_T u_j, u_j\rangle^2 \le \|\langle B\rangle_T u_j\|^2 = \langle\langle B^*\rangle_T\langle B\rangle_T u_j, u_j\rangle.$$

Hence

$$\epsilon(h) \le (2\pi h)^n \sum_{a\le E_j\le b}\langle\langle B^*\rangle_T\langle B\rangle_T u_j, u_j\rangle \tag{9.27}$$

3. Theorem 9.2 tells us that

$$\langle B\rangle_T = \langle\tilde{B}\rangle_T + O_T(h), \quad \langle\tilde{B}\rangle_T := \fint_0^T \tilde{B}_t\, dt,$$

where $\tilde{B}_t \in \Psi(M)$ and $\sigma(\tilde{B}_t) = \Phi_t^*\sigma(B)$. Hence

$$\sigma(\langle\tilde{B}\rangle_T) = \fint_0^T \sigma(B)\circ\Phi_t\, dt + O_T(h) = \langle\sigma(B)\rangle_T + O_T(h).$$

as $h \to 0$.

Since modulo $O(h)$ errors we can replace $e^{itP(h)/h}Be^{-itP(h)/h}$ by $\tilde{B}_t$, Theorem 9.1 shows that

$$\begin{aligned}\limsup_{h\to 0}\epsilon(h) &\le \iint_{\{a\le p\le b\}}\sigma(\langle(\tilde{B}^*\rangle_T\langle\tilde{B}\rangle_T)\, dxd\xi \\ &= \iint_{\{a\le p\le b\}}|\sigma(\langle B\rangle_T))|^2\, dxd\xi,\end{aligned} \tag{9.28}$$

as the symbol map is multiplicative and the symbol of an adjoint is given by the complex conjugate.

4. We can now apply Theorem 9.3 with $a = \sigma(B)$, to conclude that

$$\int_{p^{-1}[a,b]}|\langle\sigma(B)\rangle_T|^2 dxd\xi \to 0,$$

as $T \to \infty$. Since the left hand side of (9.28) is independent of T, this calculation shows that the limit must in fact be zero. $\square$

APPLICATION. The simplest and most striking application concerns the complete set of eigenfuctions of the Laplacian on a compact Riemannian manifold:

$$-\Delta_g u_j = \lambda_j u_j \qquad (j = 1, \dots),$$

normalized so that

$$\|u_j\|_{L^2(M)} = 1.$$

THEOREM 9.5 (Equidistribution of eigenfunctions). *There exists a sequence $j_k \to \infty$ of density one,*

$$\lim_{m\to\infty} \frac{\#\{k \ : \ j_k \leq m\}}{m} = 1,$$

such that for any $f \in C^\infty(M)$,

$$\int_M |u_{j_k}|^2 f \, d\mathrm{vol}_g \to \int_M f \, d\mathrm{vol}_g. \tag{9.29}$$

10. Quantizing symplectic transformations

This final chapter presents some more advanced topics, mostly concerning how (and why) to quantize symplectic transformations.

10.1 DEFORMATION AND QUANTIZATION

Throughout this chapter, we identify $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$. In this section $\boldsymbol{\kappa} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ denotes a symplectomorphism:

$$\boldsymbol{\kappa}^*\sigma = \sigma \quad \text{ for } \sigma = \sum_{j=1}^{n} d\xi_j \wedge dx_j,$$

normalized so that $\boldsymbol{\kappa}(0,0) = (0,0)$. Our goal is to *quantize* $\boldsymbol{\kappa}$ *locally*, meaning to find a unitary operator $F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that

$$F^{-1}AF = B \qquad \text{near } (0,0)$$

for $A = \mathrm{Op}(a)$, where $a \in S$ and $B = \mathrm{Op}(b)$ for

$$b = \boldsymbol{\kappa}^* a + O(h).$$

This can be useful in practice, since sometimes we can design $\boldsymbol{\kappa}$ so that $\boldsymbol{\kappa}^* a$ is more tractable than a.

The basic strategy will be (i) finding a family $\{\boldsymbol{\kappa}_t\}_{0 \le t \le 1}$ of symplectomorphisms so that $\boldsymbol{\kappa}_0 = I$ and $\boldsymbol{\kappa}_1 = \boldsymbol{\kappa}$; (ii) quantizing the functions q_t generating this flow of mappings; and then (iii) solving an associated operator ODE (10.7).

10.1.1 Deformations. We begin by deforming $\boldsymbol{\kappa}$ to the identity mapping. So assume U_0 and U_1 are simply connected neighborhoods of $(0,0)$ and $\boldsymbol{\kappa} : U_0 \to U_1$ is a symplectomorphism such that $\boldsymbol{\kappa}(0,0) = (0,0)$.

THEOREM 10.1 (Deforming symplectomorphisms). *There exists a continuous, piecewise smooth family*

$$\{\boldsymbol{\kappa}_t\}_{0\le t\le 1}$$

of local symplectomorphisms $\boldsymbol{\kappa}_t : U_0 \to U_t =: \boldsymbol{\kappa}_t(U_0)$ *such that*

(i) $\boldsymbol{\kappa}_t(0,0) = 0 \quad (0 \le t \le 1)$

(ii) $\boldsymbol{\kappa}_1 = \boldsymbol{\kappa}$, $\boldsymbol{\kappa}_0 = I$.

(iii) *Also,*

$$\frac{d}{dt}\boldsymbol{\kappa}_t = (\boldsymbol{\kappa}_t)_* H_{q_t} \qquad (0 \le t \le 1) \tag{10.1}$$

for a smooth family of functions $\{q_t\}_{0\le t\le 1}$.

REMARK. The statement (10.1) means that for each function $a \in C^\infty(U_1)$, we have

$$\frac{d}{dt}\boldsymbol{\kappa}_t^* a = H_{q_t}\boldsymbol{\kappa}_t^* a. \tag{10.2}$$

In fact,

$$\frac{d}{dt}\boldsymbol{\kappa}_t^* a = \langle da, d\boldsymbol{\kappa}_t/dt\rangle = \langle da, (\boldsymbol{\kappa}_t)_* H_{q_t}\rangle = H_{q_t}\boldsymbol{\kappa}_t^* a\,,$$

where $\langle\cdot,\cdot\rangle$ is the pairing of differential 1-forms and vectorfields on U_t. □

Proof. 1. We first consider the case that $\boldsymbol{\kappa}$ is given by a linear symplectomorphism $K : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$:

$$K^* J K = J \tag{10.3}$$

for

$$J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Since K is an invertible matrix, we have the unique polar decomposition

$$K = QP,$$

where Q is orthogonal and P is positive definite. From (10.3) we deduce that

$$Q^{*-1}P^{*-1} = K^{*-1} = JQJ^{-1}JPJ^{-1};$$

whence the uniqueness of Q and P implies

$$Q^{*-1} = JQJ^{-1}, \quad P^{*-1} = JPJ^{-1}.$$

That is, both Q and P are symplectic. Furthermore, we can write

$$P = \exp A,$$

where $A = A^*$ and $JA + AJ = 0$.

2. We identify $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ with $\mathbb{C}^n$, under the relation $(x, y) \leftrightarrow x + iy$. Since

$$\langle x + iy, x' + iy' \rangle_{\mathbb{C}^n} = \langle (x, y), (x', y') \rangle_{\mathbb{R}^n} + i\sigma((x, y), (x', y')),$$

the fact that Q is orthogonal and symplectic implies it is unitary:

$$Q = Q^{*-1} = -JQJ.$$

(Similarly, any unitary transformation on $\mathbb{C}^n$ gives an orthogonal symplectic transformation in $\mathbb{R}^n \times \mathbb{R}^n$.)

We can now write

$$Q = \exp iB,$$

where $B^* = B$ is Hermitian on $\mathbb{C}^n$. A smooth deformation to the identity is now clear:

$$K_t := \exp(itB) \exp(tA) \qquad (0 \leq t \leq 1).$$

3. For the general case that $\boldsymbol{\kappa}$ is nonlinear, set $K := \partial\boldsymbol{\kappa}(0, 0)$. Then for $1/2 \leq t \leq 1$,

$$\boldsymbol{\kappa}_t := K_{2-2t}^{-1} \circ \boldsymbol{\kappa}$$

is a piecewise smooth family of symplectomorphisms satisfying

$$\boldsymbol{\kappa}_1 = \boldsymbol{\kappa}, \quad \partial\boldsymbol{\kappa}_{1/2}(0, 0) = I.$$

For $0 \leq t \leq 1/2$, we set

$$\boldsymbol{\kappa}_t(m) := \frac{1}{2t}\boldsymbol{\kappa}_{1/2+t}(2tm).$$

4. Define $V_t := \frac{d}{dt}\boldsymbol{\kappa}_t$; we must show

$$V_t = (\boldsymbol{\kappa}_t)_* H_{q_t}$$

for some function q_t. According to Cartan's formula (Theorem C.2):

$$\mathcal{L}_{V_t}\sigma = d\sigma \lrcorner V_t + d(\sigma \lrcorner V_t).$$

But $\mathcal{L}_{V_t}\sigma = \frac{d}{dt}\boldsymbol{\kappa}_t^*\sigma = \frac{d}{dt}\sigma = 0$, since $\boldsymbol{\kappa}_t^*\sigma = \sigma$. Furthermore, $d\sigma = 0$, and consequently $d(\sigma \lrcorner V_t) = 0$. Owing to Poincaré's Lemma (Theorem C.3), we have

$$\boldsymbol{\kappa}_t^*(\sigma \lrcorner V_t) = dq_t$$

for a function q_t; and this means that $V_t = (\boldsymbol{\kappa}_t)_* H_{q_t}$. $\square$

To define our symbol classes, we hereafter consider the order function

$$m := (1 + |x|^2 + |\xi|^2)^{\frac{k}{2}}$$

for some positive integer k.

THEOREM 10.2 (Quantizing families of symplectomorphisms). *Let* $\{\boldsymbol{\kappa}_t\}_{0\le t\le 1}$ *be a smooth family of symplectomorphisms of* $\mathbb{R}^{2n}$*, such that*

$$\boldsymbol{\kappa}_0 = I, \quad \frac{d}{dt}\boldsymbol{\kappa}_t = (\boldsymbol{\kappa}_t)_* H_{q_t},$$

where $q_t \in S(m)$ *is a smooth family of real valued symbols.*

Then there exists a family of unitary operators

$$F(t) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

such that

$$F(0) = I,$$

and for all $A = \mathrm{Op}(a)$ *with* $a \in \mathcal{S}$*, we have*

$$F(t)^{-1} \circ A \circ F(t) = B(t) \qquad (0 \le t \le 1) \tag{10.4}$$

for

$$B(t) = \mathrm{Op}(b_t), \tag{10.5}$$

where

$$b_t = \boldsymbol{\kappa}_t^* a + hc_t \tag{10.6}$$

for $c_t \in \mathcal{S} \cap S$*.*

Proof. 1. We define

$$Q(t) := \mathrm{Op}(q_t) : \mathcal{S} \to \mathcal{S} \qquad (0 \le t \le 1),$$

and recall that

$$Q(t)^* = Q(t).$$

Since $Q(t)$ depends smoothly on t as an operator on $\mathcal{S}$, we can solve the operator ODE

$$\begin{cases} hD_tF(t) + F(t)Q(t) = 0 \qquad (0 \le t \le 1) \\ F(0) = I \end{cases} \tag{10.7}$$

for $F(t) : \mathcal{S} \to \mathcal{S}$. Then

$$\begin{cases} hD_tF(t)^* - Q(t)F(t)^* = 0 \qquad (0 \le t \le 1) \\ F(0)^* = I. \end{cases} \tag{10.8}$$

2. We claim that

$$F(t) \text{ is unitary on } L^2(\mathbb{R}^n).$$

To confirm this, let us calculate using (10.7) and (10.8):

$$\begin{aligned} hD_t(F(t)F(t)^*) &= hD_tF(t)F(t)^* + F(t)hD_tF(t)^* \\ &= -F(t)Q(t)F(t)^* + F(t)Q(t)F(t)^* = 0. \end{aligned}$$

Hence $F(t)F(t)^* \equiv I$. On the other hand,

$$\begin{aligned} hD_t(F(t)^*F(t) - I) &= Q(t)F(t)^*F(t) - F(t)^*F(t)Q(t) \\ &= [Q(t), F(t)^*F(t) - I]. \end{aligned}$$

with $F(0)^*F(0) - I = 0$. Since this equation for $F(t)^*F(t) - I$ is homogeneous, it follows that $F(t)^*F(t) \equiv I$.

3. Now define

$$B(t) := F(t)^{-1}AF(t). \tag{10.9}$$

We assert that

$$B(t) = \mathrm{Op}(b_t) \tag{10.10}$$

for

$$b_t = \boldsymbol{\kappa}_t^* a + O(h)_{L^2 \to L^2}. \tag{10.11}$$

To prove this, define the family of pseudodifferential operators

$$\tilde{B}(t) := \mathrm{Op}(\boldsymbol{\kappa}_t^* a).$$

We calculate

$$\begin{aligned} hD_t\tilde{B}(t) &= \frac{h}{i}\mathrm{Op}\left(\frac{d}{dt}\boldsymbol{\kappa}_t^* a\right) = \frac{h}{i}\mathrm{Op}(H_{q_t}\boldsymbol{\kappa}_t^* a) \\ &= \frac{h}{i}\mathrm{Op}(\{q_t, \boldsymbol{\kappa}_t^* a\}) = [Q(t), \tilde{B}(t)] + E(t), \end{aligned}$$

and the pseudodifferential calculus implies that

$$\|E(t)\|_{L^2 \to L^2} = O(h^2)$$

where $E(t) = \mathrm{Op}(e_t)$ for a symbol $e_t \in S^{-2}$.

Therefore

$$\begin{aligned} hD_t(F(t)\tilde{B}(t)F(t)^{-1}) &= (hD_tF(t))\tilde{B}(t)F(t)^{-1} + F(t)(hD_t\tilde{B}(t))F(t)^{-1} \\ &\quad + F(t)\tilde{B}(t)hD_t(F(t)^{-1}) \\ &= -F(t)Q(t)\tilde{B}(t)F(t)^{-1} + F(t)([Q(t), \tilde{B}(t)] \\ &\quad + E(t))F(t)^{-1} + F(t)\tilde{B}(t)Q(t)F(t)^{-1} \\ &= F(t)E(t)F(t)^{-1} = O(h^2). \end{aligned}$$

Integrating and dividing by h, we deduce

$$F(t)\tilde{B}(t)F(t)^{-1} = F(0)\tilde{B}(0)F(0)^{-1} + O(h) = A + O(h);$$

and so

$$\tilde{B}(t) = F(t)^{-1}AF(t) + O(h).$$

Hence

$$\|\tilde{B}(t) - B(t)\|_{L^2 \to L^2} = O(h). \tag{10.12}$$

4. We now look at the remainder $E(t) = \mathrm{Op}(e_t)$, the symbol e_t belonging to S^{-2}. Introduce

$$E_1(t) := \mathrm{Op}((\boldsymbol{\kappa}_t^{-1})^* e_t).$$

Noting that $hD_t e_t \in S^{-3}(1)$, we see by the same argument as in Step 3 above that

$$F(t)E(t)F(t)^{-1} = E_1(t) + O(h^3).$$

Since

$$B(t) = \tilde{B}(t) + F(t)^{-1}\left(\frac{i}{h}\int_0^t E_1(s)\,ds\right)F(t) + O(h^2),$$

we can apply the same argument again, to obtain

$$B(t) = \tilde{B}(t) + hB_1(t) + O(h^2),$$

where $B_1(t) = \mathrm{Op}(b_t^1)$ for $b_t^1 \in S$.

Iterating this procedure, we deduce that

$$B(t) = \tilde{B}(t) + hB_1(t) + \cdots h^N B_N(t) + O(h^{N+1})_{L^2 \to L^2},$$

where $B_k(t) = \mathrm{Op}(b_t^k)$, $b_t^k \in S$.

5. It remains to show that $\tilde{B}(t) - B(t)$, and hence $B(t)$ is a pseudodifferential operator. To do so, we invoke Beals's Theorem 8.9 (and the Remark after it), by showing that for any choices $a_1, \cdots, a_N \in S$, we have the estimate

$$\mathrm{ad}_{a_N} \cdots \mathrm{ad}_{a_1}(\tilde{B}(t) - B(t)) = O(h^{N+1}). \tag{10.13}$$

But this statement is clear from Step 3: for any N we can find a pseudodifferential operator $\mathrm{Op}(b_t^N)$, with $b_t^N \in S^{-1}$, such that

$$\tilde{B}(t) - B(t) = \mathrm{Op}(b_t^N) + O(h^{N+1}).$$

Since

$$\mathrm{ad}_{a_N} \cdots \mathrm{ad}_{a_1}\mathrm{Op}(b_t^N) = O(h^{N+1}),$$

and each a_j is bounded on L^2, estimate (10.13) follows. $\square$

REMARK. The argument used in Step 2 of the proof shows that if in Theorem 10.2 we have

$$a(x,\xi;h) \sim a_0(x,\xi) + ha_1(x,\xi) + \cdots + h^N a_N(x,\xi) + \cdots,$$

for $a_j \in S$, then

$$b_t(x,\xi;h) \sim \boldsymbol{\kappa}_t^* a_0(x,\xi) + hb_t^1(x,\xi) + \cdots h^N b_t^N(x,\xi) + \cdots.$$

However, the higher order terms are difficult to compute. □

10.1.2 Locally defined symplectomorphisms. The requirement that the family of symplectomorphism be global on $\mathbb{R}^{2n}$ is very strong and often invalid in interesting situations. So we now discuss quantization of *locally defined symplectomorphisms*, for which the quantization formula (10.4) holds only locally.

THEOREM 10.3 (Local quantization). *Let $\boldsymbol{\kappa} : U_0 \to U_1$ be a symplectomorphism fixing $(0,0)$ and defined in a neighbourhood of U_0.*

Then there exists a unitary operator

$$F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

such that for all $A = \mathrm{Op}(a)$ with $a \in S$, we have

$$F^{-1} A F = B, \tag{10.14}$$

where $B = \mathrm{Op}(b)$ for a symbol $b \in S$ satisfying

$$b|_{U_0} := \boldsymbol{\kappa}^*(a|_{U_1})|_{U_0} + O(h). \tag{10.15}$$

Proof. 1. According to Theorem 10.1, there exists a piecewise smooth family of symplectomorphisms $\boldsymbol{\kappa}_t : U_0 \to U_t$, $(0 \leq t \leq 1)$ such that $\boldsymbol{\kappa} = \boldsymbol{\kappa}_1$, $\boldsymbol{\kappa}_0 = I$, and

$$\frac{d}{dt}\boldsymbol{\kappa}_t = (\boldsymbol{\kappa}_t)_* H_{q_t} \qquad (0 \leq t \leq 1)$$

within U, for a smooth family $\{q_t\}_{0 \leq t \leq 1}$.

We extend q_t smoothly to be equal to 0 in $\mathbb{R}^{2n} - U_0$ and then define a family of global symplectomorphisms $\tilde{\boldsymbol{\kappa}}_t$ using the now globally defined functions q_t. Observe that

$$\tilde{\boldsymbol{\kappa}}_t|_{U_0} = \boldsymbol{\kappa}_t : U_0 \to U_t;$$

and hence

$$\tilde{\boldsymbol{\kappa}}_t^*(a)|_{U_0} = \boldsymbol{\kappa}_t^*(a|_{U_t})|_{U_0}. \tag{10.16}$$

2. We now apply Theorem 10.2, to obtain the family of operators $\{F(t)\}_{0 \leq t \leq 1}$. We observe that since the supports of the functions q_t lie in a fixed compact set, the proof of Theorem 10.2 shows that (10.4) holds for $a \in S$. That is,

$$F(t)^{-1} A F(t) = \mathrm{Op}(b(t)) = B(t)$$

for

$$b(t) = \tilde{\boldsymbol{\kappa}}_t^* a + O(h).$$

We now put

$$F := F(1), \quad B := B(1).$$

Then (10.16) shows that formula (10.14) is valid. □

10.1.3 Microlocality. It will prove useful to formulate the theorems above without reference to the global properties of the operator F.

DEFINITIONS. (i) Let U, V be open, bounded subsets of $\mathbb{R}^{2n}$, and assume

$$T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

is linear.

We say that T is *tempered* if for each seminorm $\|\cdot\|_1$ on $\mathcal{S}(\mathbb{R}^n)$, there exists another seminorm $\|\cdot\|_2$ and a constant $N \in \mathbb{R}$ such that

$$\|Tu\|_1 = O(h^{-N})\|u\|_2 \tag{10.17}$$

for all $u \in \mathcal{S}$.

(ii) Given two tempered operators T and S, we say that

$$T \equiv S \quad \textit{microlocally} \ \text{ on } \quad U \times V \tag{10.18}$$

if there exist open sets $\tilde{U} \supseteq U$ and $\tilde{V} \supseteq V$ such that

$$A(T - S)B = O(h^\infty)$$

as a mapping $\mathcal{S} \to \mathcal{S}$, for all A, B such that

$$WF_h(A) \subset \tilde{V}, \ WF_h(B) \subset \tilde{U}.$$

(iii) In particular, we say

$$T \equiv I \quad \textit{microlocally} \ \text{ near } U \times U$$

if there exists an open set $\tilde{U} \supseteq U$ such that

$$A - TA = A - AT = O(h^\infty)$$

as mappings $\mathcal{S} \to \mathcal{S}$, for all A with $WF_h(A) \subset \tilde{U}$.

(iv) We will say that T is *microlocally invertible* near $U \times U$ if there exists an operator S such that $TS \equiv I$ and $ST \equiv I$ microlocally near $U \times U$.

When no confusion is likely, we write

$$S = T^{-1}$$

and call S a *microlocal inverse* of T.

LEMMA 10.4 (Wavefront sets and composition). *If*

$$WF_h(A) \cap U = \emptyset$$

and $B = \mathrm{Op}(b)$ *for* $b \in S$*, then*

$$WF_h(BA) \cap U = \emptyset. \tag{10.19}$$

Proof. The symbol of BA is $b\#a = O(h^\infty)$ in U. □

LEMMA 10.5 (Tempered unitary transformations). *The unitary transformations* $F(t)$ *given by Theorem 10.2 are tempered.*

Proof. Up to powers of h, each seminorm on $\mathcal{S}$ is bounded from above and below by these specific seminorms:

$$u \mapsto \|A_N u\| \quad \text{for} \quad A_N := (1 + |x|^2 + |hD|^2)^N.$$

We observe that the operators A_N are invertible and selfadjoint and that, in the notation of the proof of Theorem 10.2,

$$A_N Q(t) A_N^{-1} = Q_N(t) = \mathrm{Op}(q_t^N),$$

for $q_t^N \in S(m)$ such that $q_t^N - q_t \in S^{-1}(m)$.

We then have

$$hD_t A_N F(t) A_N^{-1} = A_N F(t) A_N^{-1} Q_N(t);$$

and hence the same arguments as before show that

$$\|A_N F(t) u\|^2 = \|A_N u\|^2.$$

Consequently for any seminorm $\|\cdot\|_1$ on $\mathcal{S}$, there exists a seminorm $\|\cdot\|_2$ and N such that

$$\|F(t)u\|_1 \leq O(h^{-N}) \|u\|_2.$$

□

The previous two lemmas and Theorem 10.3 give

THEOREM 10.6 (More on local quantization). *Let* $\boldsymbol{\kappa} : U_0 \to U_1$ *be a symplectomorphism fixing* $(0,0)$ *and defined in a neighbourhood of* U_0*. Suppose* U *is open,* $\overline{U} \subset\subset U_0 \cap U_1$*.*

Then there exists a tempered operator

$$F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

such that F *is microlocally invertible near* $U \times U$ *and for all* $A = \mathrm{Op}(a)$*, with* $a \in S$*,*

$$F^{-1} A F = B \qquad \text{microlocally near } U \times U, \tag{10.20}$$

where $B = \mathrm{Op}(b)$ *for a symbol* $b \in S$ *satisfying*

$$b := \boldsymbol{\kappa}^* a + O(h). \tag{10.21}$$

In (10.21) we do not specify the neighbourhoods, as we did in (10.16), since the statement needs to make sense only locally near $U \times U$.

The last theorem has the following converse which we include for completeness:

THEOREM 10.7 (Converse). *Suppose that* $F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *is a tempered operator such that for every* $A = \mathrm{Op}(a)$ *with* $a \in S$, *we have*

$$AF \equiv FB$$

microlocally near $(0,0)$, *for*

$$B = \mathrm{Op}(b), \quad b = \boldsymbol{\kappa}^* a + O(h),$$

where $\boldsymbol{\kappa} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ *is a symplectomorphism, defined locally near* U, *with* $\boldsymbol{\kappa}(0,0) = (0,0)$.

Then there exists a pseudodifferential operator F_0, *elliptic near* U, *and a family of self-adjoint pseudodifferential operators* $Q(t)$, *such that*

$$F = F(1) \quad \text{microlocally near } U \times U,$$

where

$$\begin{cases} hD_t F(t) + F(t)Q(t) = 0 & (0 \le t \le 1) \\ F(0) = F_0. \end{cases}$$

Proof. 1. From Theorem 10.1 we know that there exists a family of local symplectomorphisms, $\boldsymbol{\kappa}_t$, satisfying $\boldsymbol{\kappa}_t(0,0) = (0,0)$, $\boldsymbol{\kappa}_1 = \boldsymbol{\kappa}$ and $\boldsymbol{\kappa}_0 = I$. Since we are working locally, there exists a function q_t so that $\boldsymbol{\kappa}_t$ is generated by its Hamiltonian vectorfield H_{q_t}.

As in the proof of Theorem 10.3 we extend this function to be zero outside a compact set. Let us now consider the dynamics

$$\begin{cases} hD_t F(t) = Q(t)F(t) & (0 \le t \le 1) \\ F(1) = CFC, \end{cases}$$

where C is a pseudodifferential operator with $WF_h(I - C) \cap U = \emptyset$.

2. We claim that $F(0)$ satisfies

$$\mathrm{Op}(a)F(0) = F(0)\mathrm{Op}(a + h\tilde{a}) \tag{10.22}$$

for $a, \tilde{a} \in \mathcal{S} \cap S^0(1)$. To establish this, let us introduce $V(t)$ satisfying

$$\begin{cases} hD_tV(t) + V(t)Q(t) = 0 \qquad (0 \le t \le 1) \\ \qquad\qquad\qquad V(0) = I. \end{cases}$$

Then using Theorem 10.3 and the assumption that $b = \boldsymbol{\kappa}^* a + 0(h)$, we deduce that

$$\begin{aligned} \mathrm{Op}(a)F(t)V(t) &= F(t)\mathrm{Op}(b)V(t) \\ &= F(t)V(t)(V(t)^{-1}\mathrm{Op}(b)V(t)) \\ &= F(t)V(t)\mathrm{Op}(a + h\tilde{a}). \end{aligned}$$

Putting $t = 0$ gives (10.22).

3. We now use Beals's Theorem to conclude that $F(0) \in \Psi^0$. We verify the hypothesis by induction: suppose we know that

$$\mathrm{ad}_{\mathrm{Op}(b_1)} \cdots ad_{\mathrm{Op}(b_N)} F(0) = O(h^N),$$

for any $b_j \in S^0(1)$. Then by (10.22)

$$\mathrm{Op}(b_{N+1})F(0) - F(0)\mathrm{Op}(b_{N+1}) = h\mathrm{Op}(\tilde{b}_{N+1})F(0);$$

and hence

$$\begin{aligned} &\|\mathrm{ad}_{\mathrm{Op}(b_1)} \cdots ad_{\mathrm{Op}(b_N)} \mathrm{ad}_{\mathrm{Op}(b_{N+1})} F(0)\|_{L^2 \to L^2} = \\ &\qquad \|h\mathrm{ad}_{\mathrm{Op}(b_1)} \cdots \mathrm{ad}_{\mathrm{Op}(b_N)}(\mathrm{Op}(\tilde{b}_{N+1})F(0))\|_{L^2 \to L^2} = O(h^{N+1}), \end{aligned}$$

according to the induction hypothesis and the derivation property

$$\mathrm{ad}_A(BC) = B(\mathrm{ad}_A C) + (\mathrm{ad}_A B)C.$$

Hence Beals's Theorem applies and shows that $F(0)$ is a pseudodifferential operator. By construction, $F(1) = CFC \equiv F$ near $(0,0)$. $\square$

10.1.4. Quantization of linear symplectic maps.

Consider first the simple linear symplectic transformation $\boldsymbol{\kappa} = J$; that is,

$$\boldsymbol{\kappa}(x, \xi) = (-\xi, x) \tag{10.23}$$

on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$.

Then we can take for $0 \le t \le 1$,

$$\boldsymbol{\kappa}_t(x, \xi) = \left(\cos\left(\frac{t\pi}{2}\right)x - \sin\left(\frac{t\pi}{2}\right)\xi, \sin\left(\frac{t\pi}{2}\right)x + \cos\left(\frac{t\pi}{2}\right)\xi\right);$$

so that

$$\frac{d\boldsymbol{\kappa}_t}{dt} = (\boldsymbol{\kappa}_t)_* H_q,$$

for

$$q := \frac{\pi}{4}(|x|^2 + |\xi|^2).$$

THEOREM 10.8 (J quantized). *The operator F associated with the transformation* (10.23) *as in Theorem 10.3 is*

$$Fu(x) := \frac{e^{-\frac{\pi}{4}i}}{(2\pi h)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{i\langle x,y\rangle}{h}}u(y)\,dy = \frac{e^{-\frac{\pi}{4}i}}{(2\pi h)^{\frac{n}{2}}}\mathcal{F}_h u. \tag{10.24}$$

Proof. 1. To verify this, we first show that for $a \in \mathcal{S}'$ we have

$$a^w(x, hD) \circ F = F \circ a^w(-hD, x); \tag{10.25}$$

that is, the conclusion of Theorem 10.2 holds without any error terms. As in the proof of that theorem, we see that

$$hD_t A_t = \frac{\pi}{4}[-h^2\Delta + |x|^2, A_t]$$

for

$$A_t := F(t)^{-1}a^w(x, hD)F(t).$$

Let $l(x,\xi)$ be a linear function on $\mathbb{R}^{2n}$ and consider the exponential symbol

$$a_t(x,\xi) := \exp(\boldsymbol{\kappa}_t^* l(x,\xi)/h)$$

and its Weyl quantization

$$a_t^w(x, hD) = \exp(\boldsymbol{\kappa}_t^* l(x, hD)/h).$$

An explicit computation reveals that

$$hD_t a_t(x, hD) = \frac{\pi}{4}[-h^2\Delta + |x|^2, a_t(x, hD)].$$

Since any Weyl operator is a superposition of exponentials of l's (recall (4.16)), assertion (10.25) follows.

2. Suppose now that $\widetilde{F}$ is another unitary operator for which (10.25) holds. Then $\tilde{F} = cF$ for $c \in \mathbb{C}$, $|c| = 1$, as follows from applying Lemma 3.3 to $L = F^*\widetilde{F}$. Since the Fourier transform satisfies (10.25) and $(2\pi h)^{-n/2}\mathcal{F}_h$ is unitary, we deduce that

$$F = \frac{c}{(2\pi h)^{\frac{n}{2}}}\mathcal{F}_h.$$

3. Thus it remains to compute the constant c. For this, let us put $u_0 = \exp(-|x|^2/2)$ and consider the ODE

$$\begin{cases} hD_t u(t) = \frac{\pi}{4}(-h^2\Delta + |x|^2)u(t), \\ \qquad u(0) = u_0. \end{cases}$$

Recalling (10.7), we see that $u(t) = F(t)^* u_0$. Since u_0 is the ground state of the harmonic oscillator with eigenvalue h, we learn that $u(t) = a(t)u_0$, where $a(t)$ solves the ODE

$$\begin{cases} \frac{d}{dt}a(t) = \frac{\pi i}{4}a(t) \\ \quad a(0) = 1; \end{cases}$$

that is, $a(t) = \exp(\pi i t/4)$. Finally, we note that

$$e^{\pi i/4}u_0 = F(1)^* u_0 = \bar{c}(2\pi h)^{-n/2}\mathcal{F}_h u_0 = \bar{c}u_0;$$

whence

$$c = \exp(-\pi i/4).$$

□

REMARK. The family of canonical transformations $\boldsymbol{\kappa}_t$ $(0 \leq t \leq 1)$, used here can be extended to a periodic family of canonical transformations: $\boldsymbol{\kappa}_{t+4} = \boldsymbol{\kappa}_t$ $(t \in \mathbb{R})$. Extending $F(t)$ using (10.7), we see that the argument above gives

$$F(4k) = (-1)^k I, \quad \boldsymbol{\kappa}_{4k} = I.$$

Consequently on the quantum level the deformation produces an additional shift in the phase. This shift has an important geometric and physical interpretation and is related to the *Maslov index*. For a brief discussion and references see [S-Z1, Sect.7]. □

REMARK: Quantizing linear symplectic mappings.

Using Step 1 in the proof of Theorem 10.1, we can in fact quantize any linear symplectic transformation. So given

$$K : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \qquad K = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$C^*A = A^*C, \quad D^*B = B^*D, \quad D^*A - B^*C = I,$$

we can construct $F_K : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ satisfying

$$F_K^* F_K = F_K F_K^* = I, \quad a^w(x, hD) \circ F_K = F_K \circ (K^*a)^w(x, hD).$$

The operator F_K is unique up to a multiplicative factor; and hence

$$F_{K_1} \circ F_{K_2} = cF_{K_1 \circ K_2}, \quad |c| = 1.$$

The association $K \mapsto F_K$ can in fact be chosen so that $c = \pm 1$; therefore it is *almost* a representation of the group of symplectic transformations. To make it a representation, one has to move to the double cover of the symplectic group, the so-called the metaplectic group. Unitary operators quantizing linear symplectic transformations are consequently called *metaplectic operators*: see Dimassi–Sjöstrand [D-S, Appendix to Chapter 7] for a self-contained presentation in the semiclassical spirit, and Folland [F, Chapter 4] for more and for references. □

EXAMPLE: A invertible. For reasons already apparent in the discussion of the Fourier transform, there cannot be a general formula for the kernel F_K in terms of the entries A, B, C, D of K.

But if $\det A \neq 0$, we have for $u \in \mathcal{S}$ the formula

$$F_K u(x) = \frac{(\det A)^{-\frac{1}{2}}}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}(\phi(x,\eta) - \langle y, \eta \rangle)} u(y)\, dy d\xi, \tag{10.26}$$

where

$$\phi(x, \eta) := -\frac{1}{2}\langle CA^{-1}x, x\rangle + \langle A^{-1}x, \eta\rangle + \frac{1}{2}\langle A^{-1}B\eta, \eta\rangle. \tag{10.27}$$

We will refer to this formula in our next example. □

10.2 SEMICLASSICAL ANALYSIS OF PROPAGATORS

In this section we consider the flow of symplectic transformations

$$\boldsymbol{\kappa}_t = \exp(tH_p), \tag{10.28}$$

generated by the real-valued symbol $p \in S(m)$.

Let $P = \mathrm{Op}(p)$. Then in the notation of Theorem 10.2, $F(t) = e^{-itP/h}$ solves

$$\begin{cases} (hD_t + P)F(t)u = 0 \\ \qquad\qquad F(0)u = u \end{cases}$$

for $u \in \mathcal{S}$. In this case, Theorem 10.2 reproduces Egorov's Theorem 9.2: if $a \in \mathcal{S}$, then

$$e^{itP/h}\mathrm{Op}(a)e^{-itP/h} = \mathrm{Op}(b_t),$$

for

$$b_t = (\exp tH_p)^* a + O(h).$$

A Fourier integral representation formula. Our goal now is to find for small times $t_0 > 0$ a microlocal representation of $F(t)$ as an oscillatory integral. In other words, we would like to find an operator $U(t)$ so that for each h dependent family, $u \in \mathcal{S}$ with $WF_h(u) \subset\subset \mathbb{R}^{2n}$, we have

$$\begin{cases} hD_t U(t)u + PU(t)u = O(h^\infty) \qquad (-t_0 \le t \le t_0) \\ \qquad\qquad\qquad U(0)u = u. \end{cases} \tag{10.29}$$

Using Duhamel's formula, we can then deduce that

$$F(t) - U(t) = O(h^\infty).$$

THEOREM 10.9 (Oscillatory integral representation). *We have the representation*

(10.30)
$$U(t)u(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}(\varphi(t,x,\eta) - \langle y,\eta \rangle)} b(t,x,\eta;h)u(y)\,dy d\eta,$$

for the phase φ and amplitude b as defined below.

The proof will appear after the following constructions of the phase and amplitude.

Construction of the phase function. We start by finding the phase function φ as a local generating function associated with the symplectomorphisms (10.28). (Recall the discussion in §2.3 of generating functions.)

Let U denote a bounded open set containing $(0,0)$.

LEMMA 10.10 (Hamilton–Jacobi equation). *If $t_0 > 0$ is small enough, there exists a smooth function*

$$\varphi = \varphi(t,x,\eta)$$

defined in $(-t_0,t_0) \times U \times U$, such that

$$\boldsymbol{\kappa}_t(y,\eta) = (x,\xi)$$

locally if and only if

$$\xi = \partial_x \varphi(t,x,\eta), \quad y = \partial_\eta \varphi(t,x,\eta). \tag{10.31}$$

Furthermore, φ solves the Hamilton–Jacobi equation

$$\begin{cases} \partial_t \varphi(t,x,\eta) + p(x, \partial_x \varphi(t,x,\eta)) = 0 \\ \varphi(0,x,\eta) = \langle x,\eta \rangle. \end{cases} \tag{10.32}$$

Proof. 1. We know that for points (y,η) lying in a compact subset of $\mathbb{R}^{2n}$, the flow

$$(y,\eta) \mapsto \boldsymbol{\kappa}_t(y,\eta) \tag{10.33}$$

is surjective near $(0,0)$ for times $0 \le t \le t_0$, provided t_0 is small enough. This is so since $\boldsymbol{\kappa}_0(y,\eta) = (y,\eta)$.

2. To show the existence of φ, consider

$$\Lambda := \{(t, p(y,\eta); \boldsymbol{\kappa}_t(y,\eta); y,\eta) : t \in \mathbb{R}, (y,\eta) \in \mathbb{R}^{2n}\},$$

This is a surface in $\mathbb{R}^2 \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$, a typical point of which we will write as $(t, \tau, x, \xi, y, \eta)$. Introduce the one-form

$$V := -\tau dt + \sum_{j=1}^{n} \xi_j dx_j + \sum_{j=1}^{n} y_j d\eta_j.$$

That $\boldsymbol{\kappa}_t$ is a symplectic implies $dV|_\Lambda = 0$. By Poincaré's Lemma (Theorem C.3), there exists a smooth function φ such that

$$d\varphi = V.$$

In view of (10.33) we can use (t, x, η) as coordinates on $\Lambda \cap ((-t_0, t_0) \times U \times U)$; and hence

$$-\tau dt + \sum_{j=1}^{n} \xi_j dx_j + \sum_{j=1}^{n} y_j d\eta_j = \partial_t \varphi dt + \sum_{j=1}^{n} \partial_{x_j} \varphi dx_j + \sum_{j=1}^{n} \partial_{\eta_j} \varphi d\eta_j.$$

Comparing the terms on the two sides gives (10.31) and (10.32). $\square$

Construction of the amplitude. The amplitude b in (10.30) must satisfy

$$(hD_t + p^w(x, hD))(e^{i\varphi(t,x,\eta)/h} b(t, x, \eta; h)) = O(h^\infty);$$

and so

$$(\partial_t \varphi + hD_t + e^{-i\varphi/h} p^w(x, hD) e^{i\varphi/h}) b(t, x, \eta; h) = O(h^\infty), \tag{10.34}$$

for (x, η) in a neighbourhood of $U \times U$, $0 \le t \le t_0$.

We will build b as an expansion in powers of h:

$$b(t, x, \eta; h) \sim b_0(t, x, \eta) + h b_1(t, x, \eta) + h^2 b_2(t, x, \eta) + \cdots . \tag{10.35}$$

Once all the terms b_j are computed, Borel's Theorem 4.15 produces the amplitude b.

LEMMA 10.11 (Calculation of $\mathbf{b_0}$). *We have*

$$b_0(t, x, \eta) = (\det \partial^2_{\eta x} \varphi(t, x, \eta))^{\frac{1}{2}}. \tag{10.36}$$

Note that $\det \partial^2_{\eta x} \varphi > 0$ for $0 \le t \le t_0$, if t_0 is sufficiently small.

Proof. 1. We first observe that

$$e^{-i\varphi/h} p^w(x, hD) e^{i\varphi/h} = q_t(x, hD; h),$$

where

$$q_t(x, \xi; h) = p(x, \partial_x \varphi + \xi) + O(h^2). \tag{10.37}$$

In fact, writing $\varphi(x) - \varphi(y) = F(x,y)(x-y)$, we easily check that

$$e^{-i\varphi/h} p^w(x, hD) e^{i\varphi/h} u =$$
$$\iint a\left(\frac{x+y}{2}, \xi + F(x,y)\right) e^{i\langle x-y, \xi\rangle/h} u(y) dy d\xi \,,$$

where

$$F(x,y) = \partial_x \varphi\left(\frac{x+y}{2}\right) + O((x-y)^2) \,.$$

Hence,

$$e^{-i\varphi/h} p^w(x, hD) e^{i\varphi/h} u = \iint (a((x+y)/2, \xi + \partial_x\varphi((x+y)/2))$$
$$+ \langle e(x,y,\xi)(x-y), (x-y)\rangle) e^{i\langle x-y,\xi\rangle/h} u(y) dy d\xi \,,$$

where the entries of the matrix valued function e are in S. Integration by parts based on (8.21) gives (10.37).

2. Recalling from Lemma 10.10 that $\partial_t \varphi = -p(x, \partial_x \varphi)$, we then deduce from (10.34) that

$$(hD_t + f_t^w(x, hD, \eta)) b(t,x,\eta) = O(h^2), \tag{10.38}$$

where

$$f_t(x,\xi) := p(x, \partial_x\varphi(t,x,\eta) + \xi) - p(x, \partial_x \varphi(t,x,\eta)) \,,$$

and where η considered as a parameter. So

$$f_t(x,\xi,\eta) = \sum_{j=1}^n \xi_j \partial_{\xi_j} p(x, \partial_x\varphi(t,x,\eta)) + O(|\xi|^2).$$

Hence for $g = g(t,x,\eta) \in S$,

$$f_t^w(x, hD, \eta) g = \frac{1}{2} \sum_{j=1}^n \left((\partial_{\xi_j} p) hD_{x_j} g + hD_{x_j}\left(\partial_{\xi_j} p\, g\right)\right) + O(h^2),$$

in which expression the derivatives of p are evaluated at $(x, \partial_x\varphi(t,x,\eta))$. Consequently b_0 satisfies:

$$hD_t b_0 + \frac{1}{2}\sum_{j=1}^n (\partial_{\xi_j} p) hD_{x_j} b_0 + hD_{x_j}(\partial_{\xi_j} p\, b_0) = 0.$$

This we rewrite as

$$(\partial_t + V_t + \frac{1}{2}\operatorname{div} V_t) b_0 = 0 \tag{10.39}$$

with

$$V_t := \sum (\partial_{\xi_j} p) \partial_{x_j}.$$

3. To understand this equation geometrically, we consider $b_0(t,\cdot,\eta)$ as a function on

$$\Lambda_{t,\eta} := \{(x, \partial_x\varphi(t,x,\eta))\}.$$

Then

$$\begin{aligned} \boldsymbol{\kappa}_{s,t} &: \Lambda_{t-s,\eta} \longrightarrow \Lambda_{t,\eta}, \\ \frac{d}{ds}\boldsymbol{\kappa}_{s,t}^* u|_{s=0} &= H_p|_{\Lambda_{t,\eta}} u = V_t u, \end{aligned}$$

for $u \in C^\infty$. But equation (10.39) can be further rewritten as

$$\frac{d}{dt}\boldsymbol{\kappa}_t^* b_0(t,\cdot,\eta) = -\frac{1}{2}\boldsymbol{\kappa}_t^*(\operatorname{div} V_t\ b_0(t,\cdot,\eta)). \tag{10.40}$$

We claim next that

$$\boldsymbol{\kappa}_t^* b_0(t,x,\eta) = |\partial\boldsymbol{\kappa}_t|^{-\frac{1}{2}}\,, \tag{10.41}$$

is the solution of (10.40) satisfying $b_0(0,x,\eta) = 1$. Here $\boldsymbol{\kappa}_t$ is considered as a function $\Lambda_{0,\eta} \to \Lambda_{t,\eta}$. In fact,

$$\begin{aligned} \frac{d}{dt}|\partial\boldsymbol{\kappa}_t|^{-\frac{1}{2}} &= \frac{d}{ds}|\partial\boldsymbol{\kappa}_t \circ \boldsymbol{\kappa}_{s,t}|_{s=0}^{-\frac{1}{2}} \\ &= \frac{d}{ds}|\partial\boldsymbol{\kappa}_t|^{-\frac{1}{2}}\boldsymbol{\kappa}_t^*\ |\partial\boldsymbol{\kappa}_{s,t}|_{s=0}^{-\frac{1}{2}} \\ &= -\frac{1}{2}\boldsymbol{\kappa}_t^* \operatorname{div}\ V_t\ |\partial\boldsymbol{\kappa}_t|^{-\frac{1}{2}}\ . \end{aligned}$$

4. To obtain an explicit formula for b_0, we recall that

$$\boldsymbol{\kappa}_t^{-1}\ :\ (x, \partial_x\varphi(t,x,\eta)) \to (\partial_\eta\varphi(t,x,\eta), \eta).$$

Hence

$$\partial(\boldsymbol{\kappa}_t^{-1}|_{\Lambda_{t,\eta}}) = \partial^2_{\eta x}\varphi(t,x,\eta),$$

and consequently, from (10.41), we see that (10.36) holds. □

Proof of Theorem 10.9 Using the same argument for the higher order terms in b, we can find its full expansion with all the equations valid in $(-t_0, t_0) \times U$. That shows that $U(t)$ given by (10.30) satisfies (10.29), and thereby completes the proof of Theorem 10.9. □

EXAMPLE. Revisiting example (10.26), we see that for the phase (10.27) the corresponding amplitude is

$$b_0 = (\det \partial^2_{x\eta}\varphi(x,\eta))^{1/2} = (\det A)^{-1/2}.$$

□

REMARK: Amplitudes as half-densities. The somewhat cumbersome derivation of the formula for b_0, the leading term of the amplitude

b in (10.30), becomes much more natural when we use half-densities, introduced earlier in Section 8.1.

We first make a general observation. If $a := u|dx|^{\frac{1}{2}}$ is a half-density, and $\boldsymbol{\kappa}_t$ is a family of diffeomorphisms generated by a family of vector-fields:

$$\frac{d}{dt}\boldsymbol{\kappa}_t = (\boldsymbol{\kappa}_t)_* V_t,$$

then

$$\mathcal{L}_{V_t} a := \frac{d}{dt}\boldsymbol{\kappa}_t^* a = (V_t u + (\operatorname{div} V_t/2)u)|dx|^{\frac{1}{2}}. \tag{10.42}$$

Indeed,

$$\boldsymbol{\kappa}_t^* a = \boldsymbol{\kappa}_t^* u |\partial \boldsymbol{\kappa}_t|^{\frac{1}{2}} |dx|^{\frac{1}{2}};$$

and if we define

$$\boldsymbol{\kappa}_{s,t}(x) := \boldsymbol{\kappa}_{t+s}(\boldsymbol{\kappa}_t^{-1}(x)), \quad \frac{d}{ds}\boldsymbol{\kappa}_{s,t}(x)|_{s=0} = V_t(x),$$

then

$$\frac{d}{dt}|\partial\boldsymbol{\kappa}_t|^{\frac{1}{2}} = \frac{d}{ds}|\partial\boldsymbol{\kappa}_t \circ \boldsymbol{\kappa}_{s,t}|^{\frac{1}{2}} = \frac{1}{2}|\partial\boldsymbol{\kappa}_t|^{\frac{1}{2}}\boldsymbol{\kappa}_t^* \operatorname{div} V_t.$$

This means that if we consider $b_0(t,x,\eta)|dx|^{\frac{1}{2}}$ as a half-density on $\Lambda_{t,\eta}$ then (10.39) becomes

$$(d/dt)\boldsymbol{\kappa}_t^*(b_0|dx|^{\frac{1}{2}}) = (\partial_t + \mathcal{L}_{V_t})(b_0(t,x,\eta)|dx|^{\frac{1}{2}}) = 0.$$

This is the same as

$$\boldsymbol{\kappa}_t^*(b_0(t,x,\eta)|dx|^{\frac{1}{2}}|_{\Lambda_{\varphi_{t,\eta}}}) = |dx|^{\frac{1}{2}}|_{\Lambda_{\varphi_{0,\eta}}}.$$

It follows that $\boldsymbol{\kappa}_t^* b_0 = |\partial\boldsymbol{\kappa}_t|^{-1/2}$, the same conclusion as before.

It is appealing that the amplitude, interpreted as a half-density, is invariant under the flow. When coordinates change, and in particular when we move to larger times at which (10.31) and (10.32) are no longer valid, the statement about the amplitude as a half-density remains simple. □

REMARK: A more general version of oscillatory integral representation.

If we examine the proof of Theorem 10.9 we notice that we did not use the fact that $P = p^w(x, hD)$ is t independent. That means that we can consider the solution of a more general problem,

$$\begin{cases} (hD_t + P(t))F(t)u = 0 \\ \qquad F(0)u = u \end{cases} \tag{10.43}$$

where

$$P(t) = p^w(t, x, hD)\,, \quad p(t, x, \xi) \in C^\infty(\mathbb{R}_t, S(\mathbb{R}^{2n}_{x,\xi}, m))\,.$$

For the approximate solution of this problem we still have the same oscillatory integral representation as the one give in Theorem 10.9. In particular that means that we have an oscillatory integral representation of the family of operators defined in Theorem 10.3 for small values of t there.

For the yet more general problem of p depending on h we refer to [S-Z1, Section 7] and references given there. Here we note that the proof works for $P(t) = p^w(t, x, hD) + h^2 p_2^w(t, x, hD)$ and that form of operators acting on half-densities is invariant (see Theorem 8.1).

10.3 APPLICATION: SEMICLASSICAL STRICHARTZ ESTIMATES AND L^p BOUNDS ON APPROXIMATE SOLUTIONS

In this section we will use Theorem 10.9 to obtain L^p bounds on approximate solutions to

Let $p = p(t, x, \xi) \in C^\infty(\mathbb{R}, S(T^*\mathbb{R}^k, m))$. We introduce the following nondegeneracy condition at (t, x, ξ):

$$\partial_\xi^2 p(t, x, \xi) \ \text{ is non-degenerate}\,. \tag{10.44}$$

REMARK. The Hessian, $\partial_\xi^2 f(\xi_0)$, of a smooth function $f(\xi)$ is not invariantly defined unless $\partial_\xi f(\xi_0) = 0$. However the statement (10.44) is invariant if only *linear* transformations in ξ are allowed. That is the case for symbol transformation induced by changes of variables in x, see Theorem 8.1.

We consider the problem which essentially the same as (10.43):

$$\begin{cases} (hD_t + P(t))F(t, r)u = 0 \\ \qquad\qquad F(t, r)u = u \end{cases} \tag{10.45}$$

where $r \in \mathbb{R}$. As discussed in the remark at the end of Section 10.2, Theorem 10.9 gives a description of $F(t, r)$ for small values of t.

THEOREM 10.12 (Semiclassical Strichartz estimates). *Suppose that $p(t) \in C^\infty(\mathbb{R}_t, S(T^*\mathbb{R}^k, m))$, is real valued, $\chi \in C_c^\infty(T^*\mathbb{R}^k)$, and that (10.44) holds in $\operatorname{spt}(\chi)$, $t \in \mathbb{R}$. With $P(t) := p^w(t, x, hD)$, let $F(t, r)$ be the solution of (10.45). Then for $\psi \in C_c^\infty(\mathbb{R})$ with support sufficiently close to 0, any $I \subset\subset \mathbb{R}$, and*

$$U(t, r) := \psi(t)F(t, r)\chi^w(x, hD) \quad \text{or} \quad U(t, r) := \psi(t)\chi^w(x, hD)F(t, r)$$

we have

$$
\begin{aligned}
&\sup_{r\in I}\left(\int_{\mathbb{R}}\|U(t,r)f\|^p_{L^q(\mathbb{R}^k)}dt\right)^{\frac{1}{p}}\leq Bh^{-\frac{1}{p}}\|f\|_{L^2(\mathbb{R}^k)}\,,\\
&\frac{2}{p}+\frac{k}{q}=\frac{k}{2}\,,\quad 2\leq p\leq\infty\,,\quad 1\leq q\leq\infty\,,\ (p,q)\neq(2,\infty)\,.
\end{aligned}
\tag{10.46}
$$

Proof: 1. In view of Theorem B.10 we need to show that

$$
\|U(t,r)U(s,r)^*f\|_{L^\infty(X,\mu)}\leq Ah^{-k/2}|t-s|^{-k/2}\,,\quad t,s\in\mathbb{R}\,,\tag{10.47}
$$

with constants independent of $r\in I$. We can put $r=0$ in the argument and drop the dependence on r in U and F.

2. We use Theorem 10.9. The construction there and the assumption that $\chi\in C^\infty_c$ show that

$$
U(t)=\widetilde{U}(t)+E(t)\,,
$$

where

$$
E(t)=O(h^\infty):\mathcal{S}'\to\mathcal{S}\,,
$$

and the Schwartz kernel of $\widetilde{U}(t)$ is

$$
\begin{gathered}
\widetilde{U}(t,x,y)=\frac{1}{(2\pi h)^k}\int_{\mathbb{R}^k}e^{\frac{i}{h}(\varphi(t,x,\eta)-\langle y,\eta\rangle)}\tilde{b}(t,y,x,\eta;h)d\eta\,,\\
\tilde{b}\in S\cap C^\infty_c(\mathbb{R}^{1+3k})\,,\quad \varphi(0,x,\eta)=\langle x,\eta\rangle\,,\\
\partial_t\varphi(t,x,\eta)+p(t,x,\partial_x\varphi(t,x,\eta))=0\,.
\end{gathered}
\tag{10.48}
$$

3. Hence we only need to prove (10.47) with U replaced by $\widetilde{U}$ and that means that we need an L^∞ bound on the Schwartz kernel of $W(t,s):=\widetilde{U}(t)\widetilde{U}(s)^*$:

$$
W(t,s,x,y)=\frac{1}{(2\pi h)^{2k}}\int_{\mathbb{R}^{3k}}e^{\frac{i}{h}(\varphi(t,x,\eta)-\varphi(s,y,\zeta)-\langle z,\eta-\zeta\rangle)}\,B\,dzd\zeta d\eta\,,
$$

where

$$
B=B(t,s,x,y,z,\eta,\zeta;h)\in S\cap C^\infty_c(\mathbb{R}^{2+6k})\,.
$$

4. The phase is nondegenerate in (z,ζ) variables and stationary for $\zeta=\eta$, $z=\partial_\zeta\varphi(s,y,\zeta)$. Hence we can apply Theorem 3.14 to obtain

$$
W(t,s,x,y)=\frac{1}{(2\pi h)^k}\int_{\mathbb{R}^k}e^{\frac{i}{h}(\varphi(t,x,\eta)-\varphi(s,y,\eta))}\,B_1(t,s,x,y,\eta;h)\,d\eta\,,
$$

where $B_1\in S\cap C^\infty_c(\mathbb{R}^{2+3k})$. We now rewrite the phase as follows:

$$
\begin{aligned}
\widetilde{\varphi}:=\varphi(t,x,\eta)-\varphi(s,y,\eta)=(t-s)p(0,x,\eta)\\
+\langle x-y,\eta+sF(s,x,y,\eta)\rangle+O(t-s)^2\,,\quad F\in C^\infty(\mathbb{R}^{1+3k})\,,
\end{aligned}
$$

where using (10.48) we wrote

$$\varphi(s,x,\eta) - \varphi(s,y,\eta) = \langle x-y,\eta\rangle + \langle x-y, sF(s,x,y,\eta)\rangle\,.$$

5. The phase is stationary when

$$\partial_\eta \widetilde{\varphi} = (I + s\partial_\eta F)(x-y) + (t-s)(\partial_\eta p + O(t-s)) = 0\,,$$

and in particular, for s small, having a stationary point implies

$$x - y = O(t-s)\,,$$

as then $(I + s\partial_\eta F)$ is invertible. The Hessian is given by

$$\begin{aligned}\partial_\eta^2 \widetilde{\varphi} &= s\partial_\eta^2 F(x-y) + (t-s)(\partial_\eta^2 p + O(t-s)) \\ &= (t-s)(\partial_\eta^2 p + O(|t| + |s|))\,,\end{aligned}$$

where $\partial_\eta^2 p = \partial_\eta^2 p(0,x,\eta)$.

6. Hence, for t and s sufficiently small, that is for a suitable choice of the support of ψ in the definition of $U(\bullet)$, the nondegeneracy assumption (10.44) implies that at the critical point

$$\partial_\eta^2 \widetilde{\varphi} = (t-s)\psi(x,y)\,.$$

Hence for $|t-s| > Mh$ for a large constant M we can use the stationary phase estimate in Theorem 3.14 to see that

$$|W(t,s,x,y)| \leq Ch^{-k/2}|t-s|^{-k/2}\,.$$

When $|t-s| < Mh$ we see that the trivial estimate of the integral gives

$$|W(t,s,x,y)| \leq Ch^{-k} \leq C'h^{-k/2}|t-s|^{-k/2}\,,$$

which is what we need to apply Theorem B.10. $\square$

We formulate the following *microhyperbolicity* assumption at $(x_0,\xi_0) \in T^*\mathbb{R}^n$, where $p(x_0,\xi_0) = 0$, and $\partial_\xi p(x_0,\xi_0) \neq 0$. By a linear change of variables assume that $\partial_\xi p(x_0,\xi_0) = (\rho, 0, \cdots, 0)$, $\rho \neq 0$. Then near (x_0,ξ_0),

$$p(x,\xi) = e(x,\xi)(\xi_1 - a(x,\xi'))\,,$$

and our assumptions reads

$$\partial_{\xi'}^2 a(x_0,\xi_0') \ \text{ is nondegenerate.} \tag{10.49}$$

As in the remark following (10.44) we note that this assumption is invariant under linear changes of coordinates in ξ. In particular (10.49) is invariant under changes of variables.

THEOREM 10.13 (L^p bounds on approximate solutions). *Suppose that $u(h)$, $\|u(h)\|_{L^2} = 1$, is an h-tempered family of functions satisfying the frequency localization condition* (8.29). *Suppose also that* (10.49) *is satisfied in $WF_h(u)$, and that*

$$p^w(x, hD)u(h) = O_{L^2}(h)\,. \tag{10.50}$$

Then for $p = 2(n+1)/(n-1)$, and any $K \subset\subset \mathbb{R}^n$,

$$\|u(h)\|_{L^p(K)} = O(h^{-1/p})\,. \tag{10.51}$$

REMARK. The first example in the remark after Theorem 8.5 shows that the microhyperbolicity condition (10.49) is in general necessary. In fact, if $P(h) = hD_{x_1}$ and $u(h) = h^{-(n-1)/2}\chi(x_1)\chi(x'/h)$ then for $p = 2(n+1)/(n-1)$,

$$\|u\|_{L^p} \simeq h^{(n-1)(1/p-1/2)} = h^{-(n-1)/(n+1)} \neq O(h^{-1/p})\,.$$

However for the simplest case in which (10.49) holds,

$$p(x,\xi) = \xi_1 - \xi_2^2 - \cdots - \xi_n^2\,,$$

the estimate (10.51) is optimal. To see that put

$$u(h) := h^{-(n-1)/4}\chi_0(x_1)\exp(-|x'|^2/2h)\,,$$

where $x = (x_1, x')$, $\chi_0 \in C_c^\infty(\mathbb{R})$. Then

$$(-h^2\Delta_{x'} + |x'|^2)u(h) = (n-1)h\,u(h)\,,$$

$\|u(h)\|_{L^2} \simeq 1$, $|x'|^{2k}u(h) = O_{L^2}(h^k)$. Hence,

$$p^w(x, hD)u(h) = O_{L^2}(h)\,,$$

and

$$\|u(h)\|_{L^p(\mathbb{R}^n)} \simeq h^{(n-1)(2/p-1)/4} = h^{-1/p}\,, \quad p = 2(n+1)/(n-1)\,.$$

Before proving Theorem 10.13 we prove a lemma which is a consequence of Theorem 10.12

LEMMA 10.14. *In the notation of Theorem 10.13 we have*

$$\|\int_0^t U(t,s)\mathbf{1}_I(s)f(s,x)ds\|_{L^p(\mathbb{R}_t\times\mathbb{R}_x^k)} \leq C\int_{\mathbb{R}}\|f(s,x)\|_{L^2(\mathbb{R}_x^k)}ds\,. \tag{10.52}$$

Proof: We apply the integral version of Minkowski's inequality and estimate (10.46):

$$\begin{aligned}
\| \int_0^t & U(t,s)\mathbf{1}_I(s) f(s,x) ds \|_{L^p(\mathbb{R}_t \times \mathbb{R}^k_x)} \\
& \leq C \int_{I\cap\mathbb{R}_+} \| \mathbf{1}_{[s,\infty)}(t) U(t,s) f(s,x) \|_{L^p(\mathbb{R}_t\times\mathbb{R}^k_x)} ds \\
& \leq C \int_{I\cap\mathbb{R}_+} \| U(t,s) f(s,x) \|_{L^p(\mathbb{R}_t\times\mathbb{R}^k_x)} ds \\
& \leq C' \int_I \| f(s,x) \|_{L^2(\mathbb{R}^k_x)} ds \,.
\end{aligned}$$

$\square$

Proof of Theorem 10.13: 1. We follow the same procedure as in the proof of Theorem 8.5. As in that case the condition (10.50) is local in phase space, that is, it implies that for any $\chi \in C^\infty_c(T^*\mathbb{R}^n)$,

$$p^w(x, hD\chi^w(x,hD)u(h)) = O(h)\,.$$

2. We factorize $p(x,\xi)$ as in (8.34) and we easily conclude that for χ with sufficiently small support,

$$(hD_{x_1} - a(x, hD_{x'}))(\chi^w u(h)) = O_{L^2}(h)\,.$$

Let

$$f(x_1, x', h) = (hD_{x_1} - a(x, hD_{x'}))(\chi^w u(h))\,.$$

Since $\|f\|_{L^2} = O(h)$, we see

$$\int_{\mathbb{R}} \|f(x_1, \bullet)\|_{L^2(\mathbb{R}^{n-1})} dt \leq C\|f\|_{L^2(\mathbb{R}^n)} = O(h)\,. \tag{10.53}$$

3. We now apply Theorem 10.12 with $t = x_1$ and x replaced by $x' \in \mathbb{R}^{n-1}$, that is $k = n-1$. The assumption (10.49) shows that $\partial^2_{\xi'} a$ is nondegenerate in the support of χ. We can choose ψ and χ in the definition of $U(t)$ in the statement of Theorem 10.12 so that

$$\chi^w(x, hD) u(x_1, x', h) = \frac{i}{h} \int_0^{x_1} U(t,s) f(s,x') ds + O_{\mathcal{S}}(h^\infty)\,.$$

Let us choose $p = q$ in (10.46) (now with n replaced by $n-1$, that is,

$$p = q = \frac{2(n+1)}{n-1}\,.$$

Then, using (10.46), (10.53), and (10.52),

$$\begin{aligned}\|\chi^w(x,hD)u\|_{L^p} &\leq \frac{1}{h}\,h^{-1/p}\int_{\mathbb{R}}\|f(s,\bullet,h)\|_{L^2(\mathbb{R}^{n-1})}ds + O(h^\infty)\\ &= O(h^{-1/p})\,.\end{aligned}$$

A partition of unity argument used in the proof of Theorem 8.5 concludes the proof. □

As a corollary we obtain Sogge's bounds on spectral clusters on Riemannian manifolds:

THEOREM 10.15 (L^p bounds on eigenfuctions). *Suppose that M is an n-dimensional compact Riemannian manifold and let Δ_g be its Laplace-Beltrami operator. If*

$$0 = \lambda_0 < \lambda_1 \leq \cdots \lambda_j \to \infty$$

is the complete set of eigenvalues of $-\Delta_g$, and

$$-\Delta_g\varphi_j = \lambda_j\varphi_j$$

are the corresponding eigenfunctions, then for any $c_j \in \mathbb{C}$, $j = 0, 1, \cdots$,

$$\|\sum_{\mu\leq\sqrt{\lambda_j}\leq\mu+1} c_j\varphi_j\|_{L^p} \leq C\mu^{\sigma(p)}\|\sum_{\mu\leq\sqrt{\lambda_j}\leq\mu+1} c_j\varphi_j\|_{L^2}\,,$$

(10.54)
$$\sigma(p) = \begin{cases} \frac{n-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right) & \textit{for}\ \ 2\leq p\leq \frac{2(n+1)}{n-1}\,,\\[2ex] \frac{n-1}{2}-\frac{n}{p} & \textit{for}\ \ \frac{2(n+1)}{n-1}\leq p\leq\infty\,.\end{cases}$$

In particular

(10.55)
$$\|\varphi_j\|_{L^p} \leq C\lambda_j^{\sigma(p)/2}\|\varphi_j\|_{L^2}\,.$$

Proof: We argue as in the proof of Theorem 8.7. All we need to check is (10.49) but that is clear since at any point (x_0,ξ_0) and for suitable coordinates

$$p(x_0,\xi) = |\xi|^2 - 1\,,\quad \xi_0 = (1,0,\cdots,0)\,.$$

Complex interpolation [H1, Theorem 7.1.12] between the estimate in Theorem 8.7, the trivial L^2 estimate and the estimate in Theorem 10.13 gives the full result. □

10.4 MORE SYMPLECTIC GEOMETRY

To further apply the local theory of quantized symplectic transformations to the study of semiclassical operators we will need two results from symplectic geometry. The first is a stronger form of Darboux's Theorem 2.8, which we state without proof.

THEOREM 10.16 (Variant of Darboux's Theorem). *Let A and B be two subsets of $\{1, \cdots, n\}$, and suppose that*

$$p_j(x,\xi) \quad (j \in A), \quad q_k(x,\xi) \quad (k \in B)$$

are smooth, real-valued functions defined in a neighbourhood of $(0,0) \in \mathbb{R}^{2n}$, with linearly independent gradients at $(0,0)$.

If

$$\begin{gathered} \{q_i, q_j\} = 0 \quad (i,j \in A), \quad \{p_k, p_l\} = 0 \quad (k,l \in B), \\ \{p_k, q_j\} = \delta_{kj} \quad (j \in A, k \in B), \end{gathered} \tag{10.56}$$

then there exists a symplectomorphism $\boldsymbol{\kappa}$, locally defined near $(0,0)$, such that $\boldsymbol{\kappa}(0,0) = (0,0)$ and

$$\boldsymbol{\kappa}^* q_j = x_j \quad (j \in A), \quad \boldsymbol{\kappa}^* p_k = \xi_j \quad (k \in B). \tag{10.57}$$

See Hörmander [H2, Theorem 21.1.6] for an elegant exposition.

The next result is less standard and comes from the work of Duistermaat and Sjöstrand: consult Hörmander [H2, Lemma 21.3.4] for the proof.

THEOREM 10.17 (Symplectic integrating factor). *Let p and q be smooth, real-valued functions defined near $(0,0) \in \mathbb{R}^{2n}$, satisfying*

$$p(0,0) = q(0,0) = 0, \quad \{p,q\}(0,0) > 0. \tag{10.58}$$

Then there exists a smooth, positive function u for which

$$\{up, uq\} \equiv 1 \tag{10.59}$$

in a neighborhood of $(0,0)$.

10.5 NORMAL FORMS FOR OPERATORS WITH REAL SYMBOLS

Operators of real principal type. Recall that we are taking our order function to be

$$m := (1 + |x|^2 + |\xi|^2)^{\frac{k}{2}}.$$

Set $P = p^w(x, hD; h)$, where

$$p(x,\xi;h) \sim p_0(x,\xi) + hp_1(x,\xi) + \cdots + h^N p_N(x,\xi) + \cdots,$$

for $p_j \in S(m)$. We assume that the *real-valued* principal symbol p_0 satisfies

$$p_0(0,0) = 0, \quad \partial p_0(0,0) \neq 0; \tag{10.60}$$

and then say that P is an operator of *real principal type* at $(0,0)$.

THEOREM 10.18 (Normal form for real principal type operators). *Suppose that $P = p^w(x, hD; h)$ is a semiclassical real principal type operator at $(0,0)$.*

Then there exist
(i) *a local canonical transformation $\boldsymbol{\kappa}$ defined near $(0,0)$, such that $\boldsymbol{\kappa}(0,0) = (0,0)$ and*

$$\boldsymbol{\kappa}^* \xi_1 = p_0; \tag{10.61}$$

and
(ii) *an operator T, quantizing $\boldsymbol{\kappa}$ in the sense of Theorem 10.6, such that*

$$T^{-1} \quad \textit{exists microlocally near } ((0,0),(0,0)) \tag{10.62}$$

and

$$TPT^{-1} = hD_{x_1} \quad \textit{microlocally near } ((0,0),(0,0)). \tag{10.63}$$

INTERPRETATION. The point is that using this theorem, we can transplant various mathematical objects related to P to others related to hD_{x_1}, which are much easier to study. A simple example is given by the following estimate:

$$\|u\| \leq \frac{C}{h} \|Pu\|,$$

when $u = u(h) \in \mathcal{S}$ has $WF_h(u)$ in a small neighbourhood of $(0,0)$. $\square$

Proof. 1. Theorem 10.16 applied with $A = \emptyset$ and $B = \{1\}$, provides $\boldsymbol{\kappa}$ satisfying (10.61) near $(0,0)$. Then Theorem 10.1 gives us a family of symplectic transfomations $\boldsymbol{\kappa}_t$ for $0 \leq t \leq 1$.

Let $F(t)$ be defined using the family $\boldsymbol{\kappa}_t$ in Theorem 10.6, and put $T_0 = F(1)$. Then

$$T_0 P - hD_{x_1} = E \;\; \text{microlocally near } (0,0). \;,$$

for $E = \mathrm{Op}(e)$, $e \in S^{-1}$.

2. We now look for a symbol $a \in S$ so that a is elliptic at $(0,0)$ and

$$hD_{x_1} + E = AhD_{x_1}A^{-1} \;\; \text{microlocally near } (0,0)$$

for $A := \mathrm{Op}(a)$. This is the same as solving

$$[hD_{x_1}, A] + EA = 0.$$

Since $P = p_0^w + hp_1^w + h^2 p_2^w + \cdots$, the Remark after the proof of Theorem 10.2 shows that

$$e(x, \xi; h) = he_0(x, \xi) + h^2 e_1(x, \xi) + \cdots .$$

Hence we can find $a_0 \in S$ such that $a_0(0,0) \neq 0$ and

$$\frac{1}{i}\{\xi_1, a_0\} + e_0 a_0 = 0$$

near $(0,0)$.

Define $A_0 := \mathrm{Op}(a_0)$; then

$$[hD_{x_1}, A_0] + EA_0 = \mathrm{Op}(r_0)$$

for a symbol $r_0 \in S^{-2}$.

3. We now inductively find $A_j = \mathrm{Op}(a_j)$, for $a_j \in S^{-j}$, satisfying

$$[hD_{x_1}, A_0 + A_1 + \cdots + A_N] + E(A_0 + A_1 + \cdots A_N) = \mathrm{Op}(r_N),$$

for $r_N \in S^{-N-2}(1)$. We then put

$$A \sim A_1 + A_2 + \cdots + A_N + \cdots ,$$

which is elliptic near $(0,0)$. Finally, define

$$T := A^{-1} T_0.$$

This operator quantizes $\boldsymbol{\kappa}$ in the sense of Theorem 10.6. $\square$

10.6 NORMAL FORMS FOR OPERATORS WITH COMPLEX SYMBOLS

Operators of complex principal type. Assume as before that $P = p^w(x, hD; h)$ has the symbol

$$p(x, \xi; h) \sim p_0(x, \xi) + hp_1(x, \xi) + \cdots + h^N p_N(x, \xi) + \cdots$$

with $p_j \in S(m)$. We now allow $p(x, \xi)$ to be *complex-valued*, and still say that P is *principal type* at $(0,0)$ if

$$p_0(0,0) = 0, \quad \partial p_0(0,0) \neq 0.$$

Discussion. If $\partial(\operatorname{Re} p_0)$ and $\partial(\operatorname{Im} p_0)$ are linearly independent, then the submanifold of $\mathbb{R}^{2n}$ where P is *not* elliptic has codimension two – as opposed to codimension one in the real-valued case. The symplectic form restricted to that submanifold is non-degenerate if $\{\operatorname{Re} p_0, \operatorname{Im} p_0\} \neq 0$.

Under this assumption a combination of Theorems 10.16 and 10.17 shows that there exists a canonical transformation $\boldsymbol{\kappa}$, defined near $(0,0)$, and a smooth positive function u such that

$$\boldsymbol{\kappa}^*(\xi_1 \pm ix_1) = up_0.$$

That is, after a multiplication by a function we obtain the symbol of the creation or annihilation operator for the harmonic oscillator in the (x_1, ξ_1) variables. (Recall the discussion of the harmonic oscillator in Section 6.1.)

THEOREM 10.19 (Normal form for the complex symplectic case). *Suppose that* $P = p^w(x, hD; h)$ *is a semiclassical principal type operator at* $(0,0)$*, with principal symbol* p_0 *satisfying*

$$p_0(0,0) = 0, \quad \pm\{\operatorname{Re} p_0, \operatorname{Im} p_0\}(0,0) > 0. \tag{10.64}$$

Then there exist
(i) *a local canonical transformation* $\boldsymbol{\kappa}$ *defined near* $(0,0)$ *and a smooth function* u *such that* $\boldsymbol{\kappa}(0,0) = (0,0)$*,* $u(0,0) > 0$*, and*

$$\boldsymbol{\kappa}^*(\xi_1 \pm ix_1) = up_0;$$

and (ii) *an operator* T*, quantizing* $\boldsymbol{\kappa}$ *in the sense of Theorem 10.6, and a pseudodifferential operator* A*, elliptic at* $(0,0)$*, such that*

$$T^{-1} \quad \text{exists microlocally near } ((0,0),(0,0)) \tag{10.65}$$

and

$$TPT^{-1} = A(hD_{x_1} \pm ix_1) \quad \text{microlocally near } ((0,0),(0,0)). \tag{10.66}$$

INTERPRETATION. We can transplant mathematical objects related to P to others related to $A(hD_{x_1} \pm ix_1)$, which are clearly much easier to study. □

Proof. 1. We start as in the proof of Theorem 10.18. To simplify the notation, let us assume

$$\{\operatorname{Re} p_0, \operatorname{Im} p_0\} > 0.$$

As noted above, using Theorems 10.16 and 10.17 we can find a smooth function u, with $u(0,0) > 0$, and a local canonical transformation $\boldsymbol{\kappa}$ such that $\boldsymbol{\kappa}(0,0) = (0,0)$ and $\boldsymbol{\kappa}^*(\xi_1 + ix_1) = up_0$.

Quantizing as before, we obtain an operator T_0 satisfying

$$T_0 P = Q(hD_{x_1} + ix_1 + E)T_0, \tag{10.67}$$

where $Q = \mathrm{Op}(q)$ for a function q satisfying

$$\boldsymbol{\kappa}^* q = 1/u$$

and $E = \mathrm{Op}(e)$ for some $e \in S^{-1}$.

2. We now need to find pseudodifferential operators B and C, elliptic at $(0,0)$, and such that

$$(hD_{x_1} + ix_1 + E)B \equiv C(hD_{x_1} + ix_1), \tag{10.68}$$

microlocally near $(0,0)$. As in the proof of Theorem 10.18, we have

$$E = \mathrm{Op}(e), \quad e = he_0(x,\xi) + h^2 e_1(x,\xi) + \cdots .$$

We will find the symbols of B and C by computing successive terms in their expansions:

$$\begin{aligned} b &\sim b_0 + hb_1 + \cdots + h^N b_N + \cdots , \\ c &\sim c_0 + hc_1 + \cdots + h^N c_N + \cdots . \end{aligned}$$

3. Let us rewrite (10.68) as

$$(hD_{x_1} + ix_1 + E)B - C(hD_{x_1} + ix_1) = \mathrm{Op}(r),$$

for

$$r(x,\xi) = r_0(x,\xi) + hr_1(x,\xi) + \cdots + h^N r_N(x,\xi) + \cdots ,$$

with

$$\begin{aligned} r_0 &= (\xi_1 + ix_1)(b_0 - c_0), \\ r_1 &= (\xi_1 + ix_1)(b_1 - c_1) + e_0 b_0 + \{\xi_1 + ix_1, b_0\}/2i - \{c_0, \xi_1 + ix_1\}/2i. \end{aligned}$$

Here we used composition formula in Weyl calculus (see Theorem 4.9).

We want to choose b and c so that $r_j \equiv 0$ for all j. For $r_0 = 0$ we simply need $b_0 = c_0$. Then to obtain $r_1 = 0$ we have to solve

$$-i(\partial_{x_1} - i\partial_{\xi_1})b_0 + e_0 b_0 + (\xi_1 + ix_1)(b_1 - c_1) = 0.$$

4. We first find b_0 such that

$$\begin{cases} -i(\partial_{x_1} - i\partial_{\xi_1})b_0 + e_0 b_0 = O(x_1^\infty) \\ \qquad\qquad b_0|_{x_1=0} = 1; \end{cases}$$

that is, the left hand side vanishes to infinite order at $x_1 = 0$, and $b_0 = 1$ there. The derivatives $\partial_{x_1}^k e_0|_{x_1=0}$ determine $\partial_{x_1}^k b_0|_{x_1=0}$. Then Borel's Theorem 4.15 produces a smooth function b_0 with these prescribed derivatives.

5. With $b_0 = c_0$ chosen that way we see that

$$t_1 := (-i(\partial_{x_1} - i\partial_{\xi_1})b_0 + e_0 b_0)/(\xi_1 + ix_1)$$

is a smooth function: the numerator vanishes to infinite order on the zero set of the denominator. If we put

$$c_1 = b_1 + t_1 \tag{10.69}$$

then $r_1 = 0$.

6. Now, using (10.69) the same calculation as before, we see that

$$r_3 = (\xi_1 + ix_1)(b_2 - c_2) + e_0 b_1 - i\{\xi_1 + ix_1, b_1\} + \tilde{r}_3,$$

where $\tilde{r}_3$ depends only on $b_0 = c_0$, t_1, and e. Hence $\tilde{r}_3$ is already determined. We proceed as in Step 4 and first solve

$$\begin{cases} -i(\partial_{x_1} - i\partial_{\xi_1})b_1 + e_0 b_1 + \tilde{r}_3 = O(x_1^\infty) \\ \hfill b_1|_{x=1} = 0. \end{cases}$$

This determines b_1 (and hence c_1). We continue in the same way to determine b_2 (and hence c_2). An iteration of the argument completes the construction of b and c, for which (10.68) holds microlocally near $(0, 0)$.

7. Finally, we put $T = B^{-1}T_0$, where B^{-1} is the microlocal inverse of B near $(0, 0)$, and $A = B^{-1}QC$, to obtain the statement of the theorem. □

10.7 APPLICATION: SEMICLASSICAL PSEUDOSPECTRA

We present in this last section an application to the so-called *semiclassical pseudospectrum.* Recall from Chapter 6 that if $P = P(h) = -h^2\Delta + V(x)$ and V is real-valued, satisfying

$$V \in S(\langle x \rangle^m), \quad |\xi^2 + V(x)| \geq (1 + |\xi|^2 + |x|^m)/C \ \text{ for } |x| \geq C, \tag{10.70}$$

then the spectrum of P is discrete. (We deduced this from the meromorphy of the resolvent of P, $R(z) = (P - z)^{-1}$.)

Quasimodes. Because of the Spectral Theorem, which is applicable as V is real, we also know that approximate location of eigenvalues is implied by the existence of approximate eigenfunctions, called *quasimodes.* Indeed suppose that

$$\|(P - z(h))u(h)\| = O(h^\infty), \ \|u(h)\| = 1. \tag{10.71}$$

Then there exist $E(h)$ and $v(h)$ such that

$$(P - E(h))v(h) = 0, \ \|v(h)\| = 1, \ |E(h) - z(h)| = O(h^\infty). \tag{10.72}$$

In other words, if we can solve (10.71), then the approximate eigenvalue $z(h)$ is in fact close to a true eigenvalue $E(h)$ (although $u(h)$ need not be close to a true eigenfunction $v(h)$.)

Nonnormal operators. But it is well known that this is not the case for nonnormal operators P, for which the commutator $[P^*, P]$ does not vanish. Now if If $p = |\xi|^2 + V(x)$, then the symbol of this commutator is

$$\frac{1}{i}\{\bar{p}, p\} = 2\{\operatorname{Re} p, \operatorname{Im} p\}; \tag{10.73}$$

and when this is nonzero we are in the situation discussed in Theorem 10.19. This discussion leads us to

THEOREM 10.20 (Quasimodes). *Suppose that $P = -h^2\Delta + V(x)$ and that*

$$z_0 = \xi_0^2 + V(x_0), \quad \operatorname{Im}\langle \xi_0, \partial V(x_0)\rangle \neq 0. \tag{10.74}$$

Then there exists a family of functions $u(h) \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ such that

$$\|(P - z_0)u(h)\|_{L^2} = O(h^\infty), \quad \|u(h)\|_{L^2} = 1. \tag{10.75}$$

Moreover, we can choose $u(h)$ so that

$$WF_h(u(h)) = \{(x_0, \xi_0)\}, \quad \operatorname{Im}\langle \xi_0, \partial V(x_0)\rangle < 0. \tag{10.76}$$

Proof. We first replace V by a compactly supported potential agreeing with V near x_0. Our function $u(h)$ will be constructed with support near x_0.

By changing the sign of ξ_0 if necessary, but without changing z_0, we can assume that

$$\{\operatorname{Re} p, \operatorname{Im} p\}(x_0, \xi_0) = 2\operatorname{Im}\langle \xi_0, \partial V(x_0)\rangle < 0.$$

According Theorem 10.19, $P - z_0$ is microlocally conjugate to $A(hD_{x_1} - ix_1)$ near $((x_0, \xi_0), (0, 0))$. Let

$$u_0(x, h) := \exp(-|x|^2/2h);$$

so that

$$(hD_{x_1} - ix_1)u_0(h) = 0, \quad WF_h(u_0(h)) = \{(0, 0)\}.$$

Following the notation of Theorem 10.19, we define $u(h) := T^{-1}u_0(h)$. Then $WF_h(u(h)) = \{(x_0, \xi_0)\}$ and

$$(P - z_0)u(h) \equiv T^{-1}A(hD_{x_1} - ix_1)T(T^{-1}u_0) \equiv 0.$$

□

REMARK. If $p(x, \xi) = |\xi|^2 + V(x)$, the potential V satisfies (10.70), and

$$\{p(x, \xi) : (x, \xi) \in \mathbb{R}^{2n}\} \neq \mathbb{C},$$

then the operator P still has a discrete spectrum. This follows from the proof of Theorem 6.7, once we have a point at which $P - z$ is elliptic. Such a point is produced if there exists z not in the set of values of $p(x, \xi)$. However, the hypothesis of Theorem 10.20 holds in a dense open subset of the interior of the closure of the range of p. $\square$

EXAMPLE. It is also clear that more general operators can be considered. As a simple one-dimensional example, take

$$P = (hD_x)^2 + ihD_x + x^2$$

with

$$p(x, \xi) = \xi^2 + i\xi + x^2, \quad \{\operatorname{Re} p, \operatorname{Im} p\} = -2x.$$

Hence there is a quasimode corresponding to any point in the interior of the range of p, namely $\{z : \operatorname{Re} z \geq (\operatorname{Im} z)^2\}$. On the other hand, since

$$e^{x/2h} P e^{-x/2h} = (hD)^2 + x^2 + \frac{1}{4},$$

P has the discrete spectrum $\{1/4 + nh : n \in \mathbb{N}\}$. Since the spectrum lies inside an open set of quasimodes, it is unlikely to have any true physical meaning. $\square$

Appendix A. Notation

A.1 BASIC NOTATION.

$\mathbb{R}^n = n$-dimensional Euclidean space

$\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$

$m = (x, \xi) =$ typical point in $\mathbb{R}^n \times \mathbb{R}^n$

$\mathbb{C} =$ complex plane

$\mathbb{C}^n =$ n-dimensional complex space

$\mathbb{T}^n = n$-dimensional flat torus $= [0,1]^n,$ with opposite faces identified

$\langle x, y \rangle := \sum_{i=1}^n x_i \bar{y}_i =$ inner product

$|x| = \langle x, x \rangle^{1/2}$

$\langle x \rangle := (1 + |x|^2)^{1/2}$

$\mathbb{M}^{m \times n} = m \times n$-matrices

$\mathbb{S}^n = n \times n$ real symmetric matrices

$A^T =$ transpose of the matrix A

I denotes both the identity matrix and the identity mapping.

$J := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$

$\sigma(u, v) = \langle u, Jv \rangle =$ symplectic inner product

A.2 ELEMENTARY OPERATORS.

Multiplication operator: $M_\lambda f(x) = \lambda f(x)$

Translation operator: $T_\xi f(x) = f(x - \xi)$

Reflection operator: $Rf(x) := f(-x)$

A.3 FUNCTIONS, DIFFERENTIATION.

The support of a function is denoted "spt", and a subscript "c" on a space of functions means those with compact support.

• Partial derivatives:

$$\partial_{x_j} := \frac{\partial}{\partial x_j}, \quad D_{x_j} := \frac{1}{i}\frac{\partial}{\partial x_j} \tag{A.1}$$

• Multiindex notation: A multiindex is a vector $\alpha = (\alpha_1, \dots, \alpha_n)$, the entries of which are nonnegative integers. The size of α is

$$|\alpha| = \alpha_1 + \cdots + \alpha_n.$$

We then write for $x \in \mathbb{R}^n$:

$$x^\alpha := {x_1}^{\alpha_1} \dots {x_n}^{\alpha_n}$$

Also

$$\partial^\alpha := \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$$

and

$$D^\alpha := \frac{1}{i^{|\alpha|}} \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}.$$

(WARNING: Our use of the symbols "D" and "D^α" differs from that in the PDE textbook [E].)

If $\varphi : \mathbb{R}^n \to \mathbb{R}$, then we write

$$\partial \varphi := (\varphi_{x_1}, \dots, \varphi_{x_n}) = \text{gradient},$$

and

$$\partial^2 \varphi := \begin{pmatrix} \varphi_{x_1 x_1} & \dots & \varphi_{x_1 x_n} \\ & \ddots & \\ \varphi_{x_n x_1} & \dots & \varphi_{x_n x_n} \end{pmatrix} = \text{Hessian matrix}$$

Also

$$D\varphi := \frac{1}{i} \partial \varphi.$$

If φ depends on both the variables $x, y \in \mathbb{R}^n$, we put

$$\partial_x^2 \varphi := \begin{pmatrix} \varphi_{x_1 x_1} & \dots & \varphi_{x_1 x_n} \\ & \ddots & \\ \varphi_{x_n x_1} & \dots & \varphi_{x_n x_n} \end{pmatrix}$$

and

$$\partial_{x,y}^2 \varphi := \begin{pmatrix} \varphi_{x_1 y_1} & \dots & \varphi_{x_1 y_n} \\ & \ddots & \\ \varphi_{x_n y_1} & \dots & \varphi_{x_n y_n} \end{pmatrix}.$$

• Jacobians: Let

$$x \mapsto y = \mathbf{y}(x)$$

be a diffeomorphism. The Jacobian matrix is

$$\frac{\partial y}{\partial x} := \left(\frac{\partial y^i}{\partial x_j} \right)_{n \times n}. \tag{A.2}$$

• Poisson bracket: If $f, g : \mathbb{R}^n \to \mathbb{R}$ are C^1 functions,

$$\{f, g\} := \langle \partial_\xi f, \partial_x g \rangle - \langle \partial_x f, \partial_\xi g \rangle \ = \sum_{j=1}^n \frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial \xi_j}. \tag{A.3}$$

• The *Schwartz space* is

$\mathcal{S} = \mathcal{S}(\mathbb{R}^n) :=$

$$\{\varphi \in C^\infty(\mathbb{R}^n) \mid \sup_{\mathbb{R}^n} |x^\alpha \partial^\beta \varphi| < \infty \text{ for all multiindices } \alpha, \beta\}.$$

We say

$$\varphi_j \to \varphi \quad \text{in } \mathcal{S}$$

provided

$$\sup_{\mathbb{R}^n} |x^\alpha D^\beta(\varphi_j - \varphi)| \to 0$$

for all multiindices α, β

We write $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ for the space of *tempered distributions*, which is the dual of $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$. That is, $u \in \mathcal{S}'$ provided $u : \mathcal{S} \to \mathbb{C}$ is linear and $\varphi_j \to \varphi$ in $\mathcal{S}$ implies $u(\varphi_j) \to u(\varphi)$.

We say

$$u_j \to u \quad \text{in } \mathcal{S}'$$

provided

$$u_j(\varphi) \to u(\varphi) \quad \text{for all } \varphi \in \mathcal{S}.$$

A.4 OPERATORS.

A^* = adjoint of the operator A

Commutator: $[A, B] = AB - BA$

$\sigma(A)$ = symbol of the pseudodifferential operator A

$\operatorname{spec}(A)$ = spectrum of A.

$\operatorname{tr}(A)$ = trace of the A.

We say that the operator B is of *trace class* if

$$\operatorname{tr}(B) := \sum \mu_j < \infty,$$

where the μ_j^2 are the eigenvalues of B^*B.

• If $A : X \to Y$ is a bounded linear operator, we define the operator norm

$$\|A\| := \sup\{\|Au\|_Y \mid \|u\|_X \leq 1\}.$$

We will often write this norm as

$$\|A\|_{X \to Y}$$

when we want to emphasize the spaces between which A maps.

The space of bounded operators is denoted by $\mathcal{B}(X, Y)$, and when $X = Y$, by $\mathcal{B}(X)$.

A.5 ESTIMATES.

• We write

$$f = O(h^\infty) \quad \text{as } h \to 0$$

if for each positive integer N there exists a constant C_N such that

$$|f| \le C_N h^N \quad \text{for all } 0 < h \le 1.$$

• If A is a bounded linear operator between the spaces X, Y, we will often write

$$A = O(h^N)_{X \to Y}$$

to mean

$$\|A\|_{X \to Y} = O(h^N).$$

A.6. SYMBOL CLASSES.

We record from Chapter 4 the various definitions of classes for symbols $a = a(x, \xi, h)$.

• Given an order function m on $\mathbb{R}^{2n}$, we define the corresponding class of symbols:

$$\begin{aligned} S(m) \ :=\ & \{a \in C^\infty \mid \text{for each multiindex } \alpha \\ & \qquad \text{there exists a constant } C_\alpha \text{ so that } |\partial^\alpha a| \le C_\alpha m\}. \end{aligned}$$

• We as well define

$$S^k(m) := \{a \in C^\infty \mid |\partial^\alpha a| \le C_\alpha h^{-k} m \ \text{ for all multiindices } \alpha\}$$

and

$$S^k_\delta(m) := \{a \in C^\infty \mid |\partial^\alpha a| \le C_\alpha h^{-\delta|\alpha| - k} m \ \text{ for all multiindices } \alpha\}.$$

The index k indicates how singular is the symbol a as $h \to 0$; the index δ allows for increasing singularity of the derivatives of a.

• Write also

$$S^{-\infty}(m) := \{a \in C^\infty \mid \text{for each } \alpha \text{ and } N,\ \ |\partial^\alpha a| \le C_{\alpha,N} h^N m\}.$$

So if a is a symbol in $S^{-\infty}(m)$, then a and all of its derivatives are $O(h^\infty)$ as $h \to 0$.

• If the order function is the constant function $m \equiv 1$, we will usually not write it:

$$S^k := S^k(1),\ \ S^k_\delta = S^k_\delta(1).$$

• We will also omit zero superscripts:

$$S := S^0 = S^0(1)$$
$$= \{a \in C^\infty(\mathbb{R}^{2n}) \mid |\partial^\alpha a| \le C_\alpha \text{ for all multiindices } \alpha\}.$$

A.7 PSEUDODIFFERENTIAL OPERATORS.

The following terminology is from Appendix D.

• A linear operator $A : C^\infty(M) \to C^\infty(M)$ is called a *pseudodifferential operator* if there exist integers m, k such that for each coordinate patch $U_{\boldsymbol{\kappa}}$, and there exists a symbol $a_{\boldsymbol{\kappa}} \in S^k(\langle\xi\rangle^m)$ such that for any $\varphi, \psi \in C^\infty_{\rm c}(U_{\boldsymbol{\kappa}})$

$$\varphi A(\psi u) = \varphi \boldsymbol{\kappa}^* a^w_{\boldsymbol{\kappa}}(x, hD)(\boldsymbol{\kappa}^{-1})^*(\psi u)$$

for each $u \in C^\infty(M)$.

• We write

$$A \in \Psi^{m,k}(M)$$

and also put

$$\Psi^k(M) := \Psi^{0,k}(M), \quad \Psi(M) := \Psi^{0,0}(M).$$

Henceforth H denotes a complex Hilbert space, with inner product $\langle \cdot, \cdot \rangle$.

THEOREM B.1 (Spectrum of self-adjoint operators). *Suppose $A : H \to H$ is a bounded self-adjoint operator. Then*

(i) $(A - \lambda)^{-1}$ *exists and is a bounded linear operator on H for $\lambda \in \mathbb{C} - \operatorname{spec}(A)$, where $\operatorname{spec}(A) \subset \mathbb{R}$ is the spectrum of A.*

(ii) *If $\operatorname{spec}(A) \subseteq [a, \infty)$, then*

$$\langle Au, u \rangle \geq a\|u\|^2 \quad (u \in A). \tag{B.1}$$

THEOREM B.2 (Approximate inverses). *Let X, Y be Banach spaces and suppose $A : X \to Y$ is a bounded linear operator. Suppose there exist bounded linear operators $B_1, B_2 : Y \to X$ such that*

$$\begin{cases} AB_1 = I + R_1 & \text{on } Y \\ B_2 A = I + R_2 & \text{on } X, \end{cases} \tag{B.2}$$

where

$$\|R_1\| < 1, \ \|R_2\| < 1.$$

Then A is invertible.

Proof. The operator $I + R_1$ is invertible, with

$$(I + R_1)^{-1} = \sum_{k=0}^{\infty} (-1)^k R_1^k,$$

this series converging since $\|R_1\| < 1$. Hence

$$AC_1 = I \quad \text{for } C_1 := B_1(I + R_1)^{-1}.$$

Likewise,

$$(I + R_2)^{-1} = \sum_{k=0}^{\infty} (-1)^k R_2^k;$$

and

$$C_2 A = I \quad \text{for } C_2 := (I + R_2)^{-1} B_2.$$

So A has a left and a right inverse, and is consequently invertible, with $A^{-1} = C_1 = C_2$. $\square$

THEOREM B.3 (Minimax and maximin principles). *Suppose $A : H \to H$ is bounded linear and is self-adjoint: $A = A^*$. Denote by $\lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots$ be the (real) eigenvalues of A. Then*

$$\lambda_j = \min_{\substack{V \subseteq H \\ dim\ V \leq j}} \max_{\substack{v \in V \\ v \neq 0}} \frac{\langle Av, v \rangle}{\|v\|^2}. \tag{B.3}$$

and

$$\lambda_j = \max_{\substack{V \subseteq H \\ codim\ V \leq j}} \min_{\substack{v \in V \\ v \neq 0}} \frac{\langle Av, v \rangle}{\|v\|^2}. \tag{B.4}$$

In these formulas, V denotes a linear subspace of H.

DEFINITIONS. (i) Let $Q : H \to H$ be a bounded linear operator. We define the *rank* of Q to be the dimension of the range $Q(H)$.

(ii)If A is an operator with real and discrete specrum, we set

$$N(\lambda) := \#\{\lambda_j \mid \lambda_j \leq \lambda\}$$

THEOREM B.4 (Estimating N($\boldsymbol{\lambda}$)). (i) *If*

$$\begin{cases} \text{for each } \delta > 0, \text{ there exists an operator } Q, \\ \text{with rank } Q \leq k, \text{ such that} \\ \langle Au, u \rangle \geq (\lambda - \delta)\|u\|^2 - \langle Qu, u \rangle \ \textit{for}\ u \in H, \end{cases} \tag{B.5}$$

then

$$N(\lambda) \leq k.$$

(ii) *If*

$$\begin{cases} \text{for each } \delta > 0, \text{ there exists a subspace } V \\ \text{with } \dim V \geq k, \text{ such that} \\ \langle Au, u \rangle \leq (\lambda + \delta)\|u\|^2 \text{ for } u \in V, \end{cases} \tag{B.6}$$

then

$$N(\lambda) \geq k.$$

Proof. 1. Set $W := Q(H)^T$. Thus codim $W =$ rank Q $\leq k$. Therefore the maximin formula $(B.3)$ implies

$$\begin{aligned}\lambda_k &= \max_{\substack{V\subseteq H\\ \text{codim } V\leq k}} \min_{\substack{v\in V\\ v\neq 0}} \frac{\langle Av, v\rangle}{\|v\|^2} \geq \min_{\substack{v\in W\\ v\neq 0}} \frac{\langle Av, v\rangle}{\|v\|^2} \\ &= \min_{\substack{v\in W\\ v\neq 0}} \left(\lambda - \delta - \frac{\langle Qv, v\rangle}{\|v\|^2}\right) = \lambda - \delta,\end{aligned}$$

since $\langle Qv, v\rangle = 0$ if $v \in Q(H)^T$. Hence $\lambda \leq \lambda_k + \delta$. This is valid for all $\delta > 0$, and so

$$N(\lambda) = \max\{j \mid \lambda_j \leq \lambda\} \leq k.$$

This proves assertion (i).

2. The minimax formula $(B.2)$ directly implies that

$$\lambda_k \leq \max_{\substack{v\in V\\ v\neq 0}} \frac{\langle Av, v\rangle}{\|v\|^2} \leq \lambda + \delta.$$

Hence $\lambda_k \leq \lambda + \delta$. This is valid for all $\delta > 0$, and so

$$N(\lambda) = \max\{j \mid \lambda_j \leq \lambda\} \geq k.$$

This is assertion (ii).

$\square$

LEMMA B.5 (Norms of powers of operators). *Let $A \in L(E, F)$. Then*

$$\|A\|^{2m} = \|(A^*A)^m\|.$$

THEOREM B.6 (Cotlar–Stein Theorem). *Let E, F be Hilbert spaces and $A_j \in L(E, F)$ for $j = 1, \dots$. Assume*

$$\sup_j \sum_{k=1}^{\infty} \|A_j^* A_k\|^{1/2} \leq C, \quad \sup_j \sum_{k=1}^{\infty} \|A_j A_k^*\|^{1/2} \leq C.$$

Then the series

$$A := \sum_{j=1}^{\infty} A_j \quad \text{converges in } L(E, F).$$

and

$$\|A\| \leq C.$$

Proof. 1. According to the previous lemma,

$$\|A\|^{2m} = \|(A^*A)^m\|.$$

Also

$$\begin{aligned}(A^*A)^m &= \sum_{j_1,\dots,j_{2m}=1}^{\infty} A^*_{j_1}A_{j_2}\dots A^*_{j_{2m-1}}A_{j_{2m}}\\ &=: \sum_{j_1,\dots,j_{2m}} a_{j_1,\dots,j_{2m}}.\end{aligned}$$

Now

$$\|a_{j_1,\dots,j_{2m}}\| \le \|A^*_{j_1}A_{j_2}\|\|A^*_{j_3}A_{j_4}\|\dots\|A^*_{j_{2m-1}}A_{j_{2m}}\|,$$

and also

$$\|a_{j_1,\dots,j_{2m}}\| \le \|A_{j_1}\|\|A_{j_2}A^*_{j_3}\|\dots\|A_{j_{2m-2}}A^*_{j_{2m-1}}\|\|A_{j_{2m}}\|.$$

Multiply these estimates and take square roots:

$$\|a_{j_1,\dots,j_{2m}}\| \le C\|A^*_{j_1}A_{j_2}\|^{1/2}\|A_{j_2}A^*_{j_m}\|^{1/2}\dots\|A^*_{j_{2m-1}}A_{j_{2m}}\|^{1/2}.$$

Consequently,

$$\begin{aligned}\|A\|^{2m} &= \|(A^*A)^m\| \le \sum_{j_1,\dots,j_{2m}=1}^{\infty}\|a_{j_1,\dots,j_{2m}}\|\\ &\le C\sum_{j_1,\dots,j_{2m}=1}^{\infty}\|A_{j_1}A^*_{j_2}\|^{1/2}\dots\|A^*_{j_{2m-1}}A_{j_{2m}}\|^{1/2}\\ &\le CC^{2m}.\end{aligned}$$

Hence

$$\|A\| \le C^{\frac{2m+1}{2m}} \to C \quad \text{as } m\to\infty.$$

2. Now take $u \in E$, and suppose $u = A^*_k v$ for some k. Then

$$\begin{aligned}\left\|\sum_{j=1}^{\infty}A_ju\right\| &= \left\|\sum_{j=1}^{\infty}A_jA^*_kv\right\|\\ &\le \sum_{j=1}^{\infty}\|A_jA^*_k\|^{1/2}\|A_jA^*_k\|^{1/2}\|v\|\\ &\le C^2\|v\|.\end{aligned}$$

Thus $\sum_{j=1}^{\infty}A_ju$ converges for $u \in \Sigma := \operatorname{span}\{A^*_k(E) \mid k = 1,\dots,n\}$ and so also for $u \in \bar{\Sigma}$. If u is orthogonal to $\bar{\Sigma}$, then $u \in \ker(A_k)$ for all k; in which case $\sum_{j=1}^{\infty}A_ju = 0$. $\square$

THEOREM B.7 (Inverse Function Theorem). *Let X, Y denote Banach spaces and assume*

$$f : X \to Y$$

is C^1. Select a point $x_0 \in X$ and write $y_0 := f(x_0)$.

(i) *(Right inverse) If there exists $A \in L(Y, X)$ such that*

$$\partial f(x_0) A = I,$$

then there exists $g \in C^1(Y, X)$ such that

$$f \circ g = I \quad \text{near } y_0.$$

(ii) *(Left inverse) If there exists $B \in L(Y, X)$ such that*

$$B \partial f(x_0) = I,$$

then there exists $g \in C^1(Y, X)$ such that

$$g \circ f = I \quad \text{near } x_0.$$

THEOREM B.8 (Schwartz Kernel Theorem). *Let $A : \mathcal{S} \to \mathcal{S}'$ be a continuous linear operator.*

Then there exists a distribution $K_A \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$Au(x) = \int_{\mathbb{R}^n} K_A(x, y) u(y)\, dy \tag{B.7}$$

for all $u \in \mathcal{S}$.

We call K_A the *kernel* of A.

THEOREM B.9 (Lidskii's Theorem). *Suppose that B is an operator of trace class on $L^2(M, \Omega^{\frac{1}{2}}(M))$, given by the integral kernel*

$$K \in C^\infty(M \times M; \Omega^{\frac{1}{2}}(M \times M)).$$

Then K_Δ, the restriction to the diagonal $\Delta := \{(m, m) \; : m \in M\}$, has a well-defined density; and

$$\operatorname{tr} B = \int_\Delta K_\Delta. \tag{B.8}$$

We will also use the following general result of Keel-Tao [K-T]:

THEOREM B.10 (Abstract Strichartz estimates). *Let* $(X, \mathcal{M}, \mu)$ *be a* σ*-finite measure space, and let* $U \in L^\infty(\mathbb{R}, \mathcal{B}(L^2(X,\mu))$ *satisfy*

$$\text{(B.9)} \qquad \begin{gathered} \|U(t)\|_{\mathcal{B}(L^2(X))} \leq A\,, \quad t \in \mathbb{R}\,, \\ \|U(t)U(s)^* f\|_{L^\infty(X,\mu)} \leq Ah^{-\mu}|t-s|^{-\sigma}\|f\|_{L^1(X,\mu)}\,, \quad t,s \in \mathbb{R}\,, \end{gathered}$$

where $A, \sigma, \mu > 0$ *are fixed.*

The for every pair p, q *satisfying*

$$\frac{2}{p} + \frac{2\sigma}{q} = \sigma\,, \quad 2 \leq p \leq \infty\,, \quad 1 \leq q \leq \infty\,, \quad (p,q) \neq (2,\infty)\,,$$

we have

$$\text{(B.10)} \qquad \left(\int_{\mathbb{R}} \|U(t)f\|^p_{L^q(X,\mu)} dt\right)^{\frac{1}{p}} \leq Bh^{-\frac{\mu}{p\sigma}}\|f\|_{L^2(X,\mu)}\,.$$

We should stress that in the application to bounds on approximate solution (Section 10.3) we only use the "interior" exponent $p = q$ which does not require the full power of [K-T] – see [S]. For the reader's convenience we present the proof of that case.

Proof of the case $p = q$*:* 1. A rescaling in time easily reduces the estimate to the case $h = 1$.

2. The estimate we want reads

$$\text{(B.11)} \qquad \|U(t)f\|_{L^p(\mathbb{R}_t \times X)} \leq B\|f\|_{L^2(X)}\,, \quad \frac{1}{p} = \frac{\sigma}{2(1+\sigma)}\,.$$

Let p' denote the exponent dual to p: $1/p + 1/p' = 1$. Then, since $L^{p'}$ is dual to L^p, (B.11) is equivalent to

$$\int_{\mathbb{R}\times X} U(t)f(x)\; G(t,x)\; d\mu(x)dt \leq \|f\|_{L^2(X)}\|G\|_{L^{p'}(\mathbb{R}\times X)}\,,$$

for all $G \in L^{p'}(\mathbb{R}\times X)$, and that in turn means that

$$\|\int_{\mathbb{R}} U(t)^* G(t) dt\|_{L^2(X)} \leq C\|G\|_{L^{p'}(\mathbb{R}\times X)}\,,$$

or in other words that

$$\text{(B.12)} \qquad T : L^{p'}(\mathbb{R}\times X) \longrightarrow L^2(X)\,, \quad TG(x) := \int_{\mathbb{R}} U(t)^* G(t,x) dt\,.$$

3. We note that $T^* f(s,x) := U(s)f(x)$, and that the mapping property (B.12) is equivalent to

$$\langle T^*TG, F\rangle_{L^2(\mathbb{R}\times X)} \leq C\|G\|_{L^{p'}(\mathbb{R}\times X)}\|F\|_{L^{p'}(\mathbb{R}\times X)}\,,$$

which is the same as

$$\text{(B.13)}\qquad \begin{aligned} &\left|\int_{\mathbb{R}}\int_{\mathbb{R}} \langle U(t)^* G(t), U(s)^* F(s)\rangle \; dtds\right| \\ &\qquad \leq C\|G\|_{L^{p'}(\mathbb{R}\times X)}\|F\|_{L^{p'}(\mathbb{R}\times X)} \,. \end{aligned}$$

4. The two *fixed time* estimates provided by the hypothesis (B.9) give:

(B.14)

$$\begin{aligned} |\langle U(t)^* G(t), U(s)^* F(s)\rangle|_{L^2(X)} &\leq A^2\|G(t)\|_{L^2(X)}\|F(s)\|_{L^2(X)} \,, \\ \langle U(t)^* G(t), U(s)^* F(s)\rangle_{L^2(X)} &\leq A|t-s|^{-\sigma}\|G(t)\|_{L^1(X)}\|F(s)\|_{L^1(X)} \,. \end{aligned}$$

Real interpolation between the estimates (B.14) gives

$$\text{(B.15)}\qquad \begin{aligned} &|\langle U(t)^* G(t), U(s)^* F(s)\rangle| \\ &\qquad \leq A^{3-2/p'}|t-s|^{-\sigma(2/p'-1)}\|G(t)\|_{L^{p'}(X)}\|F(s)\|_{L^{p'}(X)} \,, \end{aligned}$$

$1 \leq p' \leq 2$.

5. We now need to use the Hardy-Littlewood-Sobolev inequality which says that if $K_a(t) = |t|^{-1/a}$ and $1 < a < \infty$ then

$$\text{(B.16)}\qquad \begin{aligned} &\|K_a * u\|_{L^r(\mathbb{R})} \leq C\|u\|_{L^{p'}(\mathbb{R})} \,, \\ &\frac{1}{p} + \frac{1}{r} = \frac{1}{a} \,, \quad 1 < p' < r \,, \end{aligned}$$

see [H1, Theorem 4.5.3]. To obtain (B.13) from (B.15) we apply (B.16) with

$$\frac{1}{a} = \sigma\left(\frac{2}{p'} - 1\right) \,, \quad \frac{1}{p} + \frac{1}{r} = \frac{1}{a} \,, \quad p = r \,,$$

which has a unique solution

$$p = \frac{2(1+\sigma)}{\sigma} \,,$$

completing the proof. $\square$

APPENDIX C. DIFFERENTIAL FORMS

NOTATION.

(i) If $x = (x_1, \dots, x_n)$, $\xi = (\xi_1, \dots, \xi_n)$, then $dx_j, d\xi_j \in (\mathbb{R}^{2n})^*$ satisfy

$$\begin{aligned} dx_j(u) &= dx_j(x,\xi) = x_j \\ d\xi_j(u) &= d\xi_j(x,\xi) = \xi_j. \end{aligned}$$

(ii) If $\alpha, \beta \in (\mathbb{R}^{2n})^*$, then

$$(\alpha \wedge \beta)(u,v) := \alpha(u)\beta(v) - \alpha(v)\beta(u)$$

for $u, v \in \mathbb{R}^{2n}$.

(iii) If $f : \mathbb{R}^n \to \mathbb{R}$, the *differential* of f, is the 1-form

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

DEFINITION. If η is a 2-form and V a vector field, then the *contraction of* η *by* V, denoted

$$V \lrcorner \eta,$$

is the 1-form defined by

$$(V \lrcorner \eta)(u) = \eta(V, u).$$

DEFINITIONS. Let $\boldsymbol{\kappa} : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth mapping.

(i) If V is a vector field on $\mathbb{R}^n$, the *push-forward* is

$$\boldsymbol{\kappa}_* V = \partial \boldsymbol{\kappa}(V).$$

(ii) If η is a 1-form on $\mathbb{R}^n$, the *pull-back* is

$$(\boldsymbol{\kappa}^* \eta)(u) = \eta(\boldsymbol{\kappa}_* u).$$

THEOREM C.1 (Differentials and pull-backs). *We have*

$$d(\boldsymbol{\kappa}^* f) = \boldsymbol{\kappa}^*(df). \tag{C.1}$$

Proof. First of all, $d(\boldsymbol{\kappa}^* f) = d(\boldsymbol{\kappa}(f)) = \sum_{j=1}^{n} \frac{\partial y_i}{\partial x_j} \frac{\partial f}{\partial y_i} dx_j$. Furthermore,

$$\boldsymbol{\kappa}^*(df) = \boldsymbol{\kappa}^* \left(\sum_{i=1}^{n} \frac{\partial f}{\partial y_i} dy_i \right) = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i} \boldsymbol{\kappa}^*(dy_i).$$

$\square$

DEFINITION. If V is a vector field generating the flow φ_t, then the *Lie derivative* of w is

$$\mathcal{L}_V w := \frac{d}{dt}((\varphi_t)^* w)|_{t=0}.$$

Here w denotes a function, a vector field or a form.

EXAMPLE S. (i) If f is a function,

$$\mathcal{L}_V f = V(f).$$

(ii) If W is a vector field

$$\mathcal{L}_V W = [V, W].$$

THEOREM C.2 (Cartan's formula). *If w is a differential form,*

$$\mathcal{L}_V w = d(V \lrcorner w) + (V \lrcorner dw). \tag{C.2}$$

THEOREM C.3 (Poincaré's Lemma). *If α is a k-form defined in the open ball $U = B^0(0, R)$ and if*

$$d\alpha = 0,$$

then there exists a $(k-1)$ form ω in U such that

$$d\omega = \alpha.$$

Proof. 1. Let $\Omega^k(U)$ denote the space of k-forms on U. We will build a linear mapping

$$H : \Omega^k(U) \to \Omega^{k-1}(U)$$

such that

$$d \circ H + H \circ d = I. \tag{C.3}$$

Then

$$d(H\alpha) + H d\alpha = \alpha$$

and so $d\omega = \alpha$ for $\omega := H\alpha$.

2. Define $A : \Omega^k(U) \to \Omega^k(U)$ by

$$A(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \left(\int_0^1 t^{k-1} f(tp)\, dt\right) dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Set

$$X := \langle x, \partial_x \rangle = \sum_{j=1}^n x_j \frac{\partial}{\partial x_j}.$$

We claim

$$\text{(C.4)} \qquad A\mathcal{L}_X = I \quad \text{on } \Omega^k(U).$$

and

$$\text{(C.5)} \qquad d \circ A = A \circ d.$$

Assuming these assertions, define

$$H := A \circ X \lrcorner .$$

By Cartan's formula, Theorem C.2,

$$\mathcal{L}_X = d \circ (X \lrcorner) + X \lrcorner \circ d.$$

Thus

$$\begin{aligned} I = A\mathcal{L}_X &= A \circ d \circ (X \lrcorner) + A \circ X \lrcorner \circ d \\ &= d(A \circ X \lrcorner) + (A \circ X \lrcorner) \circ d \\ &= d \circ H + H \circ d; \end{aligned}$$

and this proves (C.3).

3. To prove (C.4), we compute

$$\begin{aligned} &A\mathcal{L}_X(f dx_{i_1} \wedge \cdots \wedge dx_{i_2}) \\ &\qquad = A\left[\left(kf + \sum_{j=1}^n x_j \frac{\partial f}{\partial x_j}\right)(dx_{i_1} \wedge \cdots \wedge dx_{i_k})\right] \\ &\qquad = \int_0^1 kt^{k-1} f(tp) + \sum_{j=1}^n t^{k-1} x_j \frac{\partial f}{\partial x_j}(tp) \\ &\qquad\qquad\qquad\qquad dt dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &\qquad = \int_0^1 \frac{d}{dt}(t^k f(tp))\, dt dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &\qquad = f dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \end{aligned}$$

4. To verify (C.5), note

$$
\begin{aligned}
A \circ d(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \\
&= A\left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right) \\
&= \left(\int_0^1 t^{k-1} \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(tp) dx_j dt\right) dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= d\left(\left(\int_0^1 t^{k-1} f(tp) dt\right) dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right) \\
&= d \circ A(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}).
\end{aligned}
$$

$\square$

Appendix D. Symbol calculus on manifolds

D.1 Definitions. For reader's convenience we provide here some basic definitions.

DEFINITION. An n-dimensional *manifold* M is a Hausdorff topological space with a countable basis, each point of which has a neighbourhood homeomorphic to some open set in $\mathbb{R}^n$.

We say that M is a *smooth* (or C^∞) manifold if there exists a family $\mathcal{F}$ of homeomorphisms between open sets:

$$\boldsymbol{\kappa} \,:\, U_{\boldsymbol{\kappa}} \longrightarrow V_{\boldsymbol{\kappa}}, \quad U_{\boldsymbol{\kappa}} \subset M, \quad V_{\boldsymbol{\kappa}} \subset \mathbb{R}^n,$$

satisfying the following properties:

(i)(Smooth overlaps) If $\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2 \in \mathcal{F}$ then

$$\boldsymbol{\kappa}_2 \circ \boldsymbol{\kappa}_1^{-1} \in C^\infty(V_{\boldsymbol{\kappa}_2} \cap V_{\boldsymbol{\kappa}_1}; V_{\boldsymbol{\kappa}_1} \cap V_{\boldsymbol{\kappa}_2}).$$

(ii)(Covering) The open sets $U_{\boldsymbol{\kappa}}$ cover M:

$$\bigcup_{\boldsymbol{\kappa}\in\mathcal{F}} U_{\boldsymbol{\kappa}} = M.$$

(iii) (Maximality) Let $\boldsymbol{\lambda}$ be a homeomorphism of an open set $U_{\boldsymbol{\lambda}} \subset M$ onto an open set $V_{\boldsymbol{\lambda}} \subset \mathbb{R}^n$. If for all $\boldsymbol{\kappa} \in \mathcal{F}$,

$$\boldsymbol{\kappa} \circ \boldsymbol{\lambda}^{-1} \in C^\infty(V_{\boldsymbol{\lambda}} \cap V_{\boldsymbol{\kappa}}; V_{\boldsymbol{\lambda}} \cap V_{\boldsymbol{\kappa}}),$$

then $\boldsymbol{\lambda} \in \mathcal{F}$.

We call $\{(\boldsymbol{\kappa}, U_{\boldsymbol{\kappa}}) \mid \boldsymbol{\kappa} \in \mathcal{F}\}$ an *atlas* for M. The open set $U_{\boldsymbol{\kappa}} \subset M$ is a *coordinate patch*.

DEFINITION. A C^∞ complex *vector bundle* over M with fiber dimension N consists of

(i) a C^∞ manifold V,

(ii) a C^∞ map $\pi : V \to M$, defining the *fibers* $V_x := \pi^{-1}(\{x\})$ for $x \in M$, and

(iii) local isomorphisms

$$\begin{gathered} V \supset \pi^{-1}(Y) \xrightarrow{\psi} Y \times \mathbb{C}^N, \\ \psi(V_x) = \{x\} \times \mathbb{C}^N, \quad \psi|_{V_x} \in GL(N, \mathbb{C}), \end{gathered} \tag{D.1}$$

where $GL(N,\mathbb{C})$ is the group of invertible linear transformations on $\mathbb{C}^N$.

REMARKS. (i) We can choose a covering $\{X_i\}_{i\in I}$ of M such that for each index i there exists

$$\psi_i : \pi^{-1}(X_i) \to X_i \times \mathbb{C}^N$$

with the properties listed in (iii) in the definition of a vector bundle.

Then

$$g_{ij} := \psi_i \circ \psi_j^{-1} \in C^\infty(X_i \cap X_j; GL(N, \mathbb{C}))\,.$$

These maps are the *transition matrices.*

(ii) It is important to observe that we can recover the vector bundle V from the transition matrices. To see this, suppose that we are given functions g_{ij} satisfying the identities

$$\begin{cases} g_{ij}(x) \circ g_{ji}(x) = I & \text{for } \ x \in X_i \cap X_j, \\ g_{ij}(x) \circ g_{jk}(x) \circ g_{kj}(x) = I, & \text{for } \ x \in X_i \cap X_j \cap X_k. \end{cases}$$

Now form the set $V' \subset I \times M \times \mathbb{C}^N$, with the equivalence relation $(i, x, t) \sim (i', x', t')$ if and only if $x = x'$ and $t' = g_{i'i}(x)t$. Then

$$V = V'/\sim\,.$$

□

DEFINITION. A *section* of the vector bundle V is a smooth map

$$u : M \to V$$

such that

$$\pi \circ u(x) = x \quad (x \in M).$$

We write

$$u \in \mathbb{C}^\infty(M, V).$$

EXAMPLE 1: Tangent bundle. Let M be a C^∞ manifold and let N be the dimension of M. We define the *tangent bundle* of M, denoted

$$T(M),$$

by defining the transition functions

$$g_{\boldsymbol{\kappa}_i\boldsymbol{\kappa}_j}(x) := \partial(\boldsymbol{\kappa}_i \circ \boldsymbol{\kappa}_j^{-1})(x) \in GL(n, \mathbb{R}).$$

for $x \in U_{\boldsymbol{\kappa}_i} \cap U_{\boldsymbol{\kappa}_j}$. Its sections $C^\infty(M, T(M))$ are the smooth *vectorfields* on M. □

EXAMPLE 2: Cotangent bundle. For any vector bundle we can define its dual,

$$V^* := \bigcup_{x\in X} (V_x)^*,$$

since we can take

$$g_{\boldsymbol{\kappa}_i \boldsymbol{\kappa}_j} = (g^*_{\boldsymbol{\kappa}_i \boldsymbol{\kappa}_j})^{-1}.$$

If $V = T(M)$, we obtain the *cotangent bundle*, denoted

$$T^*(M).$$

Its sections $C^\infty(M, T^*(M))$ are the differential *one-forms* on M. □

EXAMPLE 3: S-density bundles. Let M be an n-dimensional manifold and let $(U_{\boldsymbol{\kappa}}, \boldsymbol{\kappa}_\alpha)$ form a set of coordinate patches of X.

We define the *s-density bundle* over X, denoted

$$\Omega^s(M),$$

by choosing the following transition functions:

$$g_{\boldsymbol{\kappa}_i \boldsymbol{\kappa}_j}(x) := |\det \partial(\boldsymbol{\kappa}_i \circ \boldsymbol{\kappa}_j^{-1})|^s \circ \boldsymbol{\kappa}_j(x),$$

for $x \in U_{\boldsymbol{\kappa}_i} \cap U_{\boldsymbol{\kappa}_j}$.

This is a *line bundle* over M, that is, a bundle with with fibers of complex dimension one. □

D.2 Pseudodifferential operators on manifolds.

Pseudodifferential operators. In this section M denotes a smooth, n-dimensional compact Riemannian manifold without boundary. As above, we have $\{(\boldsymbol{\kappa}, U_{\boldsymbol{\kappa}}) \mid \boldsymbol{\kappa} \in \mathcal{F}\}$ for the atlas of M, where each $\boldsymbol{\kappa}$ is a smooth diffeomorphism of the coordinate patch $U_{\boldsymbol{\kappa}} \subset M$ onto an open subset $V_{\boldsymbol{\kappa}} \subset \mathbb{R}^n$.

NOTATION. Recall that

$$S^k(\langle \xi \rangle^m) = \{a \in C^\infty(\mathbb{R}^{2n}) \mid \sup_{\mathbb{R}^{2n}} |\partial^\alpha a| \le C_\alpha h^{-k} \langle \xi \rangle^m\}.$$

The index k records how singular the symbol a is as $h \to 0$, and m controls the growth rate as $|\xi| \to \infty$.

DEFINITION. A linear operator

$$A : C^\infty(M) \to C^\infty(M)$$

is called a *pseudodifferential operator* if there exist integers m, k such that for each coordinate patch $U_{\boldsymbol{\kappa}}$, there exists a symbol $a_{\boldsymbol{\kappa}} \in S^k(\langle \xi \rangle^m)$ such that for any $\varphi, \psi \in C_c^\infty(U_{\boldsymbol{\kappa}})$ and for each $u \in C^\infty(M)$

$$\varphi A(\psi u) = \varphi \boldsymbol{\kappa}^* a^w_{\boldsymbol{\kappa}}(x, hD)(\boldsymbol{\kappa}^{-1})^*(\psi u). \tag{D.2}$$

NOTATION. (i) In this case, we write

$$A \in \Psi^{m,k}(M),$$

and sometimes call A a *quantum observable*.

(ii) To simplify notation, we also put

$$\Psi^k(M) := \Psi^{0,k}(M), \quad \Psi(M) := \Psi^{0,0}(M).$$

The symbol of a pseudodifferential operator. Our goal is to associate with a pseudodifferential operator A a symbol a defined on T^*M, the cotangent space of M.

LEMMA D.1 (More on disjoint support). *Let $b \in S^k(\langle\xi\rangle^m)$ and suppose $\varphi, \psi \in C_c^\infty(\mathbb{R}^n)$. If*

$$\operatorname{spt}(\varphi) \cap \operatorname{spt}(\psi) = \emptyset, \tag{D.3}$$

then

$$\|\varphi b^w(x, hD)\psi\|_{H^{-m}\to H^m} = O(h^\infty) \tag{D.4}$$

for each m.

Proof. We have

$$\begin{aligned}
&\varphi b^w(x, hD)\psi(x) \\
&= \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} b\left(\frac{x+y}{2}, \xi\right)\psi(y)\varphi(x)e^{\frac{i\langle x-y,\xi\rangle}{h}}dxd\xi \\
&= \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} b\left(\frac{x+y}{2}, \xi\right)\frac{\psi(y)\varphi(x)}{|x-y|^m}|x-y|^m e^{\frac{i\langle x-y,\xi\rangle}{h}}dyd\xi.
\end{aligned}$$

Since $\operatorname{spt}\varphi \cap \operatorname{spt}\psi = \emptyset$, we see that $|x-y| \geq d > 0$ for $x \in \operatorname{spt}\varphi$, $y \in \operatorname{spt}\psi$. Furthermore

$$(x-y)^\alpha e^{\frac{i\langle x-y,\xi\rangle}{h}} = h^{|\alpha|}D_\xi^\alpha e^{\frac{i\langle x-y,\xi\rangle}{h}}.$$

Integrating by parts, we deduce that

$$|\varphi b^w\psi| \leq Ch^{m-n}$$

for all m. □

THEOREM D.2 (Symbol of a pseudodifferential operator). *There exist linear maps*

$$\sigma : \Psi^{m,k}(M) \to S^{m,k}/S^{m,k-1}(T^*M) \tag{D.5}$$

and

$$\mathrm{Op} : S^{m,k}(T^*M) \to \Psi^{m,k}(M) \tag{D.6}$$

such that

$$\sigma(A_1 A_2) = \sigma(A_1)\sigma(A_2) \tag{D.7}$$

and

$$\sigma(\mathrm{Op}(a)) = [a] \in S^{m,k}/S^{m,k-1}(T^*M). \tag{D.8}$$

We call $a = \sigma(A)$ the *symbol* of the pseudodifferential operator A.

REMARK. In the identity (D.8) "$[a]$" denotes the equivalence class of a in $S^{m,k}/S^{m,k-1}(T^*M)$. This means that

$$[a] = [\hat{a}] \quad \text{if and only if} \quad a - \hat{a} \in S^{m,k-1}(T^*M).$$

The symbol is therefore uniquely defined in $S^{m,k}$, up to a lower order term which is less singular as $h \to 0$. □

Proof. 1. Let U be an open subset of $\mathbb{R}$. Suppose that $B : C^\infty_{\mathrm{c}}(U) \to C^\infty(U)$ and that for all $\varphi, \psi \in C^\infty$ the mapping $u \longmapsto \varphi B \psi u$ belongs to $\Psi^{m,k}(\mathbb{R}^n)$, for all $u \in \mathcal{S}$.

We claim that there then exists a symbol $a \in S^k_{\mathrm{loc}}(U, \langle\xi\rangle^m)$ such that

$$B = a(x, D) + B_0, \tag{D.9}$$

where for all m

$$B_0 : H^{-m}_{\mathrm{c}}(U) \to H^m_{\mathrm{loc}}(U) \quad \text{is } O(h^\infty). \tag{D.10}$$

To see this, first choose a locally finite partition of unity $\{\psi_j\}_{j \in J} \subset C^\infty_{\mathrm{c}}(U)$:

$$\sum_{j \in J} \psi_j(x) \equiv 1 \qquad (x \in U).$$

Then

$$\psi_j B \psi_k = a^w_{jk}(x, hD),$$

where $a_{jk} \in S^k(\langle\xi\rangle)$ and $a_{jk}(x,\xi) = 0$ if $x \notin \mathrm{spt}\psi_j$. Now put

$$a := {\sum_{j,k}}' a_{jk}(x,\xi) \in S^k_{\mathrm{loc}}(\langle\xi\rangle^m),$$

where we are sum over those indices j, k's for which $\mathrm{spt}\psi_j \cap \mathrm{spt}\psi_k \neq \emptyset$. This sum is consequently locally finite.

2. We must next verify (D.10) for

$$B_0 := B - a(x, hD) = \sum_{j,k}{}'' \psi_j B \psi_k,$$

the sum over j, k's for which

$$\mathrm{spt}\psi_j \cap \mathrm{spt}\psi_k = \emptyset.$$

Let $K_B(x, y)$ be the Schwartz kernel of B. Then the Schwartz kernel of B_0 is

$$K_{B_0}(x, y) = \sum_{j,k}{}'' \psi_j(x) K_B(x, y) \psi_k(y), \tag{D.11}$$

with the sum locally finite in $U \times U$. The operators $\psi_j B \psi_k$ satisfy the assumptions of Lemma 7.4, and hence have the desired mapping property. Because of the local finiteness of (D.11) we get the global mapping property from H^{-m}_{loc} to H^m_{loc}.

3. For each coordinate chart $(\boldsymbol{\kappa}, U_{\boldsymbol{\kappa}})$, where $\boldsymbol{\kappa} : U_{\boldsymbol{\kappa}} \to V_{\boldsymbol{\kappa}}$, we can now use (D.9) with $X = V_{\boldsymbol{\kappa}}$ and $B = (\boldsymbol{\kappa}^{-1})^* A \boldsymbol{\kappa}^*$, to define $a_{\boldsymbol{\kappa}} \in T^*(U_{\boldsymbol{\kappa}})$.

The second part of Theorem 8.1 shows that if $U_{\boldsymbol{\kappa}_1} \cap U_{\boldsymbol{\kappa}_2} \neq \emptyset$, then

$$(a_{\boldsymbol{\kappa}_1} - a_{\boldsymbol{\kappa}_2})|_{U_{\boldsymbol{\kappa}_1} \cap U_{\boldsymbol{\kappa}_2}} \in S^{k-1}(T^*(U_{\boldsymbol{\kappa}_1} \cap U_{\boldsymbol{\kappa}_2}), \langle \xi \rangle^m). \tag{D.12}$$

Suppose now that we choose a covering of M by coordinate charts, $\{U_\alpha\}_{\alpha \in J}$, and a locally finite partition of unity $\{\varphi_\alpha\}_{\alpha \in J}$:

$$\mathrm{spt}\varphi_\alpha \subset U_{\boldsymbol{\kappa}}, \quad \sum_{\alpha \in J} \varphi_j(x) \equiv 1,$$

and define

$$a := \sum_{\alpha \in J} \varphi_\alpha a_\alpha.$$

We see from (D.12) that $a \in S^k(T^*M, \langle \xi \rangle^m)$ is invariantly defined up to terms in $S^{k-1}(T^*M, \langle \xi \rangle^m)$. We consequently can define

$$\sigma(A) := [a] \in S^k(T^*M, \langle \xi \rangle^m)/S^{k-1}(T^*M, \langle \xi \rangle^m).$$

4. It remains to show the existence of

$$\mathrm{Op} \; : \; S^k(T^*M, \langle \xi \rangle^m) \longrightarrow \Psi^{m,k}(M), \quad \sigma(\mathrm{Op}(a)) = [a].$$

Suppose that for our covering of M by coordinate charts, $\{U_\alpha\}_{\alpha \in J}$, we choose $\{\psi_\alpha\}_{\alpha \in J}$ such that

$$\mathrm{spt}\psi_\alpha \subset U_{\boldsymbol{\kappa}}, \quad \sum_{\alpha \in J} \psi_j^2(x) \equiv 1,$$

a sum which is locally finite. Define

$$A := \sum_{\alpha \in J} \psi_\alpha \boldsymbol{\kappa}_\alpha^* \mathrm{Op}(\tilde{a}_\alpha)(\boldsymbol{\kappa}_\alpha^{-1})^* \psi_\alpha,$$

where $\tilde{a}_\alpha(x,\xi) := a(\boldsymbol{\kappa}_\alpha^{-1}(x), (\partial\boldsymbol{\kappa}(x)^T)^{-1}\xi)$. Theorem 8.1 demonstrates that $\sigma(A)$ equals $[a]$. □

Pseudodifferential operators acting on half-densities. We now apply the full strength of Theorem 8.1 by making the pseudodifferential operators act on half-densities.

DEFINITION. A linear operator

$$A : C^\infty(M, \Omega^{\frac{1}{2}}(M)) \to C^\infty(M, \Omega^{\frac{1}{2}}(M))$$

is called a *pseudodifferential operator on half-densities* if there exist integers m, k such that for each coordinate patch U_α, and there exists a symbol $a_\alpha \in S^k(\langle\xi\rangle^m)$ such that for any $\varphi, \psi \in C_{\mathrm{c}}^\infty(U_{\boldsymbol{\kappa}})$

$$\varphi A(\psi u) = \varphi \boldsymbol{\kappa}_\alpha^* a_\alpha^w(x, hD)(\boldsymbol{\kappa}_\alpha^{-1})^*(\psi u) \tag{D.13}$$

for each $u \in C^\infty(M, \Omega^{\frac{1}{2}}(M))$.

NOTATION. In this case, we write

$$A \in \Psi^{m,k}(M, \Omega^{\frac{1}{2}}(M)).$$

By adapting the proof of Theorem D.2 to the case of half-densities using the first part of Theorem 8.1 we obtain

THEOREM D.3 (Symbol on half-densities). *There exist linear maps*

$$\sigma : \Psi^{m,k}(M, \Omega^{1/2}(M)) \to S^{m,k}/S^{m,k-2}(T^*M) \tag{D.14}$$

and

$$\mathrm{Op} : S^{m,k}(T^*M) \to \Psi^{m,k}(M, \Omega^{1/2}(M))) \tag{D.15}$$

such that

$$\sigma(A_1 A_2) = \sigma(A_1)\sigma(A_2) \tag{D.16}$$

and

$$\sigma(\mathrm{Op}(a)) = [a] \in S^{m,k}/S^{m,k-2}(T^*M)\,. \tag{D.17}$$

D.3 PDE on manifolds.

We revisit in this last section some of our theory from Chapters 5–7, replacing the flat spaces $\mathbb{R}^n$ and $\mathbb{T}^n$ by an arbitrary compact Riemannian manifold (M, g), for the metric

$$g := \sum_{i,j=1}^{n} g_{ij} dx_i dx_j.$$

Write

$$((g^{ij})) := ((g_{ij}))^{-1}, \quad \bar{g} := \det((g_{ij})).$$

D.3.1 Notation.

Tangent, cotangent bundles. We can use the metric to build an identification of the tangent and cotangent bundles of M. We identify

$$\xi \in T_x^* M \quad \text{with} \quad X \in T_x M,$$

written $\xi \sim X$, provided

$$\xi(Y) = g_x(Y, X)$$

for all $Y \in T_x M$.

Flows. Under the identification $X \sim \xi$, the flow of H_p on T^*M, generated by the symbol

$$p := |\xi|_g^2 = \sum_{i,j=1}^{n} g^{ij} \xi_i \xi_j = \sum_{i,j=1}^{n} g_{ij} X_i X_j = g(X, X), \tag{D.18}$$

is the geodesic flow on TM.

Laplace-Beltrami operator. The *Laplace-Beltrami operator* Δ_g on M is defined in local coordinates by

$$\Delta_g := \frac{1}{\sqrt{\bar{g}}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(g^{ij} \sqrt{\bar{g}} \frac{\partial}{\partial x_j} \right). \tag{D.19}$$

The function p defined by (D.18) is the symbol of the Laplace-Beltrami operator $-h^2 \Delta_g$.

PDE on manifolds. Given then a potential $V \in C^\infty(M)$, we can define the Schrödinger operator

$$P(h) := -h^2 \Delta_g + V(x). \tag{D.20}$$

The flat wave equation from Chapter 5 is replaced by an equation involving the Laplace-Beltrami operator:

$$(\partial_t^2 + a(x)\partial_t - \Delta_g)u = 0. \tag{D.21}$$

The unknown u is a function of $x \in M$ and $t \in \mathbb{R}$.

Half-densities. Half-densities on M can be identified with functions using the Riemannian density:

$$u = u(x)|dx|^{\frac{1}{2}} = \tilde{u}(x)\left(\bar{g}^{\frac{1}{2}}dx\right)^{\frac{1}{2}}.$$

D.3.2 Damped wave equation on manifolds. We consider this initial-value problem for the wave equation:

$$\text{(D.22)}\qquad \begin{cases} (\partial_t^2 + a(x)\partial_t - \Delta)u = 0 & \text{on } M \times \mathbb{R} \\ \quad u = 0,\ u_t = f & \text{on } M \times \{t = 0\}, \end{cases}$$

where $a \geq 0$; and, as in Chapter 6, define the energy of a solution at time t to be

$$E(t) := \frac{1}{2}\int_M (\partial_t u)^2 + |\partial_x u|^2\, dx.$$

It is then straightforward to adapt the proofs in §5.3 to establish

THEOREM D.4 (Exponential decay on manifold). *Suppose u solves the wave equation with damping* (D.21)*, with the initial conditions*

Assume also that there exists a time $T > 0$ such that each geodesic of length greater than or equal to T intersects the set $\{a > 0\}$.

Then there exist constants $C, \beta > 0$ such that

$$\text{(D.23)}\qquad E(t) \leq Ce^{-\beta t}\|f\|_{L^2}$$

for all times $t \geq 0$.

D.3.3 Weyl's Law for compact manifolds. More work is needed to generalize Weyl's Law from Chapter 6 to manifolds. We will prove it using a different approach, based on the Spectral Theorem.

First, we need to check that the spectrum is discrete and that follows from the compactness of the resolvent:

LEMMA D.5 (Resolvent on manifold). *If P is defined by* (D.20)*, then*

$$(P + i)^{-1} = O(1)\ :\ L^2(M) \to H_h^2(M)\,,$$

where the semiclassical Sobolev spaces are defined as in §7.1.

We prove this by the same method as that for Lemma 7.1.

Eigenvalues and eigenfunctions. According to Riesz's Theorem on the discreteness of the spectrum of a compact operator, we conclude that the spectrum of $(P+i)^{-1}$ is discrete, with an accumulation point at 0.

Hence we can write

$$P(h)=\sum_{j=1}^{\infty} E_j(h)u_j(h)\otimes\overline{u_j(h)}, \tag{D.24}$$

where $\{u_j(h)\}_{j=1}^{\infty}$ is an orthonormal set of all eigenfunctions of $P(h)$:

$$P(h)u_j(h)=E_j(h)u_j(h), \quad \langle u_k(h),u_l(h)\rangle=\delta_{lk},$$

and

$$E_j(h)\to\infty.$$

THEOREM D.6 (Functional calculus). *Suppose that f is a holomorphic function, such that for $|\mathrm{Im} z|\le 2$ and any N:*

$$f(z)=O(\langle z\rangle^{-N}).$$

Define

$$\begin{aligned} &f(P):= \\ &\frac{1}{2\pi i}\int_{\mathbb{R}}(t-i-P)^{-1}f(t-i)-(t+i-P)^{-1}f(t+i)\,dt. \end{aligned} \tag{D.25}$$

Then $f(P)\in\Psi^{-\infty}(M)$, with

$$\sigma(f(P))=f(|\xi|_g^2+V(x)).$$

Furthermore,

$$f(P)=\sum_{j=1}^{\infty} f(E_j(h))u_j(h)\otimes\overline{u_j(h)} \tag{D.26}$$

in L^2.

Proof. 1. The statement (D.26) follows from (D.24), which shows that

$$(P-z)^{-1}=\sum_{j=1}^{\infty}\frac{u_j\otimes\overline{u_j}}{E_j(h)-z}.$$

Since f decays rapidly as $t \to \infty$, we can compute residues in (D.25) to conclude that

$$f(E_j(h)) = \frac{1}{2\pi i}\int_{\mathbb{R}} \big((t-i-E_j(h))^{-1}f(t-i) - (t+i-E_j(h))^{-1}f(t+i)\big)\,dt.$$

2. We now use Beals's Theorem 8.9, to deduce that $f(P)$ is a pseudodifferential operator. As discussed in Appendix E all we need to show is that for $\varphi, \psi \in C_0^\infty(M)$, with supports in arbitrary coordinate patches, $\varphi f(P)\psi$ is a pseudodifferential operator. As described there it can be considered as an operator on $\mathbb{R}^n$ and, by Theorem 8.9, it suffices to check that for any linear $\ell_j(x,\xi)$ we have

$$\|\mathrm{ad}_{\ell_1(x,hD)} \circ \cdot \circ \mathrm{ad}_{\ell_N(x,hD)} f(P)\|_{L^2\to L^2} = O(h^N)\,.$$

To show this, note that according to Lemma D.5,

$$\|(P-t\pm i)^{-1}(\mathrm{ad}_{L_1} \circ \cdots \circ \mathrm{ad}_{L_k} P(P-t\pm i)^{-1}\|_{L^2\to L^2} = O(h^k)\,,$$

where $L_j \in \Psi^{0,0}(M)$. Now for a linear function ℓ on $\mathbb{R}^{2n}$,

$$\mathrm{ad}_{\ell(x,hD)}(\varphi(P-t\pm i)^{-1}\psi) = -(P-t\pm i)^{-1}(\mathrm{ad}_L P)(P-t\pm i)^{-1} + O_{L^2\to L^2}(h)\,,$$

where $L \in \Psi^{0,0}(M)$. The rapid decay of f gives

$$\|\mathrm{ad}_L \int_{\mathbb{R}} f(t)(t\pm i-P)^{-1}dt\| \leq \int_{\mathbb{R}} |f(t)|\|(P-t\pm i)^{-1}\mathrm{ad}_L P(P-t\pm i)^{-1}\|_{L^2\to L^2}dt = O(h)\,,$$

and this argument can be easily iterated.

3. Since

$$\mathrm{Op}(|\xi|_g^2 + V(x) - t \pm i)^{-1})(P-t\pm i) = I + O_{L^2\to L^2}(h),$$

it follows that

$$\mathrm{Op}(|\xi|_g^2 + V(x) - t \pm i)^{-1}) = (P-t\pm i)^{-1} + O_{L^2\to L^2}(h).$$

Hence the symbol of $(P+t\pm i)^{-1}$ (which we already know is a pseudodifferential operator) is given by $(|\xi|_g^2 + V(x) - t\pm i)^{-1}$.

A residue calculation now shows us that

$$f(P) = \mathrm{Op}(f(|\xi|_g^2 + V(x) - t\pm i)) + O_{L^2\to L^2}(h);$$

that is, the symbol of $f(P)$ is $f(|\xi|_g^2 + V(x))$. $\square$

THEOREM D.7 (Weyl's asymptotics on compact manifolds). *For any $a < b$, we have*

$$\begin{aligned} &\#\{E(h) \mid a \leq E(h) \leq b\} = \\ &\qquad \frac{1}{(2\pi h)^n}(\mathrm{Vol}_{T^*M}\{a \leq |\xi|_g^2 + V(x) \leq b\} + o(1)) \end{aligned} \tag{D.27}$$

as $h \to 0$.

Proof. 1. Let f_1, f_2 be two functions satisfying the assumptions of Theorem D.6 such that for real x

$$f_1(x) \leq \mathbf{1}_{[a,b]}(x) \leq f_2(x), \tag{D.28}$$

where $\mathbf{1}_{[a,b]}(x)$ is the characteristic function of the interval $[a, b]$.

It follows that

$$\mathrm{tr} f_1(P) \leq \#\{E(h) \mid a \leq E(h) \leq b\} \leq \mathrm{tr} f_2(P)\,.$$

2. Theorem B.9 now shows that for $j = 1, 2$

$$\mathrm{tr} f_j(P) = \frac{1}{(2\pi h)^n}\left(\int_{T^*M} f_j(|\xi|_g^2 + V(x))dxd\xi + O(h)\right).$$

We note that since $f_j(P) \in \Psi^{-\infty}(M)$, the errors in the symbolic computations are all $O(h\langle\xi\rangle^{-\infty})$, and hence can be integrated.

3. The final step is to construct f_1^ϵ and f_2^ϵ satisfying the hypotheses of Theorem D.6 and (D.28), and such that for $j = 1, 2$, we have

$$\int_{T^*M} f_j^\epsilon(|\xi|_g^2 + V(x))dxd\xi \to \mathrm{Vol}_{T^*M}\{a \leq |\xi|_g^2 + V(x) \leq b\},$$

as $\epsilon \to 0$. This is done as follows. Define

$$\chi_1^\epsilon := (1-\epsilon)\mathbf{1}_{[a+\epsilon,b-\epsilon]} - \epsilon(\mathbf{1}_{[a-\epsilon,a+\epsilon]} + \mathbf{1}_{[b-\epsilon,b+\epsilon]})\,, \quad \chi_2^\epsilon := (1+\epsilon)\mathbf{1}_{[a-\epsilon,b+\epsilon]}\,,$$

and then put

$$f_j^\epsilon(z) := \frac{1}{2\pi\epsilon}\int_{\mathbb{R}} \chi_j^\epsilon(x)\exp\left(-\frac{(x-z)^2}{2\epsilon^2}\right)dx.$$

We easily check that all the assumptions are satisfied. □

REMARKS. (i) If $V \equiv 0$, we recover the leading term in the usual Weyl asymptotics of the Laplacian on a compact manifold: let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to \infty$ be the complete set of eigenvalues of $-\Delta_g$ on M. Then

$$\#\{j \;:\; \lambda_j \leq r\} \sim \frac{\mathrm{Vol}(B_{\mathbb{R}^n}(0,1))}{(2\pi)^n}\mathrm{Vol}(M)r^{n/2}\,, \quad r \to \infty\,. \tag{D.29}$$

In fact, we can take $a = 0$, $b = 1$, and $h = 1/\sqrt{r}$, and apply Theorem D.7: the eigenvalues $-\Delta_g$ are just rescaled eigenvalues of $-h^2\Delta_g$ and the $\mathrm{Vol}(B_{\mathbb{R}^n}(0,1))$ term comes from integrating out the ξ variables.

We note also that (D.29) implies that

$$j^{2/n}/C_M \leq \lambda_j \leq C_M j^{2/n}. \tag{D.30}$$

(ii) Also, upon rescaling and applying Theorem B.4, we obtain estimates for counting *all* the eigenvalues of $P(h) = -h^2\Delta_g + V(x)$. Let $E_0(h) < E_1(h) \leq \cdots \leq E_j(h) \to \infty$ be all the eigenvalues of the self-ajoint operator $P(h)$. Then for $r > 1$,

$$\#\{j \ : \ E_j(h) \leq r\} \leq C_{M,V} h^{-n} r^{n/2} \,. \tag{D.31}$$

This crude estimate will be useful in §9.3. $\square$

Sources and further reading

Chapter 1: Griffiths [G] is a nice elementary introduction to quantum mechanics. For a modern physical perspective one may consult Heller-Tomsovic [H-T] and Stöckmann [St].

Chapter 2: The proof of Theorem 2.8 is due to Moser [M]. A PDE oriented introduction to symplectic geometry is contained in [H2, Chapter 21].

Chapter 3: Good references are Friedlander–Joshi [F-J] and Hörmander [H1]. The PDE example in §3.1 is from Hörmander [H1, Section 7.6].

Chapter 4: The presentation of semiclassical calculus is based upon Dimassi-Sjöstrand [D-S, Chapter 7]. See also Martinez [M], in particular for the Fefferman-Cordoba proof of the sharp Gårding inequality. The argument presented here followed the proof of [D-S, Theorem 7.12].

Chapter 5: Theorem 5.8 is due to Rauch–Taylor [R-T], but the proof here follows Lebeau [L] and uses also some ideas of Morawetz.

Chapter 6: The proof of Weyl asymptotics is a semiclassical version of the classical Dirichlet-Neumann bracketting proof for the bounded domains.

Chapter 7: Estimates in the classically forbidden region in §7.1 are known as Agmon or Lithner-Agmon estimates. They play a crucial role in the analysis of spectra of multiple well potential and of the Witten complex: see [D-S, Chapter 6] for an introduction and references. Here we followed an argument of [N], but see also [N-S-Z, Proposition 3.2]. The presentation of Carleman estimates in §7.2 is based on discussions with N. Burq and D. Tataru.

Chapter 8: The proof of symbol invariance is from [S-Z1, Appendix]. The semiclassical wavefront set is an analogue of the usual wavefront set in microlocal analysis – see [H2] and is closely related to the frequency set introduced in [G-S]. The discussion of semiclassical pointwise bounds is inspired by a recent article of Koch and Tataru [Ko-T].

Our presentation of Beals's Theorem follows Dimassi–Sjöstrand [D-S], where it was based on [H-S]. Theorem 8.10, in a much greater generality, was proved by Bony–Chemin in [B-C]. The self-contained proof in the simple case considered here comes from [S-Z3, Appendix].

Chapter 9: The Quantum Ergodicity Theorem 9.4 is from a 1974 paper of Shnirelman, and it is sometimes referred to as Shnirelman's

Theorem. The first complete proof, in a different setting, was provided by Zelditch. We have followed his more recent proof, as presented in [Z-Z]. The same proof applied with finer spectral asymptotics gives a stronger semiclassical version, first presented in [H-M-R].

Chapter 10: The construction of $U(t)$ borrows from the essentially standard presentation in [S-Z1, Section 7]. For the discussion of the Maslov index see [G-S] and [L-V]. Fourier Integral Operators which are closely related to our discussion of quantization and of propagators are discussed in detail in [D] and [H2, Chapeter 25].

Semiclassical Strichartz estimates for $P = -h^2\Delta_g - 1$ appeared explicitely in the work of Burq-Gérard-Tzvetkov [B-G-T] who used them to prove existence results for non-linear Schrödinger equations on two and three dimensional compact manifolds. We refer to that paper for pointers to the vast literature on Strichartz estimates and their applications. The adaptation of Sogge's L^p estimates to the semiclassical setting was inspired by discussions with N. Burq, H. Koch, C.D. Sogge, and D. Tataru, see [Ko-T] and [S].

The proofs for the theorems cited in §10.4 are in Hörmander [H2, Theorem 21.1.6] and [H2, Theorem 21.1.6]. Theorem 10.18 is a semiclassical analog of the standard C^∞ result of Duistermaat-Hörmander [H2, Proposition 26.1.3′]. Theorem 10.19 is a semiclassical adaptation of a microlocal result of Duistermaat-Sjöstrand [H2, Proposition 26.3.1].

Theorem 10.20 was proved in one dimension by Davies [D]. See [D-S-Z] for more on quasimodes and pseudospectra and for further references.

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